Theory and applications of the normal cycle

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Introductory remarks

The normal cycle is a simple-minded device that permits us to extend arguments and statements about curvature integrals to singular spaces. It is very robust: it exists even in cases we know very little about. It is very smart: its existence and properties tell us a lot about the underlying spaces.
Pioneers of this approach:

- M.Kashiwara (subanalytic category)
- P. Wintgen (PL)
- M. Zähle (generic unions of semiconvex sets)

In these lectures I will mostly talk about how the normal cycle can be used to think simply and clearly about certain problems involving analytic singularities. Many details will be omitted, but I hope to include most of the conceptual steps.

The normal bundle of a surface $\Sigma \subset \mathbb{R}^3$

$$\Sigma \subset \mathbb{R}^3 \quad \text{a smooth oriented embedded surface}$$

$$n : \Sigma \to S^2 \quad \text{its Gauss map}$$

$$N(\Sigma) := \{(x, n(x) : x \in \Sigma\} \subset S\mathbb{R}^3$$

or invariantly $$N^*(\Sigma) := \{(x, T_x\Sigma : x \in \Sigma\} \subset S^*\mathbb{R}^3$$

$$\Omega^1(S\mathbb{R}^3) \ni \alpha_{(x,v)} := \sum v_i dx_i \quad \text{the contact form}$$

Then $N(\Sigma)$ is *Legendrian*:

$$\alpha|_{N(\Sigma)} = d\alpha|_{N(\Sigma)} = 0$$

The tube formula

$$\exp : S\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3, \quad \exp(x, v; t) := x + tv$$

$$\exp(N(\Sigma) \times [0, r]) = \Sigma_r = \text{ one-sided tube of radius } r$$

$$r \text{ small} \implies |\Sigma_r| = \int_{\Sigma_r} d\,\mathrm{vol} = \int_{N(\Sigma)\times[0,r]} \exp^*(dx_1 \wedge dx_2 \wedge dx_3)$$

$$\begin{aligned} \exp^*(dx_1 \wedge dx_2 \wedge dx_3) &= d(x_1 + tv_1) \wedge d(x_2 + tv_2) \wedge d(x_3 + tv_3) \\ &\equiv dt \wedge \left[t^2\, \kappa_0 + t\, \kappa_1 + \kappa_2 \right] \end{aligned}$$

where

$$\begin{aligned} \kappa_0 &:= v_1 dv_2 \wedge dv_3 - v_2 dv_1 \wedge dv_3 + v_3 dv_1 \wedge dv_2 \\ \kappa_1 &:= v_1(dx_2 \wedge dv_3 + dx_2 \wedge dx_3) - v_2(dx_1 \wedge dv_3 + dv_1 \wedge dx_3) \\ &\quad + v_3(dx_1 \wedge dv_2 + dv_1 \wedge dx_2) \\ \kappa_2 &:= v_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2 \end{aligned}$$

... so

$$|\Sigma_r| = \frac{r^3}{3}\int_{N(\Sigma)} \kappa_0 + \frac{r^2}{2}\int_{N(\Sigma)} \kappa_1 + r\int_{N(\Sigma)} \kappa_2$$
$$= \frac{r^3}{3}\int_{\Sigma} \overline{n}^* \kappa_0 + \frac{r^2}{2}\int_{\Sigma} \overline{n}^* \kappa_1 + r\int_{\Sigma} \overline{n}^* \kappa_2$$

where $\overline{n}(x) := (x, n(x))$. Choose a point $p_0 \in \Sigma$, and coordinates so that

$$n(p_0) = e_3 \implies dn_3 = 0$$

Then

$$0 = \overline{n}^* d\alpha = dn_1 \wedge dx_1 + dn_2 \wedge dx_2$$
$$= (\gamma_{11} dx_1 + \gamma_{12} dx_2) \wedge dx_1 + (\gamma_{21} dx_1 + \gamma_{22} dx_2) \wedge dx_2$$
$$= (\gamma_{21} - \gamma_{12})\, dx_1 \wedge dx_2$$

i.e. the second fundamental form of Σ is symmetric.

