K/k Abelian

As the results for archimedean primes have already been determined, it may be assumed that the valuation is discrete. The notation is that introduced at the beginning of the chapter; specifically, χ is a quasi-character of K^*, of conductor $\mathfrak{p}^{1+m}$, which is trivial on K_N^*.

Theorem 1

a) $Q_{K/k}(\chi) = \kappa(m, K/k)$, a fourth root of unity depending only on the conductor (and the fields k, K)

b) $(Q_{K/k}(\chi))^2 = \prod_{j=1}^{n} \tau_j(-1)$.

Proof We first show that the second assertion is a consequence of the first. Indeed, all that is needed is that $Q(\bar{\chi}) = Q(\chi)$ which is certainly true if the first statement is true.

(b) It follows from the explicit formula for $\rho_k(c)$ (where c is any quasi-character of k^*) that $\rho_k(\bar{c}) = c(-1)\,\overline{\rho_k(c)}$. Hence if μ is a quasi-character of k^* whose composition with $N_{K/k}$ is χ, then $\overline{Q(\chi)} = \left[\prod_{j=1}^{n} \overline{\rho_k(\mu\tau_j)}\right] / \overline{\rho_K(\chi)}$

$= \frac{Q(\bar{\chi})}{\chi(-1)} \prod_{j=1}^{n} (\mu\tau_j)(-1)$. As $\chi(-1) = \mu(-1)^n$, it is clear that (b) follows from (a).

(a) It has already been shown that the assertion is true if n is prime. We prove it in general by induction on the number of (not necessarily distinct) prime factors of n. For given n take as the induction hypothesis the statement that (a) is valid whenever the ever the number of prime factors is less than the number of prime factors of n. We may assume that n is not prime. Hence there exists an intermediate field, K', such that the degree of K/K' is q, a non trivial divisor of n. (The proof does not require that q be prime.) Let μ be as defined in the proof of part (b) and let χ' be the quasi-character, $\mu \circ N_{K'/k}$, of K'^*. Let $n' = n/q$ be the degree of K'/k, and let $(\sigma_j)_{j=1}^{q}$ be the group of characters of K'^* which are trivial on $N_{K/K'}K^*$ and also let $(\delta_j)_{j=1}^{n'}$ be the group of characters of k^* which are trivial on $N_{K'/k}K'^*$. As this last group of characters is a subgroup of $(\tau_j)_{j=1}^{n}$ of index q, there exists q characters say, $\tau_1, \ldots, \tau_q$ such that $(\tau_j)_{j=1}^{n} = \bigcup_{s=1}^{q} \tau_s (\delta_j)_{j=1}^{n'}$ (a disjoint union). In terms of the local norm rest symbol, for $x \in K'^*$, $(x, K/K') = 1 \Leftrightarrow (N_{K'/k}x, K/k) = 1 \Leftrightarrow (\tau_j \circ N_{K'/k})(x) = 1$ for all j, $1 \leq j \leq n$. Hence as j runs through the integers $1, 2, \ldots, n$, $\tau_j \circ N_{K'/k}$ runs through the group of characters of K'^* which are trivial on $N_{K/K'}K^*$. As $\delta_j \circ N_{K'/k}$ is trivial on K'^*, it follows that $(\tau_j \circ N_{K'/k})_{j=1}^{q}$ is the group of characters of K'^* which are trivial on $N_{K/K'}K^*$ and therefore with a suitable choice of indices, $\sigma_s = \tau_s \circ N_{K'/k}$ for $1 \leq s \leq q$. Hence, $g_{K/k}(\chi) = (\rho_k(\chi))^{-1} \prod_{j=1}^{n} \rho_k(\mu \tau_j)$

$$= \frac{\prod_{j=1}^{n'} \prod_{s=1}^{q} \rho_k(\mu \tau_s \delta_j)}{\rho_k(\chi)} =$$

$$\left[\frac{\prod_{s=1}^{q} \rho_{K'}(\sigma_s\chi')}{\rho_K(\chi)}\right] \prod_{s=1}^{q} \left[\frac{\prod_{j=1}^{n'} \rho_K((\mu\tau_s)\sigma_j)}{\rho_{K'}(\chi'\sigma_s)}\right]$$

$= Q_{K/K'}(\chi) \prod_{s=1}^{q} Q_{K'/k}(\chi'\sigma_s)$. The set $(\chi'\sigma_s)_{s=1}^{q}$ of characters of K'^* is uniquely determined by the condition that the composition of each element of the set with $N_{K/K'}$ is χ. Let $\mathfrak{y}$ be the prime of K' and $(\mathfrak{y}^{1+m_s})_{s=1}^{q}$ be the set of conductors of these characters. The set $(m_1,\dots,m_q)$ is determined by m; for there exists unique integer b such that $m = g_{K/K'}(b)$ and (cf. Chapter I) $m_s = \text{Max}(b,t_s)$ for $1 \leq s \leq q$, where $\mathfrak{y}^{1+t_s}$ is the conductor of σ_s. Applying the induction hypothesis we now have $Q_{K/k}(\chi) = \kappa(m,K/K') \prod_{s=1}^{q} \kappa(\text{Max}(b,t_s),K'/k)$ whence the assertion follows. We have also proven:

<u>Corollary</u>

If K/k is abelian, m an integer admissible with respect to K/k, K' an intermediate field, $(1+s_j)_{j=1}^{r}$ the exponents of the group of characters of K'^* trivial on $N_{K/K'}K^*$, where $r = \deg(K/K')$, then $\kappa(m,K/k) = \kappa(m,K/K') \prod_{j=1}^{r} \kappa(\text{Max}(b,s_j),K'/k)$ where $m = g_{K/K'}(b)$.

This permits the computation of $\kappa(m,K/k)$ by reduction to cyclic fields of prime degree.

The next theorem will specify $(\kappa(m,K/k))^2$ in terms of the norm group, $N_{K/k}K^*$. To simplify the proof we introduce the following notation: $x \to (x,K/k)^{n/(n,2)}$ is a homomorphism of k^* into $G(K/k)$ and the image is the trivial automorphism of K/k unless the 2-sylow subgroup of G is cyclic and non-trivial, in which case the image is the unique subgroup of order 2.

We identify $(x,K/k)^{n/(n,2)}$ with $+1$ if it is the neutral element of the galois group, and with -1 if it is a non-trivial element. This being said, we assert:

Theorem 2

If K/k is abelian of degree n, $c \in k^*$, then
$\prod_{j=1}^{n} \tau_j(c) = (c,K/k)^{n/(n,2)}$ and therefore
$(\varkappa(n,K/k))^2 = (-1,K/k)^{n/(n,2)}$.

Proof. If the group is cyclic then we may set $\tau_j = \tau^j$, whence the product is $\tau(c^{n/2})$ if n is even and otherwise is 1. As $\tau(x) = 1 \iff (x,K/k) = 1$, the assertion follows for K/k cyclic and in particular for n prime. To prove the theorem in general, induction is used. Using the same notation and the same induction hypothesis as in the proof of the first part of Theorem I,

$$\prod_{j=1}^{n} \tau_j(c) = \prod_{s=1}^{q} \prod_{j=1}^{n'} (\tau_s \delta_j)(c) = \prod_{s=1}^{q} \Big(\tau_s(c)^{n'} \prod_{j=1}^{n'} \delta_j(c) \Big)$$

$$= \Big(\prod_{s=1}^{q} \sigma_s(c)\Big) \Big(\prod_{j=1}^{n'} \delta_j(c) \Big)^q \quad \Big(\text{as } \sigma_s(c) = (\tau_s \circ N_{K'/k})(c) = \tau_s(c^{n'})\Big).$$

Applying the induction hypothesis, it is found that the product under consideration is $(c,K/K')^{q/(q,2)}(c,K'/k)^{qn'/(2,n')}$. In completing the proof, the question of whether or not q is even must be considered.

(a) q even. Then $qn'/(2,n')$ is a multiple of n', whence the second factor is 1. Also, $(c,K/K')^{q/(q,2)} = (N_{K'/k}c,K/k)^{q/2}$ $= (c,K/k)^{qn'/2}$, which completes the proof if q is even.

(b) q odd. Certainly $(c, K/K')^{q/(q,2)} = 1$. Also $(n,2) = (n',2)$. Hence it is enough to show that $(c, K'/k)^{n/(n,2)} = (c, K/k)^{n/(n,2)}$. If the right side is 1 then the left side is also 1 as if the two sides are considered as elements of galois groups the left side is the restriction to K' of the right side. If the left side is one then there exists $x \in K'$ such that $N_{K'/k}x = c^{n/(n,2)}$, whence $N_{K/k}\, x = c^{qn/(n,2)}$, whence the q^{th} power of the right side is 1. But q is odd and the right side is +1 or -1 and therefore the right side is one. Hence the two sides are equal.

