TORSION AND EXCEPTIONAL UNITS

DINO LORENZINI

APPENDIX A. DATA

Let $K/\mathbb{Q}$ be a number field. Let $N \geq 5$ be prime. We say that E/K is an N-special elliptic curve if $E(K)$ contains a point of order N and the Tamagawa number $c(E/K)$ is not divisible by N. We list below the results of our search for N-special elliptic curves E/K when $[K : \mathbb{Q}] = d$ and $8 \leq d \leq 12$. When $d \leq 7$, the data appears in [4]. Our search mostly used the algorithm 4.6 proposed in [4].

A.1 Number fields of degree 8. We include below the octic fields K where we found at least one N-special elliptic curve E/K with $N = 17$, 19, or 23. When $N = 17$, we do not include the fields K that contains the quartic field of discriminant 725 since this field appears in the table of quartic fields in [4], 1.4. The notation is as in 1.5 of [4].

N	field K (degree 8 and rank 3)	rk	#exu	discr(K)
$11(12), 13(2), 17(2)^*$	$x^8 - 2x^7 + 4x^5 - 4x^4 + 3x^2 - 2x + 1$	3	438	1257728
$17(2)$	$\mathbb{Q}(\zeta_{15})$	3	440	1265625
$11(2), 13, \mathbf{23}$	$x^8 - x^7 + 2x^6 - 3x^5 + 3x^4 - 3x^3 + 3x^2 - 2x + 1$	3	432	1282789
$11(2), 17, \mathbf{19}$	$x^8 - x^7 + x^6 + x^4 - 2x^3 + 3x^2 - 3x + 1$	3	422	1342413
$11(2), 17$	$x^8 - 2x^7 + x^6 + 2x^5 - 3x^4 - 2x^3 + x^2 + 2x + 1$	3	374	1513728
$11(2), 13, 17(2)$	$x^8 - 2x^7 + 3x^5 - x^4 - 3x^3 + 2x + 1$	3	354	1578125

N	field K (degree 8 and rank 4)	rk	#exu	discr(K)
17	$x^8 - x^7 - 2x^6 + x^5 + x^3 + 2x^2 - 1$	4	1116	-4297507
$11(6), 13, 17, \mathbf{19}$	$x^8 - 3x^7 + 5x^6 - 6x^5 + 4x^4 - x^3 - x^2 + x - 1$	4	1062	-4542739
$11(2), 13, 17$	$x^8 - x^6 - 2x^5 + x^4 + x^2 + 2x - 1$	4	1044	-4663051
$11(2), 17$	$x^8 - x^7 + 2x^6 - x^5 + x^3 - 2x^2 + 2x - 1$	4	1020	-4785667
$13, \mathbf{19}$	$x^8 + 3x^6 - x^5 + 2x^4 - 3x^3 - 2x + 1$	4	1008	-4858379
$11(4), 13, \mathbf{19}$	$x^8 - x^7 - x^6 + x^5 + x^2 - x - 1$	4	966	-5118587
$11(2), 17$	$x^8 - x^7 + x^6 - 2x^4 + x^3 - x^2 - x + 1$	4	930	-5204491
$11(2), 17$	$x^8 - 2x^7 + 2x^5 + x^4 - x^3 - 2x^2 + x + 1$	4	936	-5233147
$11(2), 13, 17$	$x^8 - x^7 - x^6 + 3x^5 - x^4 - 3x^3 + 2x^2 - 1$	4	966	-5272027

N	field K (degree 8 and rank 4, continued)	rk	#exu	discr(K)
17	$x^8 - x^6 - x^5 + x^4 + 2x^3 - 2x - 1$	4	912	-5346947
$11(2), 13, 17$	$x^8 - 2x^7 + x^6 + x^5 - 2x^4 + 2x^3 + x^2 - 2x - 1$	4	810	-6483187
$11(2), 17$	$x^8 - 2x^7 + 2x^6 - 3x^4 + 4x^3 - x^2 - x + 1$	4	726	-7245127
$11(2), 17$	$x^8 - x^7 - x^6 + 4x^5 - x^4 - 3x^3 + 3x^2 - 1$	4	684	-7751827

The field of prime discriminant -4542739 in the above table is the first field in our data where four different raw form equations have a point (r, s) with r, s exceptional units.

N	field K (degree 8 and rank 5)	rk	#exu	discr(K)
$11(2), 13, 17$	$x^8 - 5x^6 - x^5 + 7x^4 + 4x^3 - 2x^2 - 4x - 1$	5	2736	16324589
$11(6), \mathbf{19}$	$x^8 - 3x^6 - x^3 + 3x^2 - 1$	5	2784	16526789
$11(6), 17$	$x^8 - 2x^7 + x^5 - x^3 + x^2 + 2x - 1$	5	2652	17129069
$11(4), 13(3), 17$	$x^8 - 2x^7 + x^6 + x^5 - 4x^4 + 3x^3 - 2x + 1$	5	2520	17949581
$11(4)^*, 17(2)^{**}$	$x^8 - 3x^6 + 3x^4 - 3x^2 + 1$	5	2430	19360000
$11(4), 13, 17$	$x^8 - x^7 + 3x^5 - 3x^4 + 2x^2 - 2x - 1$	5	2268	20493125
$11(2), \mathbf{23}$	$x^8 - 2x^7 + x^6 + 3x^5 - 4x^4 + x^3 + 2x^2 - 2x - 1$	5	2160	22131469
17	$x^8 - x^5 + x^4 + 3x^3 - 3x^2 - x + 1$	5	2022	23511413
$11(2), 17$	$x^8 - 3x^7 + 3x^6 - x^5 - 3x^4 + 6x^3 - 5x^2 + 4x - 1$	5	1932	25338125
$11(4), 17$	$x^8 - x^7 - x^6 - 2x^4 + 6x^3 - 5x + 1$	5	1836	26393125
17	$x^8 + x^6 - 2x^4 - 2x^2 + 1$	5	1770	28047616
$13, \mathbf{19}(2)$	$x^8 - 2x^7 - x^6 + 8x^5 - 8x^4 - x^3 + 7x^2 - 6x + 1$	5	1692	30348397
$\mathbf{19}$	$x^8 - x^7 - 2x^6 + 5x^5 - 2x^4 - 4x^3 + 4x^2 - x - 1$	5	1422	38169253
$13, 17$	$x^8 - 3x^7 + 4x^6 - 4x^5 + 3x^3 - 5x^2 + 4x - 1$	5	1182	45393125

We did not find examples with $N \geq 17$ when the rank of K is 6 or 7 (and K does not contain the quartic field of discriminant 725). We include the following examples for information only. The analytic rank of $X_1(11)$ over $\mathbb{Q}(\zeta_{17})^+$ is 5.

