Today:

- Calculation of normal cycles of Schubert varieties in Grassmannians (joint work with Brian Boe), and an advertisement for a problem
- When o-minimal structures collide: an advertisement for Alesker's approach to valuations

Proposition

If $V \subset M$ is a $\mathbb{C}$-analytic subvariety then

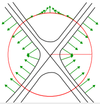
$$\mathbb{P}N^*_M(V) = \sum_{W \in \mathcal{S}} d^V_W [\![\mathbb{P}N^*_M(W)]\!] \tag{1}$$

The Schwartz-MacPherson Chern classes of V may be recovered by contracting $\mathbb{P}N^*(V)$ with canonical elements of $H^*(\mathbb{P}T^*M)$.

Theorem

If $V = \bigcap_{i=1}^N f_i^{-1}(0) \subset \mathbb{C}^n$ is a cone with vertex 0. Put $g := \sum_{i=1}^N |f_i|^2$. Then

$$\deg \left(\left. \frac{\nabla g}{|\nabla g|} \right|_{S^{2n-1}} \right) = d^V_{\{0\}}$$



A **Schubert variety** in the Grassmannian $\mathrm{Gr}_{n,m}$ of complex subspaces $P^n \subset \mathbb{C}^{n+m}$ is the subvariety determined by conditions

$$\dim(P \cap \mathbb{C}^{a_i+i}) \geq i, \quad i = 1, \ldots, n$$

where $a_1 \leq a_2 \leq \cdots \leq a_m$. The corresponding **Schubert cell** are defined by putting $=$ in these relations. Such a variety is stratified by the Schubert cells it contains.

Fixing a nondegenerate antisymmetric bilinear form ω on $\mathbb{C}^{2n}$, these things also exist in the Lagrangian Grassmannian $L_n \subset \mathrm{Gr}_{n,n}$ of Lagrangian subspaces P^n, $\omega|_P \equiv 0$, and similarly if the form is symmetric.

These are types A, C, D respectively.

They also exist in more general flag manifolds.

- Representation theorists are interested in these things: for a Schubert pair $X \supset Y$ there is a **Kazhdan-Lusztig polynomial** P_Y^X describing a certain canonical sheaf on X, with $P_X^X = 1$. Put $p_Y^X := P_Y^X(1)$ and

$$\pi_X := \sum_{Y^\circ \subset X} p_Y^X 1_{Y^\circ}$$

  Thus the π^X are a basis for the $\mathbb{Z}$-module generated by the characteristic functions of Schubert varieties.
- Lusztig conjectured that in type A flag manifolds

$$\mathbb{P}N^*(\pi_X) = \mathbb{P}N^*(X^\circ)$$

  ("the characteristic cycle of the intersection homology sheaf is irreducible")
- It turns out that this is wrong (more on this later), but Bressler, Finkelberg and Lunts (1990) showed that it is true for the type A (i.e. standard) Grassmannian by working with the actual sheaves.

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Boe and Fu (1997) computed $\mathbb{P}N^*(X) = \mathbb{P}N^*(1_X)$ for Schubert varieties in types A, C, D using the degree formula. Changing basis from 1_X to π_X in type A gives an alternative to the proof of Bressler et al.

We illustrate in the case of Schubert data with $m = n$ and

$$a_1 = a_2 = \cdots = a_{n-k} = k, \quad a_{n-k+1} = \cdots = a_n = n$$

In local coordinates this is the determinantal variety (cone)

$$D_k^n := \{\mu \in \mathbb{C}^{n\times n} : \operatorname{rank} \mu \leq k\}$$

For $\ell \leq k$ the normal slice of D_k^n to D_ℓ^n is clearly $D_{k-\ell}^{n-\ell}$. So it is enough to compute the coefficient

$$d_{\{0\}}^{D_k^n}$$

in $\mathbb{P}N^*(D_k^n)$.