The final theorem of this chapter gives an almost complete specification of $\chi(m, K/k)$.

Theorem 3

If K is an abelian extension of k of degree n, relative ramification e, relative residue class degree f, let d be the exponent of the absolute different of k and let e' be the largest integral divisor of e which is relatively prime to 2p ($\mathfrak{p}$ | p), then

(a) $$\frac{\chi(m,K/k)}{\chi(-1,K/k)} = \begin{cases} 1 & \text{if } m = 0 \\ \left(\frac{N\mathfrak{p}}{e'}\right)^{1+m} & \text{if } m \neq 0 \end{cases}$$

(b) If $2 \not| e$ then
$$\chi(-1, K/k) = (-1)^{e(f-1)d}\left(\frac{N\mathfrak{g}_{K/k}}{e'}\right)$$

Proof

Let K_0 be the inertial subfield of K/k

K_1 be the first ramification subfield of K/k

K_0' the largest field between K_1 and K_0 such that degree (

degree (K_0'/K_0) is a power of 2.

From the results for unramified extensions and the corollary to Theorem 1, it is clear that

$$\chi(m,K/k) = \chi(m,K/K_0)(-1)^{e(f-1)d}$$

therefore we may assume $k = K_0$. Furthemore from the results for cyclic extensions of degree 2 and the same corollary it follows that

$$\chi(m,K/K_0) = \chi(m,K/K_0')(\chi(-1,K_0'/K_0))^{(K/K_0')}$$

Hence the problem has been reduced to proving the theorem for:

K/k abelian, purely ramified, K_1 the first ramification subfield of K/k, $e' = K_1/k$, $2 \nmid e'$. We first assert

<u>Contention</u>: The theorem is valid if $K = K_1$.

<u>Proof</u> We are now considering the case K/k abelian, purely ramified of conductor $\mathfrak{f}$. The proof is by induction. $e' = \deg K/k$. If e' is prime then $\chi(m,K/k) = 1$ if $m = 0$ or -1

$$\left(\frac{N\mathfrak{p}}{e'}\right)^{1+m} \text{ for } m > 0$$

while, $\vartheta_{K/k} = \mathfrak{p}^{e'-1} \Rightarrow \left(\frac{N\vartheta_{K/k}}{e'}\right) = 1$ as $e'-1$ is even, whence the contention is correct if e' is prime.

If then e' is not prime, use induction on the number of prime divisors of e', and let K' be an intermediate field such that $K/K' = q$, a prime. Then in the notation of the corollary (Theorem 1) $s_1 = -1$, $s_j = 0$ for $j > 1$. Let $m = g_{K/K'}(b)$ then

$$\chi(m,K/k) = \chi(m,K/K')\chi(\mathrm{Max}(-1,b),K'/k)\left[\chi(\mathrm{Max}(0,b),K'/k)\right]^{q-1}$$

$$= \chi(m,K/K')\ \chi(\mathrm{Max}(-1,b),K'/k)$$

as $q-1$ is even.

If $m = 0$, $b = 0$

$m = -1$, $b = -1$

$m > 0$, $b = \frac{m}{q}$.

Hence $\chi(-1, K/k) = \chi(-1, K/K') \chi(-1, K'/k)$

$\chi(0, K/k) = \chi(0, K/K') \chi(0, K'/k)$

$\chi(m, K/k) = \chi(m, K/K') \chi(\frac{m}{q}, K'/k)$ for $m > 0$.

By induction hypothesis, $\chi(-1, K'/k) = \left(\frac{N\mathfrak{g}_{K'/k}}{e'/q}\right) = 1$

We know that $\chi(-1, K/K') = 1$

Hence $\chi(-1, K/k) = 1 = \left(\frac{N\mathfrak{g}_{K/k}}{e'}\right)$, which proves (b) for $K = K_1$.

As $\chi(0, K/K') = 1$, while by induction hypothesis $\chi(0, K'/k) = 1$, it follows that $\chi(0, K/k) = 1$.

For $m > 0$, let $\mathfrak{q}$ be the prime of K', then $N\mathfrak{q} = N\mathfrak{p}$ and $2|m$ $\Leftrightarrow$ $2|(m/q)$. Hence by the induction hypothesis, $\chi(m/q, K'/k) = \left(\frac{N\mathfrak{q}}{e'/q}\right)^{1+(m/q)} = \left(\frac{N\mathfrak{p}}{e'/q}\right)^{1+m}$, while we know that $\chi(m, K/K') = \left(\frac{N\mathfrak{p}}{q}\right)^{1+m}$.

It follows that $\chi(m, K/k) = \left(\frac{N\mathfrak{p}}{e'}\right)^{1+m}$ which completes the proof of the contention.

Returning to the more general case $(K \neq K_1)$ (but K/k purely ramified, $2 \nmid e'$), let K_1 be the intermediate field in the corollary to Theorem 1. Then in the notation used there, $s_1 = -1, s_j > 0$ for $j > 1$. Letting $g_{K/K_1}(b) = m$, we have

$$\chi(m, K/k) = \chi(m, K/K_1) \prod_{j=1}^{e/e'} \chi(\mathrm{Max}(s_j, b), K_1/k).$$

From the computations for the cyclic case of prime degree, it is clear that $\chi(m,K/K_1) = \chi(-1,K/k)$ and is actually 1 if K/K_1 is of odd degree (i.e. if either $p \neq 2$ or $K = K_1$).

Hence $\chi(m,K/k) = \chi(-1,K/K_1) \prod_{j=1}^{e/e'} \chi(\mathrm{Max}(s_j,K_1/k)$.

If $m = 0$ then $b = 0$, whence

$$\chi(0,K/k) = \chi(-1,K/K_1)\ \chi(0,K_1/k) \prod_{j=2}^{e/e'} \chi(s_j,K_1/k)$$

while if $m = -1$ then $b = -1$, whence

$$\chi(-1,K/k) = \chi(-1,K/K_1)\ \chi(-1,K_1/k) \prod_{j=2}^{e/e'} \chi(s_j,K_1/k),$$

so that $\chi(0,K/k) = \chi(-1,K/k)$, as $\chi(0,K_1/k) = \chi(-1,K_1/k)$.

For any $m \neq 0$ we now have (for $g_{K/K_1}(b) = m$)

$$\frac{\chi(m,K/k)}{\chi(-1,K/k)} = \prod_{j=1}^{e/e'}\left[\frac{\chi(\mathrm{Max}(b,s_j),K_1/k)}{\chi(\mathrm{Max}(-1,s_j),K_1/k)}\right]$$

$$= \prod_{j=1}^{e/e'}\left[\frac{\chi(\mathrm{Max}(b,s_j),K_1/k)}{\chi(s_j,K_1/k)}\right] = \prod_{j=1}^{e/e'}\left(\frac{N\mathfrak{p}}{e'}\right)^{\mathrm{Max}(b,s_j)-s_j}$$

$$=\left(\frac{N\mathfrak{p}}{e'}\right)^{1+m}$$ which completes the proof of (a).

For (b), we may assume that $p \neq 2$ (as for $p = 2$, $2 \nmid e \Rightarrow K = K_1$ and this case has already been taken care of). We then have

$\chi(-1,K/K_1) = 1$, whence

$$\chi(-1,K/k) = \prod_{j=2}^{e/e'} \chi(s_j,K_1/k) = \left(\frac{N\mathfrak{p}}{e'}\right)^{\sum_{j=1}^{e/e'}(1+s_j)} = \left(\frac{N\mathfrak{p}\,\vartheta_{K/K_1}}{e'}\right).$$

But $\vartheta_{K/k} = \vartheta_{K/K_1}\ \vartheta_{K_1/k}$ and $\vartheta_{K_1/k} = \mathfrak{p}_1^{e'-1}$. As e' is odd it follows that $\left(\frac{N\vartheta_{K_1/k}}{e'}\right) = 1$, which completes the proof of the theorem.

Chapter IV

The Weil L-Series

§1. Some Group-theoretical Results

Note: Many of the results of this section appear in the writeings of Mackey,(10), Shoda,(11), Takota,(12). In particular, Taketa has given a slightly weaker form of lemma 7. (below). The analogue of the Frobenius Reciprocity Law for characters of an infinite group which are induced by characters of a subgroup of finite index, which is used several times in the following, is proven in Mackey's article.

(10) Mackey,G., "On induced representations of groups", Amer.Jour. of Math,1951 (73).
(11) Shoda,K., "Uber die monomialen Darstellungen Endlichen gruppe", Proceedings of the Physico-Mathematical Society of Japan,1933 (15).
(12) Taketa,K., "Uber die Gruppen deren Darstellungen samtlich auf monomiale gestalt transformieren lassen", Proceedings of the Imperial Academy of Japan, 1930 (6).