N	field K (degree 8)	rk	#exu	discr(K)
$13(2)^{**}$	$x^8 - 4x^7 + 14x^5 - 8x^4 - 12x^3 + 7x^2 + 2x - 1$	7	15804	282300416
$11(24)$	$\mathbb{Q}(\zeta_{17})^+$	7	11700	$17^7 = 410338673$
$11(2), 13^{**}$	$x^8 - 8x^6 - 2x^5 + 19x^4 + 7x^3 - 13x^2 - 4x + 1$	7	8700	707295133

In view of this data, it is natural to wonder whether there are only finitely many octic fields with an N-special elliptic curve E/K with $N \geq 17$ and which do not contain the quartic field of discriminant 725 when a 17-special curve exists.

A.2 Number fields of degree 9.

We include below the nonic fields K where we found at least one N-special elliptic curve E/K with $N = 17, 19, 23, 29,$ or 31.

N	field K (degree 9 and rank 4)	rk	#exu	discr(K)
$11(2), \mathbf{23}$	$x^9 - 2x^8 + 2x^7 - 3x^6 + 4x^5 - 5x^4 + 5x^3 - 5x^2 + 3x - 1$	4	1266	32031161
$11(8), \mathbf{23}^*$	$x^9 - 2x^8 + 2x^7 - 2x^5 + 2x^4 - x + 1$	4	1206	33860761
$11(6), 13, \mathbf{19}$	$x^9 - x^8 - x^7 + 2x^6 - 2x^4 + x^2 - 1$	4	1194	34349041
$11(2), 17$	$x^9 - x^8 + x^7 + x^6 - x^4 + 2x^3 - 3x^2 + 2x - 1$	4	1164	34590113
$11(4), 17$	$x^9 - x^8 + 3x^6 - 3x^5 + 3x^3 - 2x^2 - x + 1$	4	1158	35678113
$11(2), 17$	$x^9 - x^8 + x^6 - x^4 + x^3 + x^2 - 2x + 1$	4	1134	36722413
$13, \mathbf{19}$	$x^9 - 2x^8 + x^7 - x^6 + x^5 + x^4 + x^2 - 2x + 1$	4	1116	37009129
$11(2), 17$	$x^9 - x^8 - x^6 - x^5 + 2x^4 + x^2 + x - 1$	4	1104	37354501
$11(4), 13(2), \mathbf{19}$	$x^9 - x^8 + 3x^7 - 3x^6 + 4x^5 - 4x^4 + 2x^3 - x^2 - x + 1$	4	1092	37732753
$13, 17$	$x^9 - 2x^8 + 4x^7 - 6x^6 + 7x^5 - 7x^4 + 7x^3 - 5x^2 + 3x - 1$	4	1068	38118173
$11(4), 17$	$x^9 - x^8 - x^7 + x^6 + x^5 - x^4 - 2x^3 + 2x + 1$	4	1098	38600453
17	$x^9 - 2x^8 + 3x^7 - 4x^6 + 5x^5 - 5x^4 + 4x^3 - 3x^2 + x - 1$	4	1038	39724513
$13, 17$	$x^9 - 2x^8 + 2x^7 + x^6 - 3x^5 + 2x^4 + x^3 - 2x^2 + 1$	4	1038	40756753
$11(4), 13, 17$	$x^9 - x^8 - x^7 + 2x^6 + x^5 - 3x^4 + x^3 + 2x^2 - 2x + 1$	4	996	42156761
$11(2), 13, 17$	$x^9 - x^8 + x^7 + 2x^6 + 2x^4 + 2x^3 + x^2 + 2x + 1$	4	990	42173713
$11(4), 13, 17$	$x^9 - x^8 - x^7 + 2x^3 + 2x^2 - x - 1$	4	996	42818653
$11(4), 17$	$x^9 + 2x^7 - 2x^6 + 2x^5 - 3x^4 + 2x^3 - 3x^2 + x - 1$	4	960	43552633
$13, 17$	$x^9 - x^8 + x^7 - 3x^6 + x^5 - x^4 + 3x^3 + x - 1$	4	978	43798753
$11(2), 17$	$x^9 - x^8 + 2x^7 - x^6 - x^5 + 3x^4 - 4x^3 + 4x^2 - 3x + 1$	4	924	45175393
$11(2), 13, 17$	$x^9 - 3x^7 - 2x^6 + 3x^5 + 3x^4 - x^2 - x - 1$	4	930	46002241
$11(2), 13, 17$	$x^9 - 2x^7 - x^6 + x^5 + 2x^4 + x^3 - x^2 - x - 1$	4	948	46094281
17	$x^9 - 2x^8 + 3x^7 - 3x^6 + 4x^5 - 4x^4 + x^3 + 3x^2 - 3x + 1$	4	906	47679193
$11(2), 17$	$x^9 - 3x^7 + 5x^5 - 3x^3 - x^2 + x + 1$	4	810	54333913
$13, 17$	$x^9 - x^8 - x^7 + 2x^6 - 3x^4 + 2x^3 + 3x^2 - x - 1$	4	846	54855733
17	$x^9 - x^8 - 2x^7 + x^6 + 3x^5 + x^4 - 3x^3 - x^2 + x + 1$	4	756	60357889
$11(2), 17$	$x^9 - 2x^8 + 4x^7 - 5x^6 + 4x^5 - 2x^4 - x^3 + 3x^2 - 2x + 1$	4	570	91446973

We searched an additional 7300 fields of degree 9 and rank 4 with discriminant larger than 91446973 in the Table [3] and did not find any additional occurrence of elliptic curves with a point of prime order $N \geq 17$.

