Technical interlude: geometric integration theory

An oriented Lipschitz submanifold is a simple kind of **integral current** (Federer and Fleming,1960).

- A **current of dimension** k in a smooth Riemannian manifold S is a functional $T \in (\Omega^k_c(S))^*$ that is continuous with respect to C^∞ convergence of forms.
- The **boundary** of T is the $(k-1)$-dimensional current $\partial T(\phi) := T(d\phi)$.
- The **mass** of T is mass $T := \sup_{\|\phi\|_\infty \le 1} T\phi$, where $\| \phi \|_\infty := \sup\{\phi_x(v_1, \dots, v_k) : x \in S, v_i \in T_x S, |v_i| \le 1\}$ is the **comass**.
- The current T is **rectifiable** if

$$T = \sum_{i=1}^{\infty} f_{i*} [\![E_i]\!] \tag{1}$$

where the $E_i \subset \mathbb{R}^k$ are measurable and the $f_i : \mathbb{R}^k \to S$ are Lipschitz.

Definition

The abelian group $\mathbb{I}_k(S)$ of **integral currents of dimension** k consists of all k-dimensional locally rectifiable currents with boundary of locally finite mass.

It turns out that the boundary of an integral current is again integral. A smooth oriented submanifold V of dimension k, with smooth boundary, defines an element $[\![V]\!] \in \mathbb{I}_k(S)$ by integration. Stokes's theorem may then be stated: $\partial[\![V]\!] = [\![\partial V]\!]$.

Theorem (Federer-Fleming (1960))

$$\mathbb{I}_k(S) = \bigcup_{C<\infty} \operatorname{clos}\Big(\{T = \sum [\![V_i]\!] : V_1, V_2, \dots \textit{ smooth}, \\ \operatorname{mass}_U T + \operatorname{mass}_U \partial T \leq C(U) \textit{ for all } U \subset\subset S\}\Big).$$

Now suppose S is the sphere bundle SM of a smooth Riemannian manifold M (e.g. $\mathbb{R}^n$) of dimension n. Let $\alpha \in \Omega^1(SM)$ be the **canonical 1-form**

$$\alpha_\xi \cdot \tau := \langle \xi, \pi_* \tau \rangle$$

$\xi \in SM, \tau \in T_\xi SM, \pi : SM \to M$ the projection.
An integral current $T \in \mathbb{I}_n(SM)$ is **Legendrian** if

$$T \llcorner \alpha = 0$$

i.e.

$$\int_T \alpha \wedge \psi = 0 \quad \text{for all } \psi \in \Omega_c^{n-2}(SM)$$

This implies also that $T \llcorner d\alpha = 0$.
The Legendrian condition is weakly closed.

Maximal generality of N

(more than we can handle actually)

Theorem

Let $\phi : S^{n-1} \times \mathbb{R} \to \mathbb{Z}$ *with*

$$\sum_{t\in\mathbb{R}} \phi(v,t) \equiv M < \infty \quad \text{for a.e. } v \in S^{n-1}.$$

Then there is at most one compactly supported Legendrian cycle $T \in \mathbb{I}_{n-1}(S\mathbb{R}^n)$ *such that for a.e.* $v \in S^{n-1}$

$$\sum_{h_v(x)=t} \operatorname{mult}_{(x,v)} \left(T \cdot (\mathbb{R}^n \times \{v\})\right) = \phi(v,t) \tag{2}$$

If $X \subset \mathbb{R}^n$ and such T exists for $\phi = \Delta\chi(X,\cdot,\cdot)$ then $N(X) := T$ is the **normal cycle** of X.

Lots of embarrassing questions:

- Must such X be topologically reasonable, e.g. a neighborhood retract?
- Does the diffeomorphic image of X admit a normal cycle? i.e. do the intersections of $N(X)$ with the graphs of the gradients of smooth functions f other than height functions yield the Euler Morse indices of $f|_X$? Are these indices even well-defined (cf. the last question)?
- Suppose $X_1, X_2, \ldots \downarrow X$ are compact smooth domains with mass $N(X_i) \leq C$. Must $N(X)$ exist? Simple examples show that $\lim N(X_i) \neq N(X)$ in general.