In the following let G be a group, H an invariant subgroup, the index not necessarily finite, X a character of G (as usual we restrict character to refer to a bounded matrix representation). If θ is a character of H and $x \in G$ then $h \to \theta(xhx^{-1})$ is also a character of H, which we shall call the x conjugate of θ and denote by θ^x. The set of all $(\theta^x)_{x\in G}$ may be referred to as the set of G conjugates of θ.

<u>Lemma 1</u> If δ is an irreducible character of H occurring in X_H, the restriction of X to H, then every conjugate of δ occurs in X_H the same number of times as δ. The set G_δ of all $x \in G$ such that $\delta^x = \delta$ is a subgroup of G containing H and of index in G equal to the number of distinct conjugates of δ. The conjugates of δ correspond to the right cosets of G_δ. Finally $G_{\delta^x} = x^{-1}(G_\delta)x$

<u>Proof</u> For $x, y \in G$, $(\delta^x)^y(h) = \delta^x(yhy^{-1}) = \delta(xyhy^{-1}x^{-1}) = \delta^{xy}(h)$ for all $h \in H$, i.e. $(\delta^x)^y = \delta^{xy}$. It follows immediately that G_δ is a subgroup of G containing H. If δ occurs in X_H, n times then for $x \in G$, δ^x occurs in X_H^x at least n times (as δ^x is also irreducible). But $X_H^x = X_H$, hence δ^x occurs in X_H at least n times. As δ is a conjugate of δ^x, δ occurs at least as often as δ^x whence δ^x occurs exactly n times. It follows that δ has only a finite set of distinct conjugates. For $x, y \in G$, $\delta^x = \delta^y \Leftrightarrow \delta^{xy^{-1}} = \delta \Leftrightarrow xy^{-1} \in G_\delta \Leftrightarrow x \in G_\delta y$, which shows that there exists a one to one correspondence between the right cosets of G_δ in G and the distinct conjugates of δ. Finally $(\delta^x)^y = \delta^x = xyx^{-1} \in G_\delta$, whence $G_{\delta^x} = x^{-1}(G_\delta)x$

<u>Lemma 2</u> If X is irreducible and δ is an irreducible character of H which occurs in the restriction, X_H, of X to H and if either

1) $(G:H)$ is finite

or 2) δ is of degree 1

then X_H is the sum of conjugates of δ.

<u>Proof</u> 1) If $(G:H)$ finite, let $t_1, \ldots, t_s$ be a set of representatives

of distinct right cosets of H in G. Let X_δ be the character of G induced by δ. Then X_δ contains X, but the restriction of X_δ to H is simply $\Sigma_{i=1}^{s} \delta^{t_i}$ which proves the assertion.

2) If δ is of degree 1, let $\delta_1, \ldots, \delta_s$ be the distinct G conjugates of δ; by lemma 1 they all occur the same number of times. Let V be the representation space of a representation ρ of G whose character is X. Let $V_i (i=1,2,\ldots,s)$ be the set of all $v \in V$ such that $\rho(h)v = \delta_i(h)v$ for all $h \in H$. Let the direct sum of the V_i be W. Let $v \in V_1$, $x \in G$ then for all $h \in H$, $\rho(h)(\rho(x)v) = \rho(hx)v = \rho(x)(\rho(x^{-1}hx)v) = \rho(x)(\delta(x^{-1}hx)v) = \delta^{x^{-1}}(h)(\rho(x)v)$. As $\delta^{x^{-1}}$ is a conjugate of δ, there exists i $(1 \leq i \leq s)$ such that $\delta^{x^{-1}} = \delta_i$, whence $\rho(x)V_1 \subset V_i$, i.e. $\rho(x)V_1 \subset W$. It follows that $\rho(x)W \subset W$, i.e. W is a subspace of V invariant under ρ. As ρ is irreducible, W must be V.

<u>Corollary</u> For X irreducible, if the restriction of X to H contains the principal representation of H then it contains only the principal representation.

<u>Lemma 3</u> Under the conditions of lemma 2, X is induced by an irreducible character σ of G_δ. The restriction of σ to H contains only δ.

<u>Proof</u>

The restriction of X to G_δ must contain an irreducible character σ of G_δ with the property that the restriction of σ to H contains δ. By lemma 2, the restriction of σ to H contains only G_δ conjugates of δ, i.e. just δ, say m times. We assert that X is induced by σ. Let X_σ be the character of G induced by σ. Certainly

X occurs in X_σ and therefore it is enough to show that the degrees are equal. Let $r = (G:G_\delta)$, then $X_\sigma(1) = r\sigma(1) = rm\delta(1)$. But by Lemma 1, r is the number of distinct G conjugates of δ. As σ occurs in the restriction of X to G_δ, and δ occurs m times in the restriction of σ to H, we may conclude that δ occurs at least m times in the restriction of X to H. Each G conjugate of δ must therefore also occur at least m times, whence $X(1) \geq rm\delta(1) = X_\sigma(1)$ which proves that X and X_σ are of the same degree.

Lemma 4 If $(G:H)$ is finite, and δ an irreducible character of H and X_δ the character of G induced by δ then X_δ is irreducible if and only if $H = G_\delta$. If in addition H is abelian then all irreducible characters of G of degree $(G:H)$ are induced by irre- characters δ of H for which $H = G_\delta$.

Proof

If X_δ is irreducible then by the induced character theorem, δ occurs just once in the restriction of X_δ to H. Hence each conjugate of δ occurs just once. By Lemma 2, only conjugates of δ occur. If r is the number of conjugates then $X_\delta(1) = r\delta(1)$. But $X_\delta(1) = (G:H)\,\delta(1)$. Hence $(G:H) = (G:G_\delta)$, whence $G_\delta = H$.

Conversely if $G_\delta = H$, then δ has $(G:H)$ distinct G conjugates. Among the irreducible characters of G occuring in X_δ, there must be one, X, whose restriction to H contains δ. X irreducible implies each conjugate of δ occurs in this restriction. Hence $X(1) \geq (G:H)\delta(1) = X_\delta(1) \geq X(1)$, whence $X_\delta = X$.

If H is abelian and X an irreducible character of G of degree $(G:H)$, let δ be an irreducible character of H appearing in the restriction of X to H. Then δ is of degree 1, whence the character of G induced by δ is of degree (G:H) and contains X and therefore is X. It follows from the above that δ must have (G:H) distinct conjugates, whence $H = G_\delta$.

<u>Lemma 5</u> If H is abelian and G/H cyclic, X an irreducible character of G and δ an irreducible character of H contained in the restriction of X to H, then X is induced by a character φ of G_δ of degree one.

<u>Proof</u>

By Lemma 3, X is induced by an irreducible character φ of G_δ whose restriction to H is just $\varphi(1)\delta$ (as δ is of degree 1). Let m be the degree of φ, then there exists a unitary matrix representation, ρ, of G_δ and of character φ such that for all $h \in H$, $\rho(h) = \delta(h)I_m$, I_m being the unit matrix of rank m. It follows that $\rho(H)$ lies in the center of $\rho(G_\delta)$. G/H cyclic $\Rightarrow G_\delta/H$ cyclic $\Rightarrow \rho(G_\delta)/\rho(H)$ cyclic; whence $\rho(G_\delta)$ is a cyclic group extension of its center and therefore must be an abelian group of unitary matrices, but it is also irreducible and therefore the degree must be one.

<u>Lemma 6</u> Under the conditions of Lemma 5, if G' is any subgroup of G which contains H and φ' is a character of degree 1 of G' which induces X then $G' = G_\delta$ and there exists an inner automorphism σ of G such that $\varphi' = \varphi \circ \sigma$.

<u>Proof</u>

As G/H is cyclic every subgroup of G containing H is an invariant subgroup of G. In particular then G' and G_δ are invariant subgroups of G. The restriction of X to G' contains φ' and therefore the restriction of φ' to H is a character of H of degree one which occurs in the restriction of X to H, whence the restriction of φ' to H is, δ', a conjugate of δ. It follows that there exists an inner automorphism σ of G such that X is induced by the character, $\varphi' \circ \sigma^{-1}$, of $\sigma(G') = G'$; this character is of degree one and its restriction to H is δ. Hence we may assume that the restriction of φ' to H is δ. For $x \in G'$, $h \in H$, $\delta^x(h) = \delta(xhx^{-1}) = \varphi'(xhx^{-1}) = \varphi'(h) = \delta(h)$, whence $G' \subset G_\delta$. But by Lemma 5, $(G:G_\delta) = X(1) = (G:G')$, whence $G' = G_\delta$. By Lemma 2, the restriction of X to G_δ contains only conjugates of φ and therefore the assertion follows.

Lemma 7 If X is irreducible and H is abelian and there exists a sequence of intermediate groups, invariant in G, $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n = H$ such that G_j/G_{j+1} is cyclic $(j=0,1,\ldots,n-1)$ and δ is an irreducible character of H appearing in the restriction of X to H then there exist a character φ of degree 1 of a group lying between G_δ and H such that φ induces X.