N	field K (degree 9 and rank 5)	rk	#exu	discr(K)
19	$x^9 - x^8 - 4x^7 + 2x^6 + 8x^5 - x^4 - 8x^3 - 2x^2 + 3x + 1$	5	3366	-110852311
$17(2)$	$x^9 - x^8 - x^7 + x^6 - 3x^4 + x^3 + 4x^2 - 1$	5	3294	-113501567
$11(6), 13, \mathbf{31}$	$x^9 - 2x^8 + x^6 - x^5 + 14x^4 - 28x^3 + 19x^2 - 2x - 1$	5	3246	-114479303
$11(6), 17$	$x^9 - 2x^7 - 3x^6 + x^5 + 3x^4 + x^3 - x^2 + 1$	5	3204	-116188367
$11(6), 13(3), \mathbf{19}$	$x^9 - 2x^8 - x^7 + 5x^6 - 2x^5 - 2x^4 + x^3 - x^2 + x + 1$	5	3222	-117283087
$11(12), 17$	$x^9 - 2x^8 + x^7 + x^6 - x^5 - x^4 + x^3 - 2x + 1$	5	3156	-119747759
$11(4), 17$	$x^9 - 2x^8 + 2x^6 - x^5 + 2x^4 - 2x^3 - 2x^2 + 2x + 1$	5	3102	-123595631
$11(2), 17$	$x^9 - x^8 - 2x^7 + 5x^6 - x^5 - 5x^4 + 6x^3 - 3x + 1$	5	3072	-123668767
$11(4), 17$	$x^9 - x^7 - 3x^6 - x^5 + 3x^4 + 4x^3 + x^2 - 2x - 1$	5	3072	-124007591
$11(6), 17$	$x^9 - 2x^7 - x^6 + 2x^5 - x^4 - 3x^3 + 3x^2 + x - 1$	5	3048	-125785223
$11(6), 13, \mathbf{19}$	$x^9 - x^8 - 3x^7 + 4x^6 + 5x^5 - 5x^4 - 5x^3 + 2x + 1$	5	2880	-133731799
$11(6), \mathbf{23}$	$x^9 - x^8 - x^7 + x^6 + 2x^5 - 4x^3 - x^2 + x + 1$	5	2808	-138182311
$11(4), \mathbf{19}$	$x^9 - x^8 + x^7 - 2x^6 - x^5 + 2x^2 + 1$	5	2724	-142603319
$13, 17$	$x^9 - x^6 + x^5 + x^2 - 2x - 1$	5	2712	-143792279
$11(4), \mathbf{19}$	$x^9 - 3x^7 - x^6 + 5x^5 - 3x^4 - 3x^3 + 4x^2 - 1$	5	2712	-144146159
$13, 17$	$x^9 - x^8 - 2x^7 + 4x^6 - 5x^4 + x^3 + 3x^2 - x - 1$	5	2676	-145894103
$11(6), 17$	$x^9 - 3x^8 + 3x^7 - 3x^6 + 4x^5 - 4x^4 + 5x^3 - 4x^2 + 3x - 1$	5	2688	-148366271
$13, \mathbf{19}^*$	$x^9 - 3x^8 + 4x^7 - 4x^6 + 3x^4 - 4x^3 + 4x^2 - x + 1$	5	2628	-150782119
$11(2), 13, 17$	$x^9 - 2x^7 - 3x^4 + 4x^3 + x^2 - 3x + 1$	5	2640	-151174007
$11(2), 17$	$x^9 - 3x^7 - 4x^6 + 2x^5 + 7x^4 + 4x^3 - 3x^2 - 4x - 1$	5	2430	-166035007
$11(6), 13, \mathbf{19}$	$x^9 - x^8 - x^7 + 3x^6 - 3x^5 + x^4 - 3x^3 + 4x^2 - 3x + 1$	5	2430	-168139799

The last field in the above table is the nonic field $K/\mathbb{Q}$ of rank 5 with the largest discriminant (in absolute value) that our search found with an 19-special elliptic curve. Fields with larger discriminants were found with a point of order $N = 17$. We include below only the largest discriminants found, as well as some fields with $N = 11, 13$, where one of the curves found has potentially good reduction.

N	field K (degree 9 rank 5, continued)	rk	#exu	discr(K)
$17(2)$	$x^9 - 3x^8 + 5x^7 - 10x^6 + 13x^5 - 12x^4 + 12x^3 - 7x^2 + 3 - 1$	5	1440	-321576947
$11(2)^*$	$x^9 - x^8 + 2x^7 - 3x^5 + 7x^4 - 9x^3 + 8x^2 - 5 + 1$	5	1326	-356722399
$11(2), 17$	$x^9 - x^7 - 4x^6 - x^5 + 3x^4 + 4x^3 + x^2 - 3 - 1$	5	1362	-364057919
17	$x^9 - 3x^8 + 4x^7 - 6x^5 + 7x^4 - 2x^3 - 2x^2 + 3 - 1$	5	1206	-398752247
$11(2)^*$	$x^9 - 2x^8 + 3x^6 - 2x^4 - 3x^3 + 3x^2 + 2 - 1$	5	1242	-403349507
17^*	$x^9 - 3x^7 - x^6 + x^5 + x^4 + 2x^3 - x^2 - 2 + 1$	5	1236	-409265327
$11(2), 13^*, 17$	$x^9 - x^8 - x^7 + 3x^6 - 2x^5 - 2x^4 + 3x^3 + x^2 - 2 - 1$	5	1188	-430320007
$11(2)^*$	$x^9 - 2x^7 - 3x^6 + 5x^4 + 5x^3 - 2x^2 - 4 - 1$	5	978	-530050543
13^*	$x^9 - 2x^8 + x^6 + x^5 - 3x^4 - 4x^3 + 9x^2 - 5 + 1$	5	894	-576500327
$13, 17$	$x^9 - 2x^8 + 5x^7 - 5x^6 + 4x^5 - 5x^3 + 5x^2 - 5 + 1$	5	984	-592732439