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The intelligence of the normal cycle: a sample result

In many ways it is easier to work in a parallel setting:

Theorem (Fu 1990, R. Jerrard 2008)

Suppose $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$*. Then there is at most one closed Lagrangian current* $\Gamma \in \mathbb{I}_n(T^*\mathbb{R}^n)$ *such that*

- $\text{mass}(\Gamma \cap \pi^{-1}K) < \infty$ *for all* $K \subset\subset \mathbb{R}^n$
- $\int_\Gamma \psi(x,y)\, dx_1 \wedge \cdots \wedge dx_n = \int_{\mathbb{R}^n} \psi(x, d_x f) d\,\text{vol}_x$ *for compactly supported* $\psi \in C^\infty(T^*\mathbb{R}^n)$*.*

In other words Γ is a completed version of the graph of df. Just as curvature integrals can be gotten from the normal cycle, so can integrals of the minors of the Hessian can be gotten from Γ, e.g.

$$\int_{\mathbb{R}^n} \det D^2 f \simeq \int_\Gamma dy_1 \wedge \cdots \wedge dy_n$$

where $x_1, \ldots, x_n, y_1, \ldots y_n$ are the standard coordinates of $T^*\mathbb{R}^n$.

- $n = 1$: such f are precisely the differences of convex functions.
- $n = 2$: any difference of convex functions admits such a Γ, but the converse is false.
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More embarrassing questions:

- **Is such f continuous?**
  — yes if $n = 2$: this is true in this dimension whenever the distributional Hessian of f is a measure (Ponce and van Schaftingen 2007).
- **If Γ exists for f, does $N(\text{graph } f)$ exist?**
  In other words, do intersections with Γ give Euler Morse data of f?

Corollary

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ *may be expressed as the locally uniform limit of a sequence* $f_1, f_2, \ldots$ *of smooth functions such that the absolute integrals of all minors of the Hessians of the* f_k *are locally bounded, i.e.*

$$\int_K \left| \det \left(\frac{\partial^2 f_k}{\partial x_i \partial x_j} \right)_{i \in I, j \in J} \right| \leq C(K), \quad k = 1, 2, \ldots \tag{3}$$

whenever K *is compact and* $I, J \subset \{1, \ldots, n\}$ *have the same cardinality. Then* $\mathrm{graph}(df_k) \rightharpoonup$ *a current* Γ *as above.*

Maybe all such f can be produced in this way?

Definition

$f : \mathbb{R}^n \to \mathbb{R}$ **is twice differentiable at** x_0 if there exists a quadratic polynomial $Q_{x_0} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{x \to x_0} \frac{f(x) - Q_{x_0}(x)}{|x - x_0|^2} = 0. \tag{4}$$

Theorem (A.D. Alexandrov 1939)

If f is convex then f is twice differentiable a.e.

Theorem

Suppose f is approximable as in the Corollary. Then f is twice differentiable a.e.

Lemma

If $V \subset\subset U$ is open and $f \in C^2$ then

$$\int_V \left|\det D^2 f\right| \geq C \left(\frac{\sup_V |f| - \sup_{\text{bdry } V} |f|}{\text{diam } V}\right)^n \tag{5}$$

Proof.

If λ is a linear function with $|\lambda| < \frac{\sup_V f - \sup_{\text{bdry } V} f}{\text{diam } V}$ then $f - \lambda$ has an interior local maximum $x \in V$, with $d_x f = \lambda$. So the image of V under df includes a ball of this radius in $\mathbb{R}^{n*}$. □

Proof of the theorem

- By induction on n. The case $n = 1$ follows from Alexandrov's theorem.
- If the theorem holds in dimension $n - 1$, then for a.e. hyperplane $P \subset \mathbb{R}^n$ the restriction $f|_P$ is twice differentiable at a.e. $x \in P$. By Fubini's theorem it follows that for a.e. $x \in \mathbb{R}^n$ the restriction $f|_P$ is twice differentiable at x for a.e. P through x. The corresponding quadratic Taylor expansions Q_P are compatible, i.e. $Q_P = Q|_P$ for some quadratic Q.
- Let ν be the limit of the sum of the measures on $\mathbb{R}^n$ gotten by integrating the absolute minors of $D^2 f$. We show that if the conclusion fails then ν has infinite density at x. We can replace f by $f - Q$ since the corresponding $\widetilde{\nu} \leq C(1 + \|Q\|)^n \nu$.
- The regions V close to x where f is too big must all have small diameter. The Lemma applies to these V, with $\nu(V)$ on the LHS, to show that $\nu(V)$ is too big. $\square$