Then $D_k^n = g^{-1}(0)$ where

$$g(\mu) := \left|\bigwedge^{k+1}\mu\right|^2$$

Lemma (*KAK* decomposition)

Put $\Lambda := \{\vec{\ell} = (\ell_1, \dots, \ell_n) : \textit{lines } \ell_i \perp \ell_j,\ i \neq j\}$ *and for* $\vec{\ell}, \vec{\lambda} \in \Lambda$ *put*

$$M_{\vec{\ell},\vec{\lambda}} := \left\{\sum_i \alpha_i \otimes \beta_i : \alpha_i \in \ell_i,\ \beta_i \in \lambda_i\right\}$$

Then

$$\mathbb{C}^{n\times n} = \bigcup_{\vec{\ell},\vec{\lambda}\in\Lambda} M_{\vec{\ell},\vec{\lambda}}$$

where a generic $\mu \in \mathbb{C}^{n\times n}$ *belongs to exactly one of the* $M_{\vec{\ell},\vec{\lambda}}$. *Furthermore*

$$\nabla g\left(M_{\vec{\ell},\vec{\lambda}}\right) \subset M_{\vec{\ell},\vec{\lambda}}$$

Put $\nabla_1 g := \frac{\nabla g}{|\nabla g|}$. It follows that

$$\deg(\nabla_1 g) = \deg\left(\nabla_1 g|_{S^{2n^2-1}\cap M_{\vec{\ell},\vec{\lambda}}}\right)$$

and we may as well take the ℓ_i, λ_i to be the coordinate axes, i.e. $M_{\vec{\ell},\vec{\lambda}} =$ diagonal matrices $(z_1, \ldots, z_n)$, with

$$g(\vec{z}) = \sum \left|z_{i_1} \ldots z_{i_{k+1}}\right|^2$$

$$\nabla g(\vec{z}) = 2\left(z_1 \sum_{1\neq j_1,\ldots j_k} \left|z_{j_1} \ldots z_{j_k}\right|^2, \ldots\right)$$

so we can even restrict to the simplex $\Delta = \Delta_+^{n-1} \subset S^{2n-1}$ of points with nonnegative real coordinates.

It's easy to check that $\nabla_1 g$ maps the interior of Δ to itself, and $\partial\Delta$ to itself in the following sense:

Lemma

- *If $\vec{x} \in \partial\Delta - D^n_k$ then $\nabla_1 g(\vec{x}) \in \partial\Delta$.*
- *$\nabla_1 g$ maps each codimension 1 face of Δ to itself.*
- *If $\Sigma \subset \partial\Delta \cap D^n_k$ is a face and $\Delta^\circ \ni \vec{x}_i \to \vec{x}_0 \in \Sigma$ then $\nabla_1 g(\vec{x}_i) \to$ the face opposite to Σ.*

Proof.

The first and second statements are easy. The second statement just means that the limiting values of $\nabla_1 g$ at a stratum of D^n_k are normal to the stratum, which can be proved using the Lojasiewicz inequality. □

Put $d_k^n := d_{\{0\}}^{D_k^n}$. We may now use the relation

$$d_k^n \partial [\![\Delta]\!] = \partial \nabla_1 g_* [\![\Delta]\!] = \nabla_1 g_* [\![\partial \Delta]\!]$$

provided we are careful with the singularities of $\nabla_1 g$: in view of the last lemma, any generic point of a codimension 1 face $F \subset \Delta$ has preimages only on F itself or on a virtual copy of F lying at the vertex opposite.

Examples:

- D_0^2:

$$g(x_1, x_2) = x_1^2 + x_2^2, \quad \nabla_1 g(x_1, x_2) = (x_1, x_2)$$
$$d_0^2 = 1$$

- D_1^2:

$$g(x_1, x_2) = x_1^2 x_2^2, \quad \nabla_1 g(x_1, x_2) = (x_2, x_1)$$
$$d_1^2 = -1$$

- D_2^3:

$$g(x_1, x_2, x_3) = x_1^2 x_2^2 x_3^2, \quad \nabla_1 g(x_1, x_2, x_3) = \frac{(x_2 x_3, x_3 x_1, x_1 x_2)}{|(x_2 x_3, x_3 x_1, x_1 x_2)|}$$

The face $x_3 = 0$ lies in the variety and maps to the opposite vertex. On the other hand the virtual copy of this face at the opposite vertex $x_3 = 1$ maps to the actual face just as in the D_1^2 case. But this virtual face is oriented negatively, so $d_2^3 = -d_1^2 = +1$.