N	field K (degree 9 and rank 6)	rk	#exu	discr(K)
$11(4), 13, 17$	$x^9 - 2x^7 - x^6 + x^5 + 4x^4 - 4x^3 - 3x^2 + 4x - 1$	6	7956	460131857
$11(2), 13, \mathbf{29}$	$x^9 - x^8 - 5x^7 + 5x^6 + 7x^5 - 8x^4 - 2x^3 + 4x^2 - x - 1$	6	7416	507490937
$11(4), 17$	$x^9 - x^8 - 3x^7 + 5x^6 - x^4 - 3x^3 - 2x^2 + 4x - 1$	6	7074	537751889
$11(2), \mathbf{19}^*$	$x^9 - 2x^8 + x^7 + 3x^6 - 6x^5 + 3x^3 - x^2 + x + 1$	6	7026	538684561
$11(6), 17$	$x^9 - 2x^8 - 2x^7 + 9x^6 - 10x^5 + 2x^4 + 5x^3 - 4x^2 + x + 1$	6	5838	677984681
$11(6)^*$	$x^9 - x^8 - 3x^7 + 3x^6 + x^5 - 4x^4 + 5x^3 + 2x^2 - 4x + 1$	6	5706	709699969
$11(2), 17$	$x^9 - 3x^7 - 2x^6 + 3x^5 + 3x^4 - x^3 + x^2 - 1$	6	5628	710045729
17	$x^9 - x^8 - 5x^7 + 4x^6 + 11x^5 - 4x^4 - 12x^3 + 4x + 1$	6	5352	755663113
$11(4), 17$	$x^9 - 2x^7 - 3x^6 + 4x^4 + 3x^3 + x^2 - 2x - 1$	6	5244	798186349
$11(2), 17$	$x^9 - 3x^8 + 4x^6 + x^4 - 4x^2 - x + 1$	6	5070	798333049
13^*	$x^9 - 3x^8 + x^7 + 7x^6 - 6x^5 - 9x^4 + 7x^3 + 6x^2 - 2x - 1$	6	5250	803996713

We ran searches through nonic fields of rank 7 and 8 with high discriminants. In both cases, our limited searches did not find instances of fields with N-special elliptic curves E/K with $N \geq 17$.

The analytic rank of $X_1(11)$ is 0 over the totally real field $\mathbb{Q}(\zeta_{19})^+$ of degree 9, with torsion subgroup reduced to $\mathbb{Z}/5\mathbb{Z}$. (The same holds for the totally real field $\mathbb{Q}(\zeta_{23})^+$ of degree 10.) The field $\mathbb{Q}(\zeta_{19})^+$ has 28398 exceptional units, a number which is slightly too large for our naive search algorithm to complete in a reasonable amount of time. We did not find any points where both coordinates are exceptional units in $\mathbb{Q}(\zeta_{19})^+$ on any plane curve $F_N(r, s) = 0$ except for the cusps when $N = 19$.

In view of this data, it is natural to wonder whether there are only finitely many nonic fields with an N-special elliptic curve E/K with $N \geq 19$.

We noted the following cluster of totally real nonic fields for $N = 13$. Every field below contains a totally real cubic field, of discriminant either 148 or 169. The 13-special elliptic curves E/K that we found all have j-invariant in the cubic subfield, and either have everywhere good reduction, or have a small set of primes of bad reduction, of surprisingly small norms. It would be interesting to determine if there is some theoretical reason for the existence of such cluster.

In the table below, the curve $X_1(13)$ has a closed point over the given totally real nonic field. The column 'prime' is the norm of the prime(s) where the unique elliptic curve found over that field has semi-stable reduction. The number 1 indicates that the associated curve has everywhere good reduction. For comparison in the size of the discriminants, we include at the end the totally real nonic field of fourth smallest discriminant, and second smallest among the nonic fields with a cubic subfield ([1], [5]). Coincidentally, over this field also, there exists a 13-special elliptic curve E/K with j-invariant in the cubic subfield, of discriminant 229.

Totally real nonic fields K and $N = 13$	#exu	discr(K)	prime
$x^9 - 4x^8 - 13x^7 + 51x^6 + 61x^5 - 190x^4 - 162x^3 + 211x^2 + 181x + 1$	660	47082882548032	23^3
$x^9 - 57x^7 - 53x^6 + 1083x^5 + 2014x^4 - 6086x^3 - 19133x^2 - 14687x - 3121$	300	36882285775168	1
$x^9 - x^8 - 17x^7 + 12x^6 + 100x^5 - 39x^4 - 243x^3 + 29x^2 + 206x + 19$	384	28614782539072	17^3
$x^9 - 2x^8 - 14x^7 + 8x^6 + 64x^5 + 40x^4 - 34x^3 - 38x^2 - 11x - 1$	396	21919316785249	$79, 79, 79$
$x^9 - x^8 - 12x^7 + 16x^6 + 30x^5 - 56x^4 + 12x^3 + 16x^2 - 8x + 1$	1350	6962194128409	1
$x^9 - 13x^7 - 9x^6 + 41x^5 + 52x^4 - 2x^3 - 23x^2 - 9x - 1$	696	5760304545088	107^3
$x^9 - x^8 - 13x^7 + 20x^6 + 26x^5 - 57x^4 + 15x^3 + 15x^2 - 8x + 1$	1080	5253417947968	29^3
$x^9 - 4x^8 - 5x^7 + 31x^6 - 17x^5 - 34x^4 + 32x^3 - x^2 - 5x + 1$	804	3222370424128	1
$x^9 - 2x^8 - 9x^7 + 11x^6 + 28x^5 - 18x^4 - 34x^3 + 8x^2 + 13x + 1$	26718	16440305941	2^6

A.3 Number felds of degree 10**.**

N	field K (degree 10 and rank 4)	rk	#exu	discr(K)
$11(20), \mathbf{31}(j=0)$	$x^{10} + 2x^8 - 3x^7 + 3x^6 - 7x^5 + 8x^4 - 7x^3 + 7x^2 - 4x + 1$	4	1502	-224415603
$11(6), 17$	$x^{10} - x^8 - 2x^7 - x^6 + 2x^5 + 2x^4 - x^2 + 1$	4	1488	-226173952
$11(2), 13, 17$	$x^{10} - 2x^9 + 2x^8 - 2x^6 + 4x^5 - 4x^4 + x^3 + 3x^2 - 3x + 1$	4	1404	-240232739
$11(2), 13, 19$	$x^{10} - x^9 + x^7 - x^6 - x^4 + x^3 + x^2 - x + 1$	4	1398	-244175707
17	$x^{10} - x^9 + x^6 - x^5 - x^4 + 3x^3 - x^2 - x + 1$	4	1382	-246944619
$11(4), 13, 19$	$x^{10} - x^8 - 2x^7 + x^6 + 3x^5 - 2x^4 - 4x^3 + x^2 + 3x + 1$	4	1328	-260270739
$11(4), 17$	$x^{10} + 3x^8 - x^7 + 4x^6 - x^5 + 4x^4 - x^3 + 2x^2 - x + 1$	4	1236	-281268947
$11(4), 13, 19$	$x^{10} - x^9 + x^8 - x^7 - x^5 + 3x^4 - 4x^3 + 4x^2 - 2x + 1$	4	1230	-285267739
$11(4), 19(2)$	$x^{10} - 3x^9 + 6x^8 - 10x^7 + 13x^6 - 13x^5 + 11x^4 - 8x^3 + 5x^2 - 2x + 1$	4	1122	-325246087
17	$x^{10} + x^8 - x^7 + x^6 - 2x^5 + x^4 - x^3 + x^2 - x + 1$	4	1068	-335570327
$13, 19$	$x^{10} - 2x^9 + 2x^8 - 5x^7 + 5x^6 - 3x^5 + 5x^4 - 2x^3 + 2x^2 - 3x + 1$	4	968	-381570507
17	$x^{10} - 2x^9 + 4x^8 - 6x^7 + 8x^6 - 8x^5 + 8x^4 - 7x^3 + 5x^2 - 3x + 1$	4	912	-417166579
$13, 19$	$x^{10} - 3x^9 + 3x^8 - 2x^7 + 2x^6 - x^5 + x^4 - 2x^3 - x^2 + 2x + 1$	4	794	-488663523
17	$x^{10} - 4x^9 + 8x^8 - 9x^7 + 6x^6 - x^5 - x^4 - x^3 + 4x^2 - 3x + 1$	4	782	-511382079
19	$x^{10} - 2x^9 + 2x^8 - 2x^6 + 3x^5 - 5x^3 + 7x^2 - 4x + 1$	4	636	-678743987
$13, 17$	$x^{10} - x^9 + x^8 - 3x^7 + 4x^6 - 3x^5 + 5x^4 - 6x^3 + 4x^2 - 2x + 1$	4	608	-750450339