Note: 1) If (G:H) is finite then the condition may be stated: the prime factor groups of the principal series of G/H are cyclic. This condition is stronger than the condition that G/H is solvable. If G/H is solvable then we may only conclude that the prime factor groups are direct products of isomorphic cyclic groups of prime order (cf Sperser, Gruppentheorie, 3rd edition Satz 32,33). Taketa has given an example of a solvable group of order 24 (a group with the quaternion group as invariant subgroup of index 3) for which the conclusions of this lemma do not hold.

Proof. As the assertion is known to be true for $n = 1$, i.e. for G/H cyclic, we may apply induction on n and assume that the assertion is valid if the chain is of length $n-1$. By lemma 3, X is induced by an irreducible character σ of G_δ whose restriction to H contains only δ. Let $G_j' = G_j \cap G_\delta$ $(0 \leq j \leq n)$, then $G_\delta = G_0' \supset G_1' \supset \dots \supset G_n' = H$ and the sequence has the property: G_j' invariant in G_0', G_j'/G_{j+1}' cyclic. There exists a matrix representation ρ of $G_0' = G_\delta$, corresponding to the character σ such that for $h \in H$, $\rho(h) = \delta(h)\, I_m$, where $m = \deg \sigma$ and I_m is the unit matrix of rank m. Let $C_j = \rho(G_j')$ $(0 \leq j \leq n)$, then $\rho(H) = C_n$ abelian, C_j invariant subgroup of C_o and C_j/C_{j+1} cyclic. Clearly C_n lies in the center of C_o, while C_{n-1}/C_n is cyclic, whence C_{n-1} is <u>abelian</u>. Let $\bar{\sigma}$ be the character $x \to \operatorname{Tr} x$ of C_o, then $\sigma = \bar{\sigma} \circ \rho$ and $\bar{\sigma}$ is irreducible. Then $C_o \supset C_1 \supset \dots \supset C_{n-1}$ is a chain of length $n-1$ to which the induction hypothesis applies, hence $\bar{\sigma}$ is induced by a character $\bar{\varphi}$ of degree 1 of a subgroup $\bar{g}$ of C_o which contains C_{n-1}. Let $g = \rho^{-1}(\bar{g})$, φ the character $\bar{\varphi} \circ \rho$ of g. We assert that X is induced by the character φ of g. We note that φ is of degree 1, that g contains the kernal of ρ and also g contains H. Clearly $(C_o : \bar{g}) = \deg \bar{\sigma}$. Hence $C_o = \bigcup_{j=1}^{m} \bar{g} c_j$ (where $m = \deg \bar{\sigma}$), a disjoint union. There exists $d_j \in G_o'$ such that $\rho(d_j) = c_j$ $(1 \leq j \leq m)$ and therefore $G_o' = \rho^{-1}(C_o) = \bigcup_{j=1}^{m} g d_j$, also a disjoint union as $g d_j \cap g d_{j'}$ not null implies $c_{j'} \in \bar{g}\, c_j$. For $z \in C_o$

$\bar{\sigma}(z) = \Sigma_{j=1}^{m} \bar{\varphi}(c_j z c_j^{-1})$, whence for $x \in G_\delta^1$, $\sigma(x) = \bar{\sigma}(\rho(x))$

$= \Sigma_{j=1}^{m} \bar{\varphi}(c_j \rho(x) c_j^{-1}) = \Sigma_{j=1}^{m} (\bar{\varphi} \circ \rho)(d_j x d_j^{-1}) = \Sigma_{j=1}^{m} \varphi(d_j x d_j^{-1})$.

It follows that σ is induced by the character φ of g. But the character σ of G_δ induces X and by the transitivity law for such relations, X is induced by φ which completes the proof.

§2. Let K be a finite normal extension of an algebraic number field k. Let C_K be the group of idele classes of K. $W_{K/k}$, the group extension of C_K by the galois group, $G(K/k)$, corresponding to the canonical cohomology class has been extensively studied by Weil, (6), and for that reason the group has been referred to as the Weil group. If $\mathfrak{y}$ is a prime of k, $k_{\mathfrak{y}}$, the corresponding completion of k, then we define, $W_{Kk_{\mathfrak{y}}/k_{\mathfrak{y}}}$, (local Weil group), to be the group extension of $(Kk_{\mathfrak{y}})^*$ by $G(Kk_{\mathfrak{y}}/k_{\mathfrak{y}})$ corresponding to the local canonical cohomology class. It is known that this group may be identified with a decomposition subgroup of $W_{K/k}$ at $\mathfrak{y}$, a subgroup which is unique to within inner automorphisms of $W_{K/k}$. This decomposition subgroup is a group extension of V by G_z, V being one of the natural images in C_K of $(Kk_{\mathfrak{y}})^*$ and G_z being the corresponding decomposition subgroup of $G(K/k)$.

If $W_{K/k}$ is a local Weil group at a discrete prime, $\mathfrak{y}$, X a character, H the inertial subgroup and t an element in the Frobenius class then we define a local zeta function, $Z(s,X,K/k)$, by setting: $\log Z(s,X,K/k) = \Sigma_{n>0} (n\,N\mathfrak{y}^{ns})^{-1} \int_H X(xt^n)\,dx$, for $\mathrm{Re}\, s > 0$, where dx is the Haar measure of H normalized so that the measure of H is 1. As the Frobenius class is a coset of H, the definition is independent of the choice of t. Trivially, $Z(s,X+X',K/k) = Z(s,X,K/k)\,Z(s,X',K/k)$. Let X be irreducible and ρ a unitary matrix representation of $W_{K/k}$ whose character is

(6) Weil, A., "Sur la Theorie du Corps de Classes", J. Math. Soc. of Japan, vol 3 (1951)

X then as H is compact, by suitable choice of ρ, $\int_H \rho(x)\,dx$ is a matrix, all of whose elements are zero except for r diagonal elements which are unity, where r is the number of times the trivial representation of H occurs in the restriction of ρ to H. Lemma 2. (§1) shows that r is either 0 or deg X. If the latter, then ρ is trivial on H and therefore $\rho(W_{K/k})$ is an irreducible cyclic group of unitary matrices, whence $r = 1$. As $\int_H X(xt^n)dx$ is the trace of $\rho(t^n)\int_H \rho(x)\,dx$, it follows that for X irreducible, the zeta function is 1 unless X is of degree 1 and trivial on H, in which case it is $(1-X(t)\,N\mathfrak{y}^{-s})^{-1}$ for Re $s > 0$.

If $W_{K/k}$ is a local Weil group at an archimedean prime the the local zeta function $Z(s,X,K/k)$ is defined to be 1. In any case the local zeta function has an obvious analytic continuation.

If $W_{K/k}$ is a global Weil group and X is a character then the Weil L-series is defined to be:

$$L(s,X,K/k) = \prod_{\mathfrak{y}} Z(s,X_{\mathfrak{y}},Kk_{\mathfrak{y}}/k_{\mathfrak{y}}) \quad \text{for Re } s > 1,$$

the product being taken over all primes of k, and $X_{\mathfrak{y}}$ being the restriction of X to the local Weil group, $W_{Kk_{\mathfrak{y}}/k_{\mathfrak{y}}}$. As X is invariant under inner automorphisms, the local zeta functions which appear are well defined. It is readily shown that for fixed s the mapping, F: $X \to L(s,X,K/k)$, of characters of $W_{K/k}$ into the complex plane has the properties:

(I) If X, X' are characters of $W_{K/k}$ then $F(X+X',K/k) = F(X,K/k)\,F(X',K/k)$.

(II) If L is an intermediate field, φ a character of $W_{K/L}$, X_{φ}

the induced character of $W_{K/k}$ then

$$F(\varphi, K/L) = F(X_\varphi, K/k) \qquad \text{(Induced character property)}$$

(III) If θ is a character of C_k, X the character of $W_{K/k}$ obtained by composing θ with the transfer homomorphism of $W_{K/k}$ into C_K (a mapping onto C_k) then $F(\theta, k/k) = F(X, K/k)$.

(IV) If L is an intermediate field, normal over k, X a character of $W_{L/k}$, $t_{K/k \to L/k}$ the homomorphism of $W_{K/k}$ onto $W_{L/k}$ described by Weil then

$$F(X, L/k) = F(X \circ t_{K/k \to L/k}, K/k).$$

By means of the first three properties and an extension of Brauer's theorem to characters of the Weil group, Weil proved that his L-series is a product of Hecke L-series and therefore is a meromorphic function in s. By extending the Artin conductor to characters of the Weil group it will be easy to extend the functional equation of the Artin L-series to a functional equation of the Weil L-series. A more difficult problem is that of decomposing the root number appearing in this functional equation into root numbers depending upon the local behavior of the global character. It will be shown that this problem may be reduced to the problem of extending in a specified manner ρ_k, Tate's function on local quasi-characters (which may be interpreted with s fixed as a function on local characters) to a function defined on the characters of local Weil groups.

