We give below some details regarding an assertion used in the paper [5] of Liu-Lorenzini-Raynaud.

Let V/k be a proper smooth geometrically connected curve over a finite field. Let $K := k(V)$ denote the function field of V. Let X/k be a smooth proper and geometrically connected surface endowed with a proper flat map $f\colon X \to V$ such that the generic fiber X_K/K is a proper smooth geometrically connected curve of genus g.

Assertion: *Let A_K/K denote the Jacobian of X_K/K. Denote the Shafarevich-Tate group of A by $Ш(A)$, and let $\mathrm{Br}(X)$ be the Brauer group of X. Then $Ш(A)$ is finite if and only if $\mathrm{Br}(X)$ is finite.*

A proof of this assertion can be obtained from the existing literature as follows (see the remark in [6] at the bottom of the page 1141).

Let $P = \mathrm{Pic}_{X_K/K}$. The assertion that $Ш(A)$ *is finite if and only if* $Ш(P)$ *is finite* is standard and follows from the exact sequence $0 \to A \to P \to \mathbb{Z} \to 0$ (see [2], exact sequence (7) on page 405).

Similarly, the assertion that $Ш(P)$ *is finite implies that* $\mathrm{Br}(X)$ *is finite* is standard (see [2], 2.4).

Assume now that $\mathrm{Br}(X)$ is finite. Let $\mathbf{P} = \mathrm{Pic}_{X/V}$ ([3], (4.3)). Then [3], Corollary 4.4, shows that $\mathrm{Br}(X)$ *is finite if and only if* $H^1(V, \mathbf{P})$ *is finite.* We can then use the exact sequence (4.17) to deduce that *if* $H^1(V, \mathbf{P})$ *is finite, then* $Ш(V, \underline{B})$ *is finite.*

It remains to compare $Ш(V, \underline{B})$ and $Ш(P)$, and this is done on page 122 of [3], Complement 4.9. It turns out that these groups are equal. The orginal definition of Shafarevich and Tate, used for $Ш(P)$, defines $Ш(P)$ as the set of elements of $H^1(K, P)$ whose images in each 'completion' $H^1(K_v, P)$ is trivial. The elements of $Ш(V, \underline{B})$ consist in the elements of $H^1(K, P)$ whose images in each 'henselization' $H^1(\tilde{K}_v, P)$ is trivial.

Let K be the field of fractions of a henselian discrete valuation ring with finite residue field. It is shown for instance in [4], page 59, remark 3.10 (ii), that when A/K is an abelian variety, the natural map $H^1(K, A) \to H^1(\hat{K}, A)$ is an isomorphism. To show that the same result holds for P, we use the exact sequence $0 \to A \to P \to \mathbb{Z} \to 0$ (first exact sequence on page 405 of [2]), and the fact that if a connected component of P has a point over $\hat{K}$, then it has a point over K (see [1]).

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