The fields above are among the first 700 decic fields of rank 4 found in the database [3]. We searched through an additional $90,000$ fields of degree 10 and unit rank 4 and did not find other occurrences of N-special elliptic curves E/K with $N \geq 17$ and such that K does not contain any degree 5 subfield appearing in the quintic fields listed in [4], 6.6, and 6.7.

N	field K (degree 10 and rank 5)	rk	#exu	discr(K)
$17, 19(2)$	$x^{10} - 3x^9 + 3x^8 - x^7 + x^5 - 3x^4 + 4x^3 - x^2 - x + 1$	5	3900	801214577
$11(24), 17$	$x^{10} - 2x^8 - x^7 - x^6 + x^5 + 3x^4 - 2x^3 - 2x^2 + x + 1$	5	3894	801589013
$11(6), 13, 17, \mathbf{37}$	$x^{10} - 3x^9 + 7x^8 - 11x^7 + 12x^6 - 8x^5 + 2x^4 + 4x^3 - 6x^2 + 4x - 1$	5	3876	803282693
$11(6), 13, 19$	$x^{10} + x^8 - 2x^7 + x^4 + 2x^3 - x^2 - 1$	5	3828	817298432
$11(2), 13, 17$	$x^{10} - 2x^9 + 2x^7 - x^6 + x^2 - x - 1$	5	3810	819649013
$13, 17(2), \mathbf{29}$	$x^{10} - 2x^9 + 2x^8 - 5x^7 + 7x^6 - 4x^5 + 4x^4 - 8x^3 + 5x^2 - 1$	5	3780	820675277
$11(2), 17$	$x^{10} - x^9 + 4x^8 - 4x^7 + 7x^6 - 8x^5 + 7x^4 - 7x^3 + 4x^2 - 3x + 1$	5	3792	827119069
$17(2)$	$x^{10} - x^9 + x^8 + 2x^7 - 4x^6 + 3x^4 - 2x^3 - 2x^2 + 2x + 1$	5	3768	828364813
$11(6), 17, \mathbf{23}$	$x^{10} - x^9 - 2x^8 + x^7 + x^6 + 2x^5 + x^4 - 2x^3 + x - 1$	5	3786	830597309