§3.

For simplicity we shall refer to the properties I - IV of the previous section as the Artin conditions. We are interested in mappings $F_{\mathfrak{y}}$: $X_{\mathfrak{y}} \to F_{\mathfrak{y}}(X_{\mathfrak{y}}, Kk_{\mathfrak{y}}/k_{\mathfrak{y}})$ of local characters (i.e. of characters of local Weil groups) into the complex plane which have the property that if $\mathfrak{y}_0$ is some fixed prime of some lower field k_0 then the function on global characters:

$$F(X,K/k) = \prod_{\mathfrak{y}|\mathfrak{y}_0} F_{\mathfrak{y}}(X_{\mathfrak{y}}, Kk_{\mathfrak{y}}/k_{\mathfrak{y}}) \qquad (A)$$

satisfies the Artin conditions ($X_{\mathfrak{y}}$ being related to X as defined in the previous section).

This product is not well defined unless $F_{\mathfrak{y}}(X_{\mathfrak{y}}, Kk_{\mathfrak{y}}/k_{\mathfrak{y}})$ has the property of being independent of the choice of replica of $W_{Kk_{\mathfrak{y}}/k_{\mathfrak{y}}}$ as decomposition subgroup of $\mathfrak{y}$ in $W_{K/k}$. Interpreting the Artin conditions as conditions on a mapping F of local characters into the complex plane, (with C_K replaced by K^*) we are thus led to the following (local) condition which is certainly satisfied if F(X,K/k) is defined by intrinsic properties.

(V) K,K' isomorphic normal extentions of a local number field k, σ an isomorphism of K onto K' leaving each element of k fixed, then certainly σ may be considered as an isomorphism of $W_{K/k}$ with $W_{K'/k}$. If X' is a character of $W_{K'/k}$ then the condition is that $F(X',K'/k) = F(X' \circ \sigma, K/k)$.

We assert that the local Artin conditions I-V for the local functions $F_{\mathfrak{y}}$ imply that the global function F, defined by formula (A) satisfies the global Artin conditions I - IV. Except for the induced character property the proof is immediate. The proof of the assertion then follows from the following lemma.

<u>Lemma</u> k an algebraic number field, K a normal extension of finite degree, L an intermediate field, φ a character of $W_{K/L}$, X the induced character of $W_{K/k}$, $\mathfrak{y}$ a prime of k, $\mathfrak{y}_1, \ldots, \mathfrak{y}_r$ the extensions of $\mathfrak{y}$ to L and $\mathfrak{P}_i$ an extension of $\mathfrak{y}_i$ to K. Let $W^{(i)}_{K/L}$ be a decomposition subgroup of $W_{K/k}$ at $\mathfrak{P}_i$ and then $W^{(i)}_{K/L} = W_{K/L} \cap W^{(i)}_{K/k}$ is a decomposition subgroup of $\mathfrak{P}_i$ in $W_{K/L}$. Let $t_1, \ldots, t_r$ be representatives in $W_{K/k}$ of elements of $G(K/k)$ which permute the primes $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ transitively. Let φ_i be the restriction of φ to $W^{(i)}_{K/L}$ and X_i the character of $W^{(i)}_{K/k}$ induced by φ_i. Then $t_i\, W^{(1)}_{K/k}\, t_i^{-1} = W^{(i)}_{K/k}$ $(1 \leq i \leq r)$, and for $w \in W^{(1)}_{K/k}$ we have $X(w) = \Sigma^r_{i=1} X_i(t_i w t_i^{-1})$.

<u>Proof</u> Let U be the galois group of K/L, then $G = G(K/k) = \bigcup^r_{i=1} \bigcup^{n_i}_{j=1} U z_{ij} b_i$ where $b_i \mathfrak{P}_1 = \mathfrak{P}_i$, $n_i = \deg(L_{\mathfrak{y}_i}/k_{\mathfrak{y}})$, and $(z_{ij})_{j=1,\ldots,n_i}$ are representatives of the right cosets of the decomposition subgroup of $\mathfrak{P}_1$ in U in the decomposition subgroup of $\mathfrak{P}_1$ in G. The t_i are chosen so as to lie in the class modulo C_K corresponding to b_i. Pick $(s_{ij})_{j=1,\ldots,n_i}$ such that $s_{ij} \in W^{(1)}_{K/k}$ and such that z_{ij} corresponds to the class of s_{ij} modulo C_K. Then

$$W_{K/k} = \bigcup^r_{i=1} \bigcup^{n_i}_{j=1} W_{K/L} s_{ij} t_i,$$ whence for $w \in W_{K/k}$

$X(w) = \Sigma_{i=1}^{r} \Sigma_{j=1}^{n_i} X(s_{ij} t_i w t_i^{-1} s_{ij}^{-1})$. But for $w_i \in W_{K/k}^{(i)}$,

$X_i(w_i) = \Sigma_{j=1}^{n_i} \varphi_i(s_{ij} w_i s_{ij}^{-1})$ while for $w \in W_{K/k}^{(1)}$,

$t_i w t_i^{-1} \in W_{K/k}^{(i)}$, whence $X_i(t_i w t_i^{-1}) = \Sigma_{j=1}^{n_i} \varphi_i(s_{ij} t_i w t_i^{-1} s_{ij}^{-1})$

which proves the lemma.

§4. The Functional Equation

If θ is an irreducible character of C_k then the Weil L-series, $L(s, \theta, k/k)$, coincides with Hecke's and its functional equation may be expressed as:

$L(s, \theta, k/k) C(s, \theta, k/k) A(s, \theta, k/k) W(\theta, k/k) =$

$L(1-s, \overline{\theta}, k/k) C(1-s, \overline{\theta}, k/k) A(1-s, \overline{\theta}, k/k)$,

where $f(\theta, k/k)$ = conductor of θ

$W(\theta, k/k)$ is a unimodular complex number

$A(s, \theta, k/k) = (d_k N_{k/Q} f(\theta, k/k))^{s/2}$, Q = field of rationals numbers

n = absolute degree of k

d_k = absolute discriminant of k

$C(s, \theta, k/k) = \prod_{y | p_\infty} \gamma(s, \theta_y, k_y/k_y)$

θ_y = restriction of θ to the natural image of k_y^* in C_k

and

$\gamma(s, \theta_y, k_y/k_y) = \Gamma((s+it_y)/2) \pi^{-s/2}$ if y real, and $\theta_y(x) = |x|^{it_y}$

$= \Gamma((1+s+it_y)/2) \pi^{-s/2}$ if y real, and $\theta_y(x) = |x|^{it}$ (sign x)

$= \pi^{-s} \Gamma((s+it_y+|a_y|)/2) \Gamma((1+s+it_y+|a_y|)/2)$,

if y complex and $\theta_y(re^{i\theta}) = r^{2it_y} e^{i2a_y}$ (t_y real, $2a_y$ an integer).

From Weil's extension of Brauer's theorem to characters of Weil groups it follows that the functional equation of the Weil L series, $L(s,X,K/k)$, X a character of the global Weil group $W_{K/k}$ is

$$L(s,X,K/k)\, C(S,X,K/k)A(s,X,K/k)W(X,K/k) =$$
$$L(1-s,\bar{X},K/k)\, C(1-s,\bar{X},K/k)A(1-s,\bar{X},K/k)$$

where $W(X,K/k)$ is a unimodular complex number, provided that we may define $C(s,X,K/k)$, and $A(s,X,K/k)$ as functions on characters of Weil groups which satisfy the Artin conditions and in addition assume the form indicated above in the special case when $\begin{cases} X \text{ of degree } 1 \\ K = k. \end{cases}$ (It is already known that $L(s,X,K/k)$ has this property). We consider the gamma function factors and the term involving the conductor separately.

a) Conductor Factor

The conductor of a character of a local Weil group has already been defined. The $\mathfrak{y}$ inertial subgroup of $W_{K/k}$ is a group extension of the units of $(K k_{\mathfrak{y}})^*$ by the inertial subgroup of $G(Kk_{\mathfrak{y}}/k_{\mathfrak{y}})$. As almost all primes of k are unramified in K it follows that for almost all $\mathfrak{y}$ the inertial subgroup is just the group of units of $(Kk_{\mathfrak{y}})^*$. The restriction of X to C_K is a finite sum of characters of degree 1 and at almost all primes these characters are unramified. It follows that X is just $X(1)$ on almost all inertial subgroups of $W_{K/k}$. We now define the conductor of X to be