The coefficients of the tube formula may be expressed as the integrals over Σ of

$$
\begin{aligned}
\overline{n}^*\kappa_2 &= dx_1 \wedge dx_2 = dA_\Sigma \\
\overline{n}^*\kappa_1 &= dx_1 \wedge dn_2 + dn_1 \wedge dx_2 = (\gamma_{11} + \gamma_{22})\, dA_\Sigma = H\, dA_\Sigma \\
\overline{n}^*\kappa_0 &= dn_1 \wedge dn_2 = (\gamma_{11}\gamma_{22} - \gamma_{12}^2)\, dA_\Sigma = K\, dA_\Sigma
\end{aligned}
$$

Morse theory of Σ

The normal vector bundle of Σ is

$$\vec{N}(\Sigma) := \{(x, tv) : t \in \mathbb{R}, (x, v) \in N(\Sigma)\}$$

If $f : \mathbb{R}^3 \to \mathbb{R}$ is smooth then

$$x \in \operatorname{crit}(f|_\Sigma) \iff \nabla_x f \in \vec{N}_x \Sigma$$

i.e. graph$(\nabla f), \vec{N}(\Sigma)$ intersect above x.
What is the Morse index of $f|_\Sigma$ at x? Suppose $x = 0$,

$$T_x \Sigma = \mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3, \quad \nabla_0 f = c e_3$$

$$\varphi : \mathbb{R}^2 \to \Sigma \quad \text{local coordinates at } 0,\ \varphi(0) = 0,\ D_0 \varphi = \mathrm{id}$$

Put $\mathrm{II}_0 = D_0^2 \varphi_3$ for the second fundamental form at 0. By the chain rule

$$D_0^2 (f \circ \varphi) = D_0^2 f \Big|_{\mathbb{R}^2} + c\, \mathrm{II}_0$$

Put σ for the index of this bilinear form.

On the other hand compute the intersection multiplicity

$$m := \text{mult}_{(0,ce_3)}(\text{graph}(\nabla f) \cdot \vec{N}(\Sigma))$$

These are both 3-folds in $T\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3$, with

$$T_{(0,ce_3)}\, \text{graph}(\nabla f) = \left\langle (e_1, D_0^2 f \cdot e_1)\ , (e_2, D_0^2 f \cdot e_2), (e_3, D_0^2 f \cdot e_3) \right\rangle$$

$$T_{(0,ce_3)} \vec{N}(\Sigma) = \langle (e_1, c\, \mathrm{II}_0 \cdot e_1)\ , (e_2, c\, \mathrm{II}_0 \cdot e_2), (0, e_3) \rangle$$

$$\implies m = \operatorname{sgn}\det \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & D_0^2 f & \\ 0 & 0 & 1 & & & \\ \hline 1 & 0 & 0 & & & 0 \\ 0 & 1 & 0 & & cII_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] = \operatorname{sgn}\det \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & D_0^2 f\big|_{\mathbb{R}^2} \\ \hline 1 & 0 & \\ 0 & 1 & cII_0 \end{array} \right]$$

$$= \operatorname{sgn}\det\left(D_0^2 f\Big|_{\mathbb{R}^2} + c\, \mathrm{II}_0 \right)$$

$$= (-1)^\sigma$$

= the “Euler-Morse index"

Σ is determined by the Euler-Morse data of height functions

Morse theory says: Σ compact, $f \in C^2(\Sigma)$, $p \in \mathrm{crit}(f)$, $\mathrm{index}(f,p) = \sigma$, $c := f(p) \neq f(q)$ for $p \neq q \in \mathrm{crit}(f) \implies$

$$\Delta\chi(f,c) := \chi(f^{-1}(-\infty, c+]) - \chi(f^{-1}(-\infty, c-]) = (-1)^\sigma$$