N	field K (degree 10, rank 5, continued)	rk	#exu	discr(K)
$11(2), 17$	$x^{10} - 2x^9 + 4x^8 - 7x^7 + 8x^6 - 8x^5 + 6x^4 - 3x^3 + x - 1$	5	3798	834771989
$13, 17(2)$	$x^{10} - x^9 - 2x^8 + 3x^7 + x^6 - x^5 - x^4 - x^3 + 2x^2 + x - 1$	5	3774	838803593
$11(12), 13, 17(2)$	$x^{10} - x^7 - 3x^6 + 3x^5 + 3x^4 - x^3 - 1$	5	3690	848429441
$11(6), 13, 19$	$x^{10} - x^8 - 2x^7 + x^6 + 2x^5 - x^4 - 2x^3 + 2x + 1$	5	3708	856749056
$11(2), 13, 19$	$x^{10} - 2x^9 + 4x^8 - 5x^7 + 6x^6 - 7x^5 + 7x^4 - 8x^3 + 5x^2 - 3x + 1$	5	3690	859848569
$11(6), 13(3), 17, 19$	$x^{10} - 2x^8 - 3x^7 + 5x^5 + 5x^4 - 3x^2 - 3x - 1$	5	3660	867798793
$11(2), 13(3), 17$	$x^{10} - x^9 - x^8 + 3x^7 - 3x^6 - x^5 + 5x^4 - x^3 - 3x^2 + x + 1$	5	3654	875003125
$11(10), 13(3), 17$	$x^{10} - 2x^9 + 4x^7 - 4x^6 + 2x^4 - x^3 + x^2 - x + 1$	5	3612	888484253
$11(6), 17(2)$	$x^{10} - x^9 + 3x^8 - 5x^7 + 5x^6 - 7x^5 + 5x^4 - 5x^3 + 3x^2 - x + 1$	5	3612	889458133
$11(12), 17$	$x^{10} - x^8 - x^7 + x^6 - 2x^4 + x^2 + x - 1$	5	3570	894158693
$11(2), 19$	$x^{10} - 2x^9 + 2x^8 - 2x^7 + x^6 + 2x^5 - 9x^4 + 12x^3 - 9x^2 + 4x - 1$	5	3576	894453953
$11(12), 17$	$x^{10} + x^8 - 3x^7 - x^6 - x^5 + x^4 + 3x^3 + x - 1$	5	3576	896659541
$11(2), 13, \mathbf{23}$	$x^{10} - x^9 - x^8 + 3x^7 - x^6 - 3x^5 + 3x^4 + 2x^3 - x - 1$	5	3504	914778125
$13, \mathbf{29}$	$x^{10} - 2x^7 + x^5 + 2x^4 + 3x^3 - 3x^2 - 2x + 1$	5	3510	917035997
$11(6), \mathbf{23}$	$x^{10} - 2x^9 + 2x^7 + x^6 - 5x^4 + 2x^3 + 4x^2 - x - 1$	5	3450	936901549
$11(4), 17(2)$	$x^{10} - 2x^8 - x^7 - x^6 - x^5 + 4x^4 + 4x^3 - x^2 - 3x - 1$	5	3468	938671133
$11(2), 13, 17$	$x^{10} - x^9 - x^8 + 3x^7 - x^6 - 2x^5 + 3x^4 - x^3 - x^2 - 1$	5	3468	941778433
$11(2), 17$	$x^{10} - 3x^9 + 6x^8 - 8x^7 + 9x^6 - 8x^5 + 5x^4 - 2x^3 - 1$	5	3426	942600293
$11(2), 17(2)^*$	$x^{10} - x^9 - 2x^8 + 5x^7 - 3x^6 + x^5 - 3x^4 + 5x^3 - 2x^2 - x + 1$	5	3408	946838273
$11(6), \mathbf{23}$	$x^{10} - x^9 - x^8 + 3x^7 - 2x^5 + 2x^4 - 1$	5	3408	951955141
$13, 17$	$x^{10} - x^8 - x^7 + 2x^6 + 2x^5 - x^3 - x^2 + x - 1$	5	3372	959628125
$11(2), 13, 17, 19$	$x^{10} - x^9 - 2x^8 + 2x^6 + 2x^5 + x^4 - 3x^3 - 2x^2 + 2x + 1$	5	3390	962143313
$11(6), 17$	$x^{10} - 2x^9 + 2x^8 - 3x^6 + 2x^5 - 2x^4 + 4x^3 - 6x^2 + 4x - 1$	5	3348	963051893
$11(6), 17$	$x^{10} - 2x^8 - x^7 + 3x^6 + 3x^5 - 5x^4 + 3x^2 - 2x + 1$	5	3348	967461881
$11(2), 13, 19$	$x^{10} - 2x^9 + 3x^8 - x^7 - 2x^6 + 5x^5 - 4x^4 - 2x^3 + 4x^2 - 11$	5	3342	971937941
$11(2), 13(6), 17$	$x^{10} + 3x^8 - 5x^7 - 11x^5 - 5x^3 + 3x^2 + 1$	5	3324	976008137
$11(2), 13, 17$	$x^{10} - x^9 - 2x^8 + 3x^7 + 2x^6 - 7x^5 + 4x^4 + 2x^3 - 5x^2 + 3x - 1$	5	3324	982657741
$11(6), 19$	$x^{10} - 2x^9 + 3x^7 - x^6 - 5x^5 + 5x^4 - 4x^2 + 3x - 1$	5	3318	988598813
$11(2), 13(2), 17$	$x^{10} - x^9 + 2x^7 - x^6 - 3x^5 + 3x^4 - 3x^2 + 2x - 1$	5	3294	994689209
$11(2), 13, 19$	$x^{10} - 3x^9 + 5x^8 - 6x^7 + 6x^6 - 4x^5 + 3x^4 - 2x^3 + x^2 - x - 1$	5	3270	995628125
$11(2), 13, 17$	$x^{10} - x^9 + 2x^7 - 3x^6 + 3x^5 + x^4 - 4x^3 + 3x^2 - 1$	5	3282	997436953
$11(2), 17$	$x^{10} + x^8 - x^7 - 3x^6 - 2x^4 + 2x^3 + 2x^2 + 1$	5	2946	1111067821
$11(2), 17$	$x^{10} - x^9 - x^7 + 2x^5 - 2x^4 + 2x^2 - x - 1$	5	3006	1117913089

N	field K (degree 10, rank 5, continued)	rk	#exu	discr(K)
$11(4), 13, 17$	$x^{10} - 2x^9 + 3x^8 - 3x^7 + 3x^6 - 2x^5 + x^4 - x^2 - 1$	5	2958	1120006561
$13(3), 19$	$x^{10} - 3x^9 + 7x^8 - 10x^7 + 11x^6 - 9x^5 + 5x^4 - x^3 - x^2 + 2 - 1$	5	2988	1129862893

A.4 Number fields of degree 11.