$f(X,K/k) = \prod_{\mathfrak{y}} f(X_{\mathfrak{y}}, Kk_{\mathfrak{y}}/k_{\mathfrak{y}})$, the product being taken over all discrete primes of k, $X_{\mathfrak{y}}$ being the restriction of X to the local Weil group $W_{Kk_{\mathfrak{y}}/k_{\mathfrak{y}}}$ and $f(X_{\mathfrak{y}}, Kk_{\mathfrak{y}}/k_{\mathfrak{y}})$ being the conductor of X_y as defined in Chapter I. The global conductor is well defined as by the above argument the local contribution is just 1 for almost all primes and because the local conductor is defined by intrinsic properties. Deviating slightly from Artin, we now set:

$$A(s,X,K/k) = (d_k^{X(1)} N_{k/Q} f(X,K/k))^{s/2}$$

Certainly this assumes the desired form when $K = k$, deg $X = 1$. To verify conditions I-IV, it is enough to consider the expression within the brackets, i.e. $A(X,K/k) = \prod_y A(X_y, Kk_y/k_y)$, where

$$A(X_y, Kk_y/k_y) = d_{k_y}^{X_y(1)} N_{k_y/Q_p} f(X_y, Kk_y/k_y) \text{ , } d_{k_y}$$

being the discriminant of k_y. As this function on local characters has been shown to satisfy the local Artin conditions, it follows that $A(s,X,K/k)$ has the desired global properties.

b) Gamma Function Factors

For these terms it is enough to extend the functions, γ, defined above for characters of R^* and C^* to characters of the local Weil group, $W_{C/R}$. (R the field of real numbers and C the field of complex numbers). If X is an irreducible character of $W_{C/R}$ then it is contained in the character of $W_{C/R}$ induced by any of the irreducible characters of C^* appearing in the restriction of X to C^*. Hence X is at most of degree 2. If X is linear then it is trivial on the commutator subgroup and therefore may be obtained by transfer from a linear character of R^*. If X is of degree 2 then (Lemma 4, §1) it is induced by a linear character, δ, of C^* which is not invariant under the galois group, $(1,\sigma)$, of C/R and therefore the restriction of X to C^* is $\delta + \delta^\sigma$. Hence δ^σ also induces X but no other character of C^* has this property. It follows that if X is linear then we may define $\gamma(s,X,C/R) = \gamma(s,\delta,R/R)$, where δ is the character of R^* from which X is obtained by transfer, while if X is irreducible of degree 2 we define $\gamma(s,X,C/R) = \gamma(s,\delta,C/C)$, where δ is a character of C^* which induces X. In this second case, $\gamma(s,\delta,C/C) = \gamma(s,\delta^\sigma,C/C)$ and therefore in any case $\gamma(s,X,C/R)$ is well defined if X is irreducible and this definition may be extended to reducible characters in the manner suggested by condition I. It is immediate that only condition II, the induced character relation, requires verification. For this it is enough to let δ be a linear character of C^* which is invariant under σ. We may set $\delta(r\,e^{i\theta}) = r^{2iv}$, v real. Then X, the character induced by δ is reducible and is readily seen to be $X_1 + X_2$, X_j is obtained by transfer from the character, e_j, of R^* $(j = 1,2)$, e_1 being the character $x \to |x|^{iv}$, e_2 being the character $x \to |x|^{iv}(\text{sign } x)$, whence by definition

$$\gamma(s,X,C/R) = \gamma(s,X_1,C/R)\,\gamma(s,X_2,C/R) = \pi^{-s}\,\Gamma((s+iv)/2)\,\Gamma((1+s+iv)/2)$$

which is $\gamma(s,\delta,C/C)$. It follows that the function γ on characters of the archimedean local Weil groups satisfy the desired conditions and therefore the function $X \to C(s,X,K/k) = \prod_{y|p_\infty} \gamma(s,X_y,K_y/k_y)$ satisfies the global Artin conditions.

This completes the demonstration of the functional equation of the Weil L series. For the problem of giving a local decomposition of the root number appearing in this equation, we consider the Weil L-series as a function defined on the characters of $W_{K/k}$ and rewrite the functional equation of the Hecke L-series in terms of the coefficients appearing in the functional equations of the local zeta functions studied by Tate.

§5

If X is a character of a local Weil group, $W_{E/F}$, let $Z'(s,X,E/F) = Z(1-s,\bar{X},E/F)/Z(s,X,E/F)$, a ratio between local zeta functions of the type discussed in §2.

If k is a global number field and θ is a character of C_k, then Tate's formulation of the functional equation of the Hecke L-series, L(s,θ,k/k) may be written

$$L(1-s,\bar{\theta},k/k) = L(s,\theta,k/k) \prod_{\mathfrak{y}}[\rho(s,\theta_{\mathfrak{y}},k_{\mathfrak{y}}/k_{\mathfrak{y}})\, Z'(s,\theta_{\mathfrak{y}},k_{\mathfrak{y}}/k_{\mathfrak{y}})]$$

where the product is over all primes of k and $\rho(s,\theta_{\mathfrak{y}},k_{\mathfrak{y}}/k_{\mathfrak{y}})$ is used to denote $\rho_{k_{\mathfrak{y}}}(\theta_{\mathfrak{y}}|\,|_{\mathfrak{y}}^{s})$, i.e. the factor associated by Tate with the quasi character $x \to |x|_{\mathfrak{y}}^{s}\,\theta_{\mathfrak{y}}(x)$ of $k_{\mathfrak{y}}^{*}$ (the valuation, $|\,|_{\mathfrak{y}}$, being the normalised valuation). It should be noted that the infinite product appearing in this functional equation causes no difficulty as for almost all $\mathfrak{y}$ the term in the brackets is 1.

Suppose that it were possible to extend the function $\rho(s,\theta_{\mathfrak{y}},k_{\mathfrak{y}}/k_{\mathfrak{y}})$ on characters of the multiplicative groups of local fields to a function $\rho(s,X_{\mathfrak{y}},Kk_{\mathfrak{y}}/k_{\mathfrak{y}})$ on characters of local Weil groups such that conditions I-V hold. As the local

Weil zeta functions have these properties, it would follow that the functional equation of the Weil L-series could be written $L(1-s,\overline{X},K/k) = L(s,X,K/k)\prod_{y}[\rho(s,X_y,Kk_y/k_y)Z'(s,X_y,Kk_y/k_y)]$ a decomposition into local factors of the global factor which appears in the functional equation. (From the similar observation for the Hecke L-series, it would follow that almost all the brackets appearing in the last equation are 1). From this decomposition and the fact that the expression,

$$\frac{C(s,X,K/k)}{C(1-s,\overline{X},K/k)}\ \frac{A(s,X,K/k)}{A(1-s,\overline{X},K/k)},$$

Also has a natural decomposition into local factors, it would follow that the root number may be decomposed into purely local factors.

Thus it is possible to state conditions of a local nature which insure the existance of a decomposition of the root number. Nothing has been said concerning the necessity of these conditions. Certainly condition IV is not necessary.

Actually, it is not possible to extend the Tate factors in such a manner that conditions I, II, III hold as the following simple counter example shows :

Let K be an abelian extension of a local number field k, θ an arbitrary character of k^*, let $\varphi = \theta \circ N_{K/k}$, a character of K^* and let X be the character of $W_{K/k}$ induced by φ. If the extension exists, then by condition II we must have

$\rho(s,X,K/k) = \rho(s,\varphi,K/K) = \rho_K(\varphi|\,|_K^s)$, $|\,|_K$ being the normalized valuation of K. But as shown in chapter I, X is obtained by transfer from the character, $\Sigma_{j=1}^{n} \theta\tau_j$, of k^*; $\tau_1, \cdots, \tau_n$ being the group of characters of k^* trivial on the norm group. It follows by conditions I and III that

$$\rho(s,X,K/k) = \prod_{j=1}^{n} \rho(s,\theta\tau_j,k/k) = \prod_{j=1}^{n} \rho_k(\theta||_k^s,k/k).$$

We thus have a contradiction as the ratio between these two expressions for $\rho(s,X,K/k)$ is $\varkappa(m,K/k)$, where $m+1$ is the exponent of the conductor of φ.

However we do know that $(\varkappa(m,K/k))^4 = 1$ and thus we are led to the conjecture:

<u>It is possible to extend the fourth power of the Tate function to a complex valued function defined on the characters of local Weil groups which satisfies the local Artin conditions.</u>

This conjecture concerning the characters of local Weil groups has been verified only in the cyclic case, i.e. only for characters of local groups $W_{K/k}$, where K is a cyclic extension of k. This will be proven subsequently. From this it follows that if $W_{K/k}$ is a global Weil group and every decomposition subgroup of G(K/k) is cyclic and if X is a character of $W_{K/k}$ then the fourth power of the root number W(X,K/k) has a canonical local decomposition. We may also conclude that if k is an algebraic number field, L a normal extension of k such that every decomposition subgroup of the galois group G(L/k) is cyclic and if K is an abelian extension of L normal over k and X a character of G(K/k) then the fourth power of the root number appearing in the functional equation of the Artin L-series, $L_A(s,X,K/k)$, has a canonical decomposition into local factors.