For $v \in S^2$ put $h_v(x) := \langle v, x \rangle$,

$$\Delta\chi(\Sigma, v, c) := \Delta\chi(\,h_v|_\Sigma\,, c) \tag{1}$$

Theorem

$\Sigma_1, \Sigma_2 \subset \mathbb{R}^3$ *compact* C^2 *surfaces,*
$\Delta\chi(\Sigma_1, v, \cdot) \equiv \Delta\chi(\Sigma_2, v, \cdot)$ *for a.e.* $v \in S^2 \implies \Sigma_1 = \Sigma_2$.

We'll actually need only that the supports of these functions are equal.

Proof: part 1

By Sard's theorem, a.e. $v_0 \in S^2$ is a regular value of both Gauss maps n_i. The preimages $n_i^{-1}(v)$ vary continuously in a neighborhood of any such v_0. Assuming (1), we show first that for a.e. $v \in S^2$

$$n_1^{-1}(v) = n_2^{-1}(v) \tag{2}$$

Put $P \subset S^2$ for the set of all regular values v for which (2) fails. Thus

$$v_0 \in P \implies \text{there are } p \in \Sigma_1, q \in \Sigma_2 \text{ with}$$
$$h_{v_0}(p) = h_{v_0}(q), \quad n_1(p) = n_2(q) = v_0, \quad p \neq q.$$

We show that the **density** of P at v_0 is zero, which is enough to establish (2) a.e.:

Let $P \ni v_1, v_2, \cdots \to v_0$, with

$$\Sigma_1 \ni p_1, p_2, \cdots \to p, \quad \Sigma_2 \ni q_1, q_2, \cdots \to q$$
$$p_i, q_i \in n_1^{-1}(v_i) \cap n_2^{-1}(v_i)$$
$$0 = h_{v_i}(p_i) - h_{v_i}(q_i) = \langle p_i - q_i, v_i \rangle$$

v_0 is a regular value $\implies p_i - p, q_i - q = O(|v_i - v_0|) \implies$

$$\langle p_i - p, v_i - v_0 \rangle = o(|v_i - v_0|) = \langle q_i - q, v_i - v_0 \rangle$$

and by definition of n (or the Legendrian condition on $N(\Sigma_i)$)

$$\langle p_i - p, v_0 \rangle = o(|p_i - p|) = o(|v_i - v_0|) = \langle q_i - q, v_0 \rangle$$

$$\implies \langle p_i - p, v_i \rangle = o(|v_i - v_0|) = \langle q_i - q, v_i \rangle$$
$$\implies o(|v_i - v_0|) = \langle p - q, v_i \rangle = \langle p - q, v_i - v_0 \rangle$$

— i.e. the sequence v_i is asymptotic to the great circle $\perp p - q$.

Remark

This argument also shows: for a.e. $v \in S^2$, the height function $h_v|_\Sigma$ has distinct critical values.

Proof: part 2

Define the 2-dimensional cycle

$$C := N(\Sigma_1) - N(\Sigma_2)$$

given by integration over the open C^1 manifold $N(\Sigma_1) \triangle N(\Sigma_2)$. Put $\pi : C \to S^2$ for the projection. By part 1, rank $D_x\pi \leq 1$ for all $x \in C$. We want to show that $C = 0$. Put

$$C^* := \{x \in C : \operatorname{rank} D_x\pi = 1\}$$

If $C^* = \emptyset$ then $\pi(C)$ is countable, and since C is a 2-dimensional cycle so is each $\pi^{-1}(v)$, $v \in \pi(C)$. Say $v = e_3$. Then

$$0 = \alpha|_{\pi^{-1}(e_3)} = dx_3$$

so any component of $\pi^{-1}(e_3)$ is a subset of some

$$\{x_3 = const\} \times \{e_3\} \subset \mathbb{R}^3 \times S^2$$

Since these components are themselves 2-cycles, in fact they are zero. So $C = 0$.