N	field K (degree 11 and rank 5)	rk	#exu	discr(K)
17	$x^{11} - 2x^{10} + x^9 + 3x^8 - 5x^7 + 2x^6 + 4x^5 - 6x^4 + 2x^3 + 3x^2 - 3x + 1$	5	4446	-5901091967
$13, 17$	$x^{11} + x^9 - 2x^8 - 2x^7 - x^6 + 3x^4 + x^3 + x^2 - 1$	5	4422	-5939843699
$11(2), 13, 17$	$x^{11} - 3x^{10} + 6x^9 - 10x^8 + 13x^7 - 13x^6 + 11x^5 - 6x^4 + 2x^3 + x^2 - 2x + 1$	5	4422	-5953134259
$11(10), 13, 17$	$x^{11} - 2x^{10} + x^9 + 3x^8 - 5x^7 + x^6 + 5x^5 - 5x^4 + x^3 + 2x^2 - 2x + 1$	5	4386	-5999947987
$11(2), 13, 17(2)$	$x^{11} - 4x^{10} + 8x^9 - 13x^8 + 17x^7 - 17x^6 + 15x^5 - 11x^4 + 7x^3 - 5x^2 + 2x - 1$	5	4398	-6052527179
$11(2), 13, 17(2)$	$x^{11} - 2x^{10} + 3x^9 - 2x^8 + 2x^7 - 3x^6 + 6x^5 - 9x^4 + 9x^3 - 7x^2 + 4x - 1$	5	4344	-6079177579
$11(12), 17$	$x^{11} - 2x^9 - 2x^8 + x^7 + 4x^6 + 2x^5 - x^4 - x^3 - x^2 - x - 1$	5	4242	-6229245439
$11(2), 17$	$x^{11} - 3x^{10} + 6x^9 - 8x^8 + 8x^7 - 6x^6 + 3x^5 - 2x^3 + 2x^2 - x + 1$	5	4272	-6260560763
$11(10), 19$	$x^{11} + 3x^9 - 2x^8 + 3x^7 - 5x^6 + 2x^5 - 4x^4 + 2x^3 - x^2 + x - 1$	5	4266	-6313976939
19	$x^{11} - 2x^{10} + 2x^9 - 4x^7 + 7x^6 - 3x^5 - 4x^4 + 6x^3 - 2x^2 - x + 1$	5	4200	-6337515647
$11(2), 13, 17$	$x^{11} - 3x^{10} + 5x^9 - 3x^8 - 3x^7 + 8x^6 - 6x^5 - x^4 + 5x^3 - 2x^2 - x + 1$	5	4200	-6385658759
$\mathbf{29}$	$x^{11} - 3x^{10} + 2x^9 + 3x^8 - 6x^7 + 2x^6 + 4x^5 - 4x^4 + 2x^2 - x - 1$	5	4146	-6472354879
$11(2), 13, 17$	$x^{11} - 3x^{10} + 2x^9 + 4x^8 - 8x^7 + 3x^6 + 4x^5 - 4x^4 + x - 1$	5	4104	-6526790267
$11(6), \mathbf{23}$	$x^{11} - 2x^9 - x^8 + x^7 + 3x^6 - x^5 - 3x^4 + x^2 + x - 1$	5	4110	-6558115807
$11(2), 13(10)$	$x^{11} - 3x^{10} + 6x^9 - 7x^8 + 6x^7 - 3x^6 - x^5 + 3x^4 - 3x^3 + x^2 - 1$	5	4086	-6612476603
$11(2), 13, 17, 19$	$x^{11} - x^{10} - 2x^9 + 3x^8 + x^7 - 5x^6 + 3x^5 + 4x^4 - 4x^3 - 2x^2 + 2x + 1$	5	4086	-6642253999
$11(2), 17$	$x^{11} - 3x^{10} + 6x^9 - 8x^8 + 6x^7 - 2x^6 - 2x^5 + 3x^4 - x^3 - x^2 + x - 1$	5	4032	-6683439479
$11(4), 19$	$x^{11} - x^{10} - x^9 + x^7 + x^4 - x^2 + 1$	5	2718	-11009745739
$11(2), 19$	$x^{11} - x^{10} + 2x^9 - 2x^6 + 6x^5 - 11x^4 + 11x^3 - 8x^2 + 4x - 1$	5	2736	-11077432423
17	$x^{11} - 3x^{10} + 6x^9 - 10x^8 + 15x^7 - 18x^6 + 20x^5 - 19x^4 + 15x^3 - 9x^2 + 4x - 1$	5	2628	-11079834679
$11(2), 17$	$x^{11} - x^{10} - x^9 + 3x^8 - x^6 - 2x^5 + x^4 + 4x^3 - 3x^2 - x + 1$	5	2808	-11091632459
17	$x^{11} - x^{10} + 2x^9 - 4x^8 + 2x^7 - 4x^6 + 4x^5 - x^4 + 3x^3 - x^2 + x - 1$	5	2592	-11309793311
$13(3), 19$	$x^{11} - 3x^{10} + 4x^9 - 2x^8 - x^7 + 2x^6 - x^5 - 2x^4 + 4x^3 - 4x^2 + 2x - 1$	5	2694	-11374313387
$11(2), 17$	$x^{11} - 4x^{10} + 8x^9 - 10x^8 + 6x^7 + 4x^6 - 11x^5 + 9x^4 - x^3 - 3x^2 + 3x - 1$	5	2634	-11377562459
$11(2), 17$	$x^{11} - 2x^{10} + 5x^9 - 7x^8 + 9x^7 - 9x^6 + 7x^5 - 4x^4 + x^3 + x^2 - 2x + 1$	5	2604	-11879585159

N	field K (degree 11 and rank 5, continued)	rk	#exu	discr(K)
$11(2), 17$	$x^{11} - 2x^{10} + x^8 + 2x^6 - x^5 - 2x^4 - x^3 + 2x + 1$	5	2502	-11901264959
$11(2), 19$	$x^{11} - 2x^{10} + x^9 + 2x^8 - 4x^7 + 4x^6 - 4x^4 + 4x^3 - 2x + 1$	5	2070	-16166694263

The three fields below with $N = 31$ are taken from points on $X_1(31)$ found in [6].

N	field K (degree 11 and rank 6)	rk	#exu	discr(K)
$11(4), 13, 19$	$x^{11} - x^{10} - 3x^9 + 7x^7 + 2x^6 - 8x^5 - x^4 + 4x^3 + x^2 - 2x - 1$	6	10836	23882289781
19	$x^{11} - 2x^{10} - x^8 + 3x^7 + 3x^6 - 5x^5 + x^4 + 2x - 1$	6	10710	24313061389
$11(2), 13, 17$	$x^{11} - 3x^{10} + 6x^9 - 8x^8 + 6x^7 - 6x^5 + 10x^4 - 7x^3 + 3x^2 - 1$	6	10626	24350040329
$13, 17$	$x^{11} - 2x^8 + 4x^4 - x^3 - 3x^2 + x + 1$	6	10512	24963663301
31	$x^{11} - x^{10} + 2x^8 + x^6 - 7x^5 + x^4 + 4x^3 - x^2 + 2x - 1$	6	8688	32252418553
$11(2), 13, 19$	$x^{11} - 3x^{10} + 3x^9 + 3x^8 - 10x^7 + 9x^6 + x^5 - 9x^4 + 8x^3 - x^2 - 2 + 1$	6	8622	32338529681
$11(4), 13, 19$	$x^{11} - x^{10} - 2x^7 - x^6 - 3x^5 - 4x^4 - x^3 - 2x^2 - 3 - 1$	6	8496	32363454073

N	field K (degree 11 and rank 7)	rk	#exu	discr(K)
$11(6), \mathbf{31}^{**}$	$x^{11} - 8x^{10} + 23x^9 - 25x^8 + 8x^6 + 14x^5 - 10x^4 - 8x^3 + 10x^2 - 5x + 1$	7	20070	-145065457639
$11(2), \mathbf{31}$	$x^{11} - 2x^{10} - 6x^9 + 8x^8 + 9x^7 - x^6 - 6x^5 - 13x^4 - x^3 + 7x^2 + 4 + 1$	7	18060	-167720394923

A.5 Number fields of degree 12.