In the abelian local example cited above we note that for the character X appearing there, we could have used property III to define $\rho(s,X,K/k)$. It then follows that $\kappa(m,K/k)$ is a "measure" of the extent to which this $\rho(s,X,K/k)$ fails to have the induced character property. As the global root numbers do have the induced character property we may expect some kind of product formula involving these local "error" terms $\kappa(m,K/k)$. An example of such a product formula shall now be given:

Theorem

Let k be an algebraic number field, K an abelian extension field, δ a character of C_k and φ the character $\delta \circ N_{K/k}$ of C_K. If $\mathfrak{y}$ is a prime of k then there exists an integer $m_{\mathfrak{y}}$ such that $\mathfrak{p}^{1+m_{\mathfrak{y}}}$ is the conductor of $\varphi_{\mathfrak{p}}$ for each prime $\mathfrak{p}$ of K which extends $\mathfrak{y}$. Let $r_{\mathfrak{y}}$ be the number of primes of K which extend $\mathfrak{y}$. Then $\prod_{\mathfrak{y}}[\kappa(m_{\mathfrak{y}}, K k_{\mathfrak{y}}/k_{\mathfrak{y}})]^{r_{\mathfrak{y}}} = 1$

the product being taken over all valuations of k.

Proof

Let X be the character of $W_{K/k}$ induced by φ. Let G = $\tau_1, \dots, \tau_n$ be the group characters of C_k which are trivial on $N_{K/k}C_K$. Precisely as in the local case (of chapter I),

$$X = \Sigma_{j=1}^{n}(\delta\,\tau_j) \circ \psi_{W_{K/k} \to C_K}$$

and therefore, using a slightly simpler notation for Hecke L-series,

$$\frac{L(1-s,\bar{\varphi},K)}{L(s,\varphi,K)} = \prod_{j=1}^{n}\left[\frac{L(1-s,\bar{\delta}\bar{\tau}_j,k)}{L(s,\delta\,\tau_j,k)}\right]$$

From the functional equation of the Hecke L-series, it follows that

$$1 = \prod_{\mathfrak{y}} \left\{ \frac{\prod_{j=1}^{n} [\rho(s,(\delta\tau_j)_{\mathfrak{y}},k_{\mathfrak{y}})Z'(s,(\delta\tau_j)_{\mathfrak{y}},k_{\mathfrak{y}})]}{\prod_{\mathfrak{p}|\mathfrak{y}}[\rho(s,\varphi_{\mathfrak{p}},K_{\mathfrak{p}})Z'(s,\varphi_{\mathfrak{p}},K_{\mathfrak{p}})]} \right\}.$$

It only remains to show that the term $\{\ \}$ is $[\chi(m_{\mathfrak{y}},Kk_{\mathfrak{y}}/k_{\mathfrak{y}})]^{r_{\mathfrak{y}}}$. It follows from the isomorphism between completions of K at primes which extend $\mathfrak{y}$ that the denominator is $[\rho(s,\varphi_{\mathfrak{p}},K_{\mathfrak{p}})Z'(s,\varphi_{\mathfrak{p}},K_{\mathfrak{p}})]^{r_{\mathfrak{y}}}$ where $\mathfrak{p}$ now denotes a fixed extension of $\mathfrak{y}$. Let L be the decomposition subfield of K at $\mathfrak{y}$, and let H be the group of characters of C_k which are trivial on $N_{L/k}C_L$. Arranging G in cosets of H, let $\tau_1,\ldots,\tau_{n_{\mathfrak{y}}}$ be representatives of the distinct cosets. The restriction of each element of H to $k_{\mathfrak{y}}^*$ is trivial, whence the $\{\}$ may be written

$$\left[\frac{\prod_{j=1}^{n_{\mathfrak{y}}} [\rho(s,(\delta\tau_j)_{\mathfrak{y}},k_{\mathfrak{y}})Z'(s,(\delta\tau_j)_{\mathfrak{y}},k_{\mathfrak{y}})]}{\rho(s,\varphi_{\mathfrak{p}},K_{\mathfrak{p}})Z'(s,\varphi_{\mathfrak{p}},K_{\mathfrak{p}})}\right]^{r_{\mathfrak{y}}}$$

Letting $X_{\mathfrak{y}}$ be the character of the local Weil group, $W_{K_{\mathfrak{p}}/k_{\mathfrak{y}}}$, induced by $\varphi_{\mathfrak{p}}$ of $K_{\mathfrak{p}}^*$ and noting that $\{(\tau_j)_{\mathfrak{y}}\}_{j=1}^{n_{\mathfrak{y}}}$ is the group of characters of $k_{\mathfrak{y}}^*$ which are trivial on $N_{K_{\mathfrak{p}}/k_{\mathfrak{y}}} K_{\mathfrak{p}}^*$, it follows that

$$X_{\mathfrak{y}} = \Sigma_{j=1}^{n_{\mathfrak{y}}}(\delta\tau_j) \circ \mathcal{W}_{W_{K_{\mathfrak{p}}/k_{\mathfrak{y}}}} \rightarrow: K_{\mathfrak{p}}^* \quad \text{and that}$$

$\prod_{j=1}^{n} Z(s,(\delta\tau_j)_{\mathfrak{y}},k_{\mathfrak{y}})$ is just the local zeta function, $Z(s,X_{\mathfrak{y}},K_{\mathfrak{p}}/k_{\mathfrak{y}})$. As these zeta functions satisfy the local induced character condition, it follows that the $\{\ \}$ is

$$\left[\frac{\prod_{j=1}^{n_{\mathfrak{P}}} \rho(s,(\delta\,\tau_j)_{\mathfrak{P}}, k_{\mathfrak{P}})}{\rho(s,\varphi_{\mathfrak{P}}, K_{\mathfrak{P}})}\right]^{r_{\mathfrak{P}}}$$

, from which the theorem follows.

The question of whether a product formula of this type exists for general normal extensions which are locally abelian remains unanswered.

§6 The Partial Extension of Tate's Function

If k is a local number field, K a normal extension, then the local Weil group $W_{K/k}$ has abelian invariant subgroup K^* of finite index. The factor group $W_{K/k}/K^*$ is isomorphic to the galois group G(K/k) which is always solvable. As indicated above, this is not enough to satisfy the conditions of lemma 7 (§1). If G(K/k) is "super-solvable", i.e. has a principal series whose prime factor groups are cyclic and if X is an irreducible character of the Weil group then there exists a character φ of degree 1 of a subgroup, g, which contains K such that φ induces X. Unfortunately φ is not necessarily even essentially unique, i.e. there may exist another subgroup g' of $W_{K/k}$ with character φ' of degree 1 such that φ' also induces X and such that there exists no inner automorphism of $W_{K/k}$ which maps g onto g' and relates φ with φ'. This may occur even if G(K/k) is abelian. It is only when G(K/k) is cyclic that g is determined uniquely by X (cf Lemmas 5 and 6) and the determination in that case follows from the fact that if a group has a cyclic factor group over its center then the group is abelian.

The problem of extending Tate's function to characters of local Weil groups may now be considered. This function is defined by Tate for all irreducible characters of Weil groups $W_{k/k}$ ($= k^*$). If X is a linear character of $W_{K/k}$ then X is trivial on the closure of the commutator subgroup and therefore there exists a unique linear character θ (of degree 1) of k^* such that $X = \theta \circ \wp_{W_{K/k}} \to K^*$. It is then natural to define $\rho(s,X,K/k) = \rho(s,\theta,k/k)$, for characters X of degree 1, and this definition may be extended in a natural way to characters of $W_{K/k}$ which are sums of such characters. If X is an irreducible character of $W_{K/k}$ and G(K/k) is "super-solvable" then there exists a subgroup $W_{K/L}$ containing K^* with character φ of degree 1 which induces X. $\rho(s,\varphi,K/L)$ being already defined, it would be natural to set $\rho(s,X,K/k)$ equal to it (that is to use the Induced Character property to define the extended Tate coefficient of X) but φ and L are not uniquely determined by X and therefore this would not in general be a canonical definition of the extended Tate coefficient of X. Indeed even if K/k is abelian, $\rho(s,\varphi,K/L)$ would actually not be uniquely determined by X (however there are good reasons to believe that the fourth power of $\rho(s,\varphi,K/L)$ is uniquely determined for K/k abelian and furthermore if deg(K/k) is odd then even the square of $\rho(s,\varphi,K/L)$ is probably uniquely determined by X). If however G(K/k) is cyclic then (Lemma 6, §1) $W_{K/L}$ is uniquely determined by X and φ is unique to within inner automorphisms of $W_{K/k}$. The remainder of this discussion is restricted to the cyclic case.