Examples:

- D^2_0:

$$g(x_1, x_2) = x_1^2 + x_2^2, \quad \nabla_1 g(x_1, x_2) = (x_1, x_2)$$
$$d^2_0 = 1$$

- D^2_1:

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$$d^2_1 = -1$$

- D^3_2:

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- D_1^3:

$$g(x_1, x_2, x_3) = x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2,$$
$$\nabla_1 g(x_1, x_2, x_3) = \frac{(x_1(x_2^2 + x_3^2), x_2(x_3^2 + x_1^2), x_3(x_1^2 + x_2^2))}{|(x_1(x_2^2 + x_3^2), x_2(x_3^2 + x_1^2), x_3(x_1^2 + x_2^2))|}$$

$\nabla_1 g$ maps $x_3 = 0$ to itself exactly like the D_1^2 case, and maps the virtual face $x_3 = 1$ to the true face just like the D_0^2 case. So

$$d_1^3 = d_1^2 - d_0^2 = -1 - 1 = -2$$

- Generally

$$d_k^n = d_k^{n-1} - d_{k-1}^{n-1}, d_k^k := 0, \quad d_0^k = 1 \implies d_k^n = (-1)^k \binom{n-1}{k}$$

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The general rule for type A Grassmannians is:

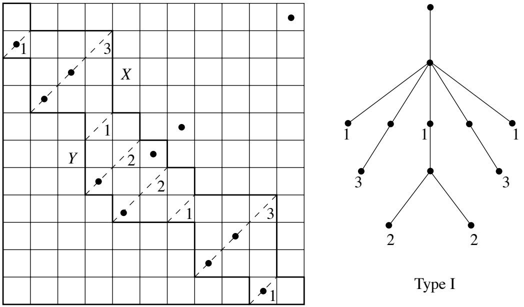

FIGURE 4.2. The trees T_Y^X.

Theorem (Kashiwara-Saito)

Consider the conic variety X of all 4×4 matrices where all four dominoes have rank ≤ 1:

$$\left[\begin{array}{cc|cc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \hline \times & \times & \times & \times \\ \times & \times & \times & \times \end{array}\right]$$

Then $d_0^X \neq 2$.

The point is that this is Schubert singularity in a type A flag manifold, and the conclusion is equivalent to the statement that the multiplicity of 0 in $\mathbb{P}N^*(\pi_X)$ is nonzero.

Problem: What is the actual value of d_0^X?

This is the bottleneck (at least psychologically) in the way of extending our method beyond the Grassmannian.

Tom Braden has a complicated algorithm for computing the multiplicities of $\mathbb{P}N^*(\pi_X)$, and after several tries computed— without confidence— that the multiplicity of 0 is 1. This would imply that $d_0^X = 3$.

Definition

A **smooth valuation** on an oriented manifold M is an operation of the form

$$\nu_{\beta,\gamma} = \int_{N(\cdot)} \beta + \int_{\cdot} \gamma$$

The space of all such things is denoted $\mathcal{V} = \mathcal{V}(M)$, and $\mathcal{V}_c$ is the subspace of those with compact support.

Theorem (Alesker)

- *The valuations* $[Y, \{F_t\}, m] \in \mathcal{V}$ *and their span is dense in* $\mathcal{V}$.
- *There is a natural continuous and commutative multiplication* $\mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$, *extending the multiplication of the* $[Y, \{F_t\}, m]$.
- *Let* M *be a vector space* W^n *and put* $\mathcal{V}^W$ *for the subalgebra of translation-invariant valuations on* W. *This algebra is naturally* $\mathbb{Z}$*-graded by degree of homogeneity*

  $$\mu \in \mathcal{V}^W_k(W) \iff \mu(tA) = t^k \mu(A) \text{ for } t > 0, \quad k = 0, \dots n$$

  and $\mathbb{Z}_2$*-graded by parity*

  $$\mu \in \mathcal{V}^W_{\pm}(W) \iff \mu(-A) \equiv \pm\mu(A).$$

  $\mathcal{V}^W_0 = \langle \chi \rangle$ *and* $\mathcal{V}^W_n = \langle \mathrm{vol} \rangle$.
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- The pairing $\mathcal{V} \times \mathcal{V}_c \to \mathbb{R}$

$$(\mu, \nu) := (\mu \cdot \nu)(M)$$

is perfect. If M is a vector space as above then the pairing

$$(\mu, \nu) := \text{ degree } n \text{ component of } \mu \cdot \nu$$

is perfect.
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