N	field K (degree 12 and rank 5)	#exu	discr(K)
$11(2), 13(5), 17$	$x^{12} - 2x^{11} + 2x^{10} - x^9 + 2x^8 - 5x^7 + 8x^6 - 7x^5 + 4x^4 - 3x^3 + 4x^2 - 3x + 1$	5334	41223887921
$11(2), 17(2), 19(2)$	$x^{12} + 2x^{10} - x^9 + x^8 - 5x^7 + 6x^6 - 10x^5 + 13x^4 - 10x^3 + 7x^2 - 4x + 1$	5270	42058512657
$11(14), 13^*, \mathbf{43}$	$x^{12} - x^9 + 5x^8 - 2x^7 - x^6 - 7x^5 + 8x^4 - 4x^3 + 5x^2 - 4x + 1$	5204	42553255797
$11(12), 13(2), 17$	$x^{12} - x^{11} - 2x^{10} + 3x^9 + x^8 - x^7 - x^6 - 2x^5 + 3x^4 + 2x^3 - 2x^2 - x + 1$	5124	42925852301
17	$x^{12} - 2x^{11} + 4x^{10} - 6x^9 + 7x^8 - 10x^7 + 9x^6 - 9x^5 + 8x^4 - 5x^3 + 4x^2 - x + 1$	5082	43643638373
$11(12), 17$	$x^{12} - 3x^{11} + 7x^{10} - 13x^9 + 21x^8 - 28x^7 + 34x^6 - 35x^5 + 30x^4 - 21x^3 + 12x^2 - 5x + 1$	5084	44376802641
$11(6), 13, \mathbf{37}$	$x^{12} - 2x^{11} + 7x^{10} - 7x^9 + 14x^8 - 6x^7 + 13x^6 - 2x^5 + 10x^4 - 2x^3 + 5x^2 - x + 1$	4986	44533203125
$11(6), 17$	$x^{12} - 4x^{11} + 7x^{10} - 6x^9 + 2x^8 + x^6 - x^5 + x^2 - x + 1$	4926	45441432661
$11(6), \mathbf{23}$	$x^{12} - 2x^{11} + x^{10} + 3x^9 - 5x^8 + 2x^7 + 3x^6 - 4x^5 + x^4 + 2x^3 - x^2 - x + 1$	4824	46334726117
$11(6), 19$	$x^{12} - x^{11} - 2x^{10} + 4x^9 - 5x^7 + 4x^6 + x^5 - 3x^4 + 2x^3 - x + 1$	4776	47103726121
$11(6), 17$	$x^{12} - x^{11} + x^{10} - 2x^9 + 4x^8 - 4x^7 + 2x^6 - 3x^5 + 4x^4 - 2x^3 + 1$	4704	48231112297
$11(6), 13(2), 17$	$x^{12} - 2x^{11} + 2x^{10} - x^9 + 2x^6 - 4x^5 + 6x^4 - 6x^3 + 4x^2 - 2x + 1$	4656	48612725813
$13(2), \mathbf{37}(j=0)$	$x^{12} - 4x^{11} + 11x^{10} - 21x^9 + 32x^8 - 40x^7 + 45x^6 - 46x^5 + 40x^4 - 26x^3 + 12x^2 - 4x + 1$	4634	50551744653
$11(2), 13, 17(2)$	$x^{12} - 3x^{10} - 3x^9 + 3x^8 + 8x^7 + 4x^6 - 6x^5 - 9x^4 - 3x^3 + 4x^2 + 4x + 1$	4520	51152452317
$11(12), 17$	$x^{12} - x^{10} - 2x^9 + x^8 + 3x^7 - x^5 - x^4 + x^3 - x + 1$	4440	52090836797
$17(2)$	$x^{12} - 2x^{11} + x^{10} + x^9 - x^8 + x^7 - 3x^6 + 4x^5 - x^4 - 2x^3 + 3x^2 - 2 + 1$	4416	52765663201
$11(4), 19$	$x^{12} - x^{11} - x^{10} + x^9 - x^8 + 2x^7 + x^6 - x^5 - x^4 - x^3 + x^2 + 1$	4338	53996413117
$11(2), 17$	$x^{12} - x^{11} + x^{10} - x^8 + 3x^7 - 3x^6 + 2x^5 - 2x^3 + 2x^2 - 2 + 1$	4212	55151314501
$11(6), 13(2), 19$	$x^{12} - 3x^{11} + 6x^{10} - 7x^9 + 5x^8 - x^7 - x^6 - x^5 + 6x^4 - 8x^3 + 7x^2 - 4x + 1$	4164	55591687817
$11(4), 17$	$x^{12} - 2x^{11} + x^{10} + 2x^9 - x^8 - 3x^7 + 4x^6 + 2x^5 - 7x^4 + 4x^3 + 2x^2 - 3x + 1$	4092	56633340437
$11(2), 19$	$x^{12} - 3x^{11} + 6x^{10} - 8x^9 + 8x^8 - 5x^7 + 3x^6 - 3x^5 + 5x^4 - 7x^3 + 7x^2 - 4 + 1$	4104	57354370889
$13(3), 17$	$x^{12} - x^{11} + 2x^9 - x^7 + x^6 + 3x^5 + 2x^4 + 2x^2 + 3x + 1$	3978	59661667817
$11(10), 17$	$x^{12} - 4x^{11} + 6x^{10} - 3x^9 - x^8 + 5x^6 - 9x^5 + 10x^4 - 8x^3 + 5x^2 - 2x + 1$	3978	60109653997
$11(2), 13(2), 17$	$x^{12} - x^{11} + 3x^{10} - 5x^9 + 7x^8 - 8x^7 + 10x^6 - 9x^5 + 6x^4 - 6x^3 + 3x^2 - x + 1$	3938	60707892717
$13, 19$	$x^{12} - 2x^{10} - x^9 + 2x^8 + 3x^7 - 3x^5 + x^3 - x^2 + 1$	3882	60974918537

References

[1] H. Fujita, *The minimum discriminant of totally real algebraic number fields of degree 9 with cubic subfields*, Math. Comp. **60** (1993), no. 202, 801–810. 6

[2] J. Jones, http://hobbes.la.asu.edu/NFDB/

[3] J. Jones and D. Roberts, *A database of number fields*, LMS J. Comput. Math. **17** (2014), no. 1, 595–618. 3, 7

[4] D. Lorenzini, *Torsion and Exceptional Units*, preprint. 1, 7

[5] K. Takeuchi, *Totally real algebraic number fields of degree* 9 *with small discriminant*, Saitama Math. J. **17** (1999), 63–85 (2000). 6

[6] M. van Hoeij, *Low Degree Places on the Modular Curve* $X_1(N)$, `https://arxiv.org/abs/1202.4355`. 10

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GA 30602, USA

Email address: `lorenzin@uga.edu`