K/k cyclic

If X is an irreducible character of $W_{K/k}$ then there exists a unique subgroup $W_{K/L}$ containing K^* for which there exists a linear character which induces X. If φ and φ' are characters of $W_{K/L}$ which induce X then by lemma 6 there exists $x \in W_{K/k}$ such that $\varphi' = \varphi^x$. Let θ be the unique character of L^* such that $\varphi = \theta \circ \mathcal{W}_{W_{K/L} \to K^*}$. As $W_{K/k}/W_{K/L} \simeq G(L/k)$ we may identify the class of x modulo $W_{K/L}$ with its natural image σ in $G(L/k)$. It then follows that $\varphi' = \theta^\sigma \circ \mathcal{W}_{W_{K/L} \to K^*}$, θ^σ being the character, $w \to \theta(\sigma(w))$, of L^*. As σ certainly leaves the elements of the closure, R, of Q in k fixed, it follows from the definition of the Tate coefficients that $\rho(s,\theta,L/L) = \rho(s,\theta^\sigma,L/L)$, whence by the previous definitions (for linear characters of Weil groups) $\rho(s,\varphi,K/L) = \rho(s,\varphi',K/L)$ and therefore $\rho(s,X,K/k)$ may be <u>well</u> defined by setting it equal to $\rho(s,\varphi,K/L)$.

Having extended the definition to irreducible characters of the local Weil group, the statement of property I is used to <u>define</u> $\rho(s,X,K/k)$ for reducible characters of $W_{K/k}$. Properties I and V then hold trivially; it only remains to verify properties II, III, IV. Property III can cause no difficulty, for if θ is a character of k^* and $X = \theta \circ \mathcal{W}_{W_{K/k} \to K^*}$, then θ is the sum of characters of k^* of degree one and likewise X is the sum of such characters, whence from the definitions, $\rho(s,X,K/k) = \rho(s,\theta,k/k)$.

For property IV, we may assume φ an irreducible character of $W_{L/k}$, X the character $\varphi \circ t_{K/k \to L/k}$ of $W_{K/k}$. Then there exists a unique subgroup $W_{L/k'}$ of $W_{L/k}$ $(L \supset k' \supset k)$ with linear character σ which induces φ. The kernal of $t_{K/k \to L/k}$ being the closure of the commutator subgroup of $W_{K/L}$, it follows that $W_{K/k'} = t^{-1}_{K/k \to L/k}(W_{L/k'})$. Let σ' be the character $\sigma \circ t_{K/k \to L/k}$ of $W_{K/k'}$. It follows by an argument similar to that of the proof of Lemma 7 (§1) that X is induced by σ'. As σ is linear there exists θ, a character of k'^* such that $\sigma = \theta \circ \mathcal{W}_{W_{L/k'} \to L^*}$, whence $\sigma' = \theta \circ \mathcal{W}_{W_{K/k'} \to K^*}$. As X is irreducible, the property follows from the definitions,

$$\rho(s,X,K/k) = \rho(s,\sigma',K/k') = \rho(s,\theta,k'/k') = \rho(s,\sigma,L/k') = \rho(s,\varphi,L/k).$$

Thus the definition of the extension of the Tate function to characters of $W_{K/k}$ has been formulated in such a manner that the conditions I, III, IV, and V are satisfied almost trivially. Condition II cannot be expected to be satisfied as the previous illustration shows that cyclic counter examples exist. The final lemma completes the proof the main conjecture for K/k cyclic.

Lemma

If $K \supset L \supset k$, φ a character of $W_{K/L}$, X the induced character of $W_{K/k}$ then

$$\rho(s,X,K/k) \equiv \rho(s,\varphi,K/L) \bmod \gamma$$

where $\gamma =$ group of the fourth roots of 1 if $(-1,K/k) = -1$
group of square roots of 1 if $(-1,K/k) = +1$.

<u>Proof</u> Using property I, it may be assumed that φ is irreducible. Then there exists a subgroup $W_{K/F}$ of $W_{K/L}$ for which there exists a linear character σ which induces X. Furthermore there exists θ, a character of F^* such that $\sigma = \theta \circ \mathcal{W}_{W_{K/F}} \rightarrow K^*$. Let X' be the character of $W_{F/k}$ induced by θ. As $W_{K/F} = t^{-1}_{K/k} \rightarrow F/k(F^*)$, an argument similar to the one involved in the proof of Lemma 7 shows that $X = X' \circ t_{K/k} \rightarrow F/k$. Hence by the definition and known properties of the extension of the Tate coefficients, it follows that $\frac{\rho(s,X,K/k)}{\rho(s,\varphi,K/L)} = \frac{\rho(s,X', F/k)}{\rho(s,\sigma,K/F)} = \frac{\rho(s,X',F/k)}{\rho(s,\theta,F/F)}$.

As $(-1,K/k) = 1$ implies $(-1,F/k) = 1$, it is enough to show that if δ is a character of K^* of degree 1 and X is the induced character of $W_{K/k}$ then $\rho(s,\delta,K/K) \equiv \rho(s,X,K/k) \bmod \gamma$.

Let G(K/L) be the subgroup of G(K/k) whose elements leave δ fixed. Let φ be the character of $W_{K/L}$ induced by δ. Let σ be a generator of the cyclic group G(K/L). $\delta^{\sigma-1} = 1$ implies that δ is trivial on the kernel in K^* of $N_{K/L}$, whence $\delta = \mu \circ N_{K/L}$, μ a character of L^*. Then $\varphi = \Sigma^{r}_{j=1}(\mu\tau_j) \circ \mathcal{W}_{W_{K/L}} \rightarrow K^*$, where $\tau_1, \ldots, \tau_r$ are the characters of L^* trivial on $N_{K/L}K^*$. For simplicity, let $\varphi_j = (\mu\tau_j) \circ \mathcal{W}_{W_{K/L}} \rightarrow K^*$. Then $\varphi = \Sigma^{r}_{j=1}\varphi_j$,

$$\frac{\rho(s,\varphi,K/L)}{\rho(s,\delta,K/K)} = \frac{\prod^{r}_{j=1} \rho(s, \mu\tau_j, L/L)}{\rho(s,\delta,K/K)} = \varkappa(m,K/L)$$

where $\mathfrak{p}^{1+m}$ is the conductor of θ. Let X_j be the character of $W_{K/k}$ induced by φ_j, then $X = \Sigma_{j=1}^{r} X_j$. If X' is any irreducible character of $W_{K/k}$ occuring in X then θ occurs in the restriction of X' to K^*, whence by Lemma 5, X' is induced by a character of $W_{K/L}$ of degree one, whence $X'(1) = \deg L/k = X_j(1)$. Hence X is the sum of r irreducible characters of degree $X_1(1)$ and therefore $X_1 + \ldots + X_j$ is the decomposition of X into irreducible characters. Hence by definition, $\rho(s,X,K/k) = \prod_{j=1}^{r} \rho(s,X_j,K/k) = \prod_{j=1}^{r} \rho(s,\varphi_j,K/L) = \rho(s,\varphi,K/L)$. As $\kappa(m,K/L) \in \gamma$, the assertion is proven.

§ 7. Reformulation of the problem

The basic approach to the problem of determining a canonical factorization of the root numbers of characters of Weil groups has been to start with Tate's canonical factorization (§5) of the ratio $L(1-s,\bar{\theta},k/k)/L(s,\theta,k/k)$ between associated Hecke L-series and to attempt to extend these local factors to a function on characters of local Weil groups which satisfy the local Artin conditions. (It should be noted that for given θ, character of C_k, almost all of the local factors are trivial). It has been shown that this factorization cannot lead in this way to the full solution of the problem. This suggests the question: (1) Does there exist another canonical factorization of $L(1-s,\bar{\theta},k/k) / L(s,\theta,k/k)$ (with the property that almost all local factors are trivial) which could lead to a full solution of the problem? This suggests the questions:

(2) What are the set of all canonical functions T on multiplicative characters of local fields with the property that if θ is a character of C_k then

(a) $T(\theta_y) = 1$ for almost all primes of k.

(b) $\prod_y T(\theta_y) = 1$, (the product being over all primes of k)?

(If a is a fixed rational number then the function $\theta_y \to \theta_y(a)$ has this property).

(3) Does there exist a function T, satisfying the conditions of (2) such that the function $\theta_y \to T(\theta_y)\, \rho_k(\theta_y)$ can be extended to a function on characters of local Weil groups which satisfies the local Artin conditions? (This is equivalent to question (1).)

The results of §5 show that if T satisfies the conditions of (3) then it must satisfy the condition:

If K/k is a local abelian extension, X a character of K^* obtained by composition with $N_{K/k}$ from a character of k^* then in the notation of chapter III,

$1 = \kappa(m, K/k)\left(\prod_\mu T(\mu)\right)/T(X)$, the product being taken over all characters μ of k^* for which $X = \mu \circ N_{K/k}$ and where $1+m$ is the exponent of the conductor of X.

These critera may provide a means of disposing of question 1 (particularly if the answer to question (3) is negative).