1. Alesker Product

Recall that we have defined a multiplication on valuations based on the following heuristic principle. For a nicely shaped object $A$ in a manifold $M$, consider the (highly discontinuous, non-invariant) valuation $\mu_A$

$$\mu_A(C) := \chi(A \cap C).$$

Then we put

$$\mu_A \cdot \mu_B := \mu_{A \cap B}.$$ 

To define the Alesker product for nicer valuations we impose the condition that it be continuous and distributive. For instance, we may think of $\chi$ itself as the limit of $\mu_A$ as $A$ grows to include all of $M$. Therefore

$$\chi \cdot \mu_B = \lim_{A \uparrow M} \mu_A \cdot \mu_B = \lim_{A \uparrow M} (\mu_A \cdot \mu_B) = \lim_{A \uparrow M} \mu_{A \cap B} = \mu_B$$

since $B \subset A$ for $A$ large enough. Therefore $\chi$ is the multiplicative identity element.

More generally, observe that if we put $\phi := \mu_B$ then (1), (2) imply that

$$\phi \cdot \mu_A(C) = \phi(A \cap C).$$

We are mainly interested in valuations $\phi$ that are continuous, and invariant with respect to the action of some group $G$ on $M$, i.e. such that $\phi(gA) = \phi(A)$ for all nice $A \subset M$. If $G$ has an invariant measure and $A \subset M$ then we can construct such invariant valuation by

$$\mu_A^G := \int_G \chi(gA \cap \cdot) dg.$$ 

If we view the integral as, say, a limit of Riemann sums, then by (3) and the continuity and distributivity principles,

$$\phi \cdot \mu_A^G = \int_G \phi(gA \cap \cdot) dg.$$ 

If $M$ is a real vector space $V$