Proof: part 3

So we may assume that $C^* \neq \emptyset$. It's clear that $\pi(C^*) \subset S^2$ is a countable union of C^1 arcs $\gamma \subset S^2$.

Lemma (Slicing/coarea lemma)

For a.e. $v \in \gamma$ the preimage

$$\pi^{-1}(v) \supset \Gamma \supset (\pi^{-1}(v) \cap C^*)$$

where Γ is a countable union of rectifiable loops, and $\pi^{-1}(v) - C^*$ has 1-dimensional measure zero. □

Underlying point: Even though the set of critical values t of a C^1 map $f : \Sigma^2 \to \mathbb{R}$ may have nonzero length (Whitney 1935), for a.e. such t the set $\mathrm{crit}(f) \cap f^{-1}(t)$ has 1-dimensional measure zero (coarea formula, Federer 1959).

For a.e. $v \in \gamma$ the tangent line ℓ_v to γ may be thought of as the tangent line to all of $\pi(C^*)$: the set of transverse double points is countable. Let $v \in S^2$ as in the slicing/coarea lemma. We may suppose that $v = e_3$ and $\ell_v = \langle e_2 \rangle$. For $(x, e_3) \in \pi^{-1}(e_3) \cap C^*$

$$0 = \alpha|_{T_{x,e_3} C^*} = dx_3, \quad 0 = d\alpha|_{T_{x,e_3} C^*} = dv_1 \wedge dx_1 + dv_2 \wedge dx_2 = dv_2 \wedge dx_2.$$

It follows that $dx_2 = dx_3 = 0$ along $\pi^{-1}(v)$, which is therefore a 1-cycle contained in (a countable union of lines parallel to the x_1-axis)$\times\{e_3\}$. This can only be zero. $\square$

The moral of the story

I. For surfaces $\Sigma \subset \mathbb{R}^3$, if we know $N(\Sigma)$

- as a cycle
- as a current, i.e. a functional on differential forms on $\mathbb{R}^3 \times S^2$

then we know a lot about Σ itself.

II. $N(\Sigma)$ is characterized by

- its Euler Morse data
- its Legendrian nature

III. It is natural, and possible, to take the fundamental characteristics of the cycle/current $N(\Sigma)$ as axioms for the **normal cycle** $N(X)$ of more general “singular subspaces" $X \subset \mathbb{R}^n$.

Case study: convex and polyconvex sets

Give $\mathbb{R}^n$ the usual orientation. Put $\mathcal{K} = \mathcal{K}(\mathbb{R}^n)$ for the family of compact convex subsets of $\mathbb{R}^n$ and $\mathcal{L} = \mathcal{L}(\mathbb{R}^n)$ for the ring of **polyconvex** subsets. For $A \in \mathcal{K}$ and $r > 0$ put

$$A_r =: \{x \in \mathbb{R}^n : \mathrm{dist}(x, A) \leq r\} \text{ (always a } C^1 \text{ submanifold of } \mathbb{R}^n)$$

$$\vec{N}(A) := \{(x, v) \in A \times \mathbb{R}^n : \langle v, x - y \rangle \geq 0 \text{ for all } y \in A\}$$

$$N(A) := \vec{N}(A) \cap (\mathbb{R}^n \times S^{n-1})$$

Fact

$N(A), \vec{N}(A)$ are oriented Lipschitz submanifolds. In fact $N(A)$ is biLipschitz equivalent to ∂A_r via

$$\Pi_A : p \mapsto \left(\pi_A(p), \frac{p - \pi_A(p)}{|p - \pi_A(p)|}\right), \quad (x, v) \mapsto x + rv$$

Radememacher's theorem

A Lipschitz function $\mathbb{R}^k \to \mathbb{R}^l$ is differentiable a.e.