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We've used the fact the Lemma holds with the LHS replaced by $\nu(V)$. But does it still hold if it is replaced by

$$\text{mass}((\pi^{-1}(V) \cap \Gamma) \llcorner dy_1 \wedge \cdots \wedge dy_n) \quad ?$$

This is related to the question of the existence of the normal cycle for the graph: a positive answer would mean that intersections with Γ detect critical points of f of index zero.

The tame setting

But everything works (i.e. the embarrassing questions have generally positive answers) in any reasonable category, e.g. polyconvex or subanalytic (or constructible with respect to some o-minimal structure) sets. Using local coordinates we can also make sense of (co)normal cycles of subsets of smooth manifolds.

Theorem

Let $X, A_1, A_2, \ldots \subset\subset \mathbb{R}^n$, such that all $N(A_j)$ exist, with mass $N(A_i) \leq C < \infty$. Put $H_{v,t} := h_v^{-1}(-\infty, c]$. Suppose that for a.e. $v \in S^{n-1}$

$$\lim_{j\to\infty} \chi(H_{v,c} \cap A_j) = \chi(H_{v,c} \cap X)$$

Then $N(X)$ exists, with $\lim_{j\to\infty} N(A_j) = N(X)$. $\square$

Corollary

If $X \subset \mathbb{R}^n$ is subanalytic and compact then $N(X)$ exists. If $X_t \downarrow X$ is a proper nested subanalytic family as $t \downarrow 0$ then

$$N(X) = \lim_{t \downarrow 0} N(X_t) \tag{6}$$

Proof.

- Let $g := \mathrm{dist}(.X)$. Then g is **semiconcave** on $\mathbb{R}^n - X$, i.e. is locally expressible as $f - k$ where f is smooth and k is convex. Furthermore g has no small critical values. Therefore $\mathbb{R}^n - g^{-1}[0, r)$ is **semiconvex** for small $r > 0$, i.e. each point admits a neighborood that is the diffeomorphic image of an open subset of a convex set $\subset \mathbb{R}^n$. So $N(\mathbb{R}^n - g^{-1}[0, r))$ exists.
- Therefore the $N(g^{-1}[0, r])$ are the images of $N(\mathbb{R}^n - g^{-1}[0, r))$ under the map $(x, v) \mapsto (x, -v)$, and constitute a proper subanalytic family of sets. Thus their masses are uniformly bounded. Furthermore

$$\lim_{r \downarrow 0} \chi(H_{v,c} \cap g^{-1}[0, r]) = \chi(H_{v,c} \cap X)$$

for generic v, t. So the last Theorem applies, and $N(X)$ exists.
- To prove (6) we apply the last paragraph again, since the $N(X_t)$ exist and themselves constitute a proper subanalytic family.

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Again Groemer's Integral Theorem $\implies$ any compactly supported subanalytically constructible function $\alpha = \sum c_i 1_{X_i}$ admits a normal cycle, with

$$N(\alpha) = \sum c_i N(X_i)$$

if the X_i are compact.

Theorem (Specialization formula)

Let $\{X_t \subset \mathbb{R}^n\}_{t>0}$ be a proper subanalytic family. Then

$$\lim_{t \downarrow 0} N(X_t) = N(\alpha) \tag{7}$$

where α is the constructible function

$$\alpha(p) = \lim_{\epsilon \downarrow 0} \lim_{t \downarrow 0} \chi(X_t \cap B(p, \epsilon)) \tag{8}$$

Proof.

This follows from

- If $\alpha = \sum c_i 1_{X_i}$, X_i compact, then for small $t > 0$

$$\chi(X_t) = \sum c_i \chi(X_i)$$

- Put $\alpha_{v,c}$ for the limiting constructible function for the family $\{X_t \cap H_{v,c}\}$. Then for generic v, c

$$\alpha_{v,c} = \alpha|_{H_{v,c}}$$

□