Thus $N(A)$ defines a current of dimension $n-1$ in $S\mathbb{R}^n := \mathbb{R}^n \times S^{n-1}$:

$$\int_{N(A)} \varphi := \int_{\partial A_r} \Pi_A^* \varphi \iff \Pi_{A*}[\![\partial A_r]\!] = N(A) \tag{3}$$

for $\varphi \in \Omega^{n-1}(S\mathbb{R}^n)$, $r > 0$.

Theorem

As an operator $\mathcal{K} \to$ currents, N satisfies the **inclusion-exclusion identities** *and is* **continuous***:*

$$A_1, \ldots, A_m,\ B := A_1 \cup \cdots \cup A_m \in \mathcal{K} \implies \tag{4}$$

$$N(B) = N(A_1) + \cdots + N(A_m) - \sum_{i<j} N(A_i \cap A_j) + \ldots$$

$$A_i \to A \text{ in the Hausdorff metric } \implies N(A_i) \rightharpoonup N(A) \tag{5}$$

Corollary

Taking $N(1_A) := N(A)$ *for* $A \in \mathcal{K}$, N *extends by linearity to the* $\mathbb{Z}$*-module of* **polyconvex-constructible functions** *generated by such* 1_A, *and in particular to* $\mathcal{L}$ *by identifying* $C \leftrightarrow 1_C$.

Proof.

Groemer's Integral Theorem (cf. Klain & Rota): if L is a lattice of subsets of a set S, K is a generating set for L, G is an abelian group and $\varphi : K \to G$ satisfies (4), then φ extends uniquely to all of L, and also to a homomorphism from the abelian group of "L-constructible functions" on S to G. □

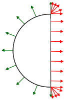
Proof of (5).

Generalizing (3), $N(A) = \Pi_{A*}[\![M]\!]$ for *any* convex C^1 hypersurface enclosing A, and $\Pi_{A_i} \to \Pi_A$ if $A_i \to A$. Taking M to be a large sphere,

$$N(A_i) = \Pi_{A_i*}[\![M]\!] \rightharpoonup \Pi_{A*}[\![M]\!] = N(A)$$

□

First proof of (4) ($m = 2$).



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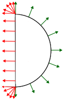

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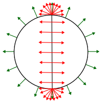

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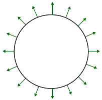
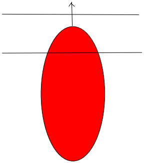


□

Second proof of (4).



Evidently for $A \in \mathcal{K}$

$$\Delta\chi(A, v, c) = \text{mult}_{h_v(x)=c}(N(A) \cdot \text{graph}(\nabla h_v))$$

Since χ obeys the inclusion-exclusion identities, so does $\Delta\chi(\,\cdot\,, v, c)$ for each v, c. Now repeat (or extend) the proof that the Euler Morse data of a compact surface Σ determines Σ. □

Corollary

Given $\varphi \in \Omega^{n-1}(S\mathbb{R}^n)$ *the functional* $\nu_\varphi : \mathcal{K} \to \mathbb{R}$*,*

$$\nu_\varphi(A) := \int_{N(A)} \varphi$$

is Euler additive and continuous , i.e. it is a **continuous valuation***. Again,* ν_φ *extends by linearity to all polyconvex-constructible functions.*

What is the full generality of this approach? Is it just stratified Morse theory? No:

- It's weaker because it only gives Euler Morse data
- It's stronger because it can handle the singularities of a convex set, which generally aren't stratified:
  Let $f : [0,1] \to \mathbb{R}$ be the Cantor function and $c(x) := \int_0^x f(t)\, dt$.
  Since $f \uparrow$, c is convex. There is $C \in \mathcal{K}(\mathbb{R}^{n+1})$ where
  $\partial C \supset \text{graph}((x_1, \ldots, x_n) \mapsto c(x_1) + \cdots + c(x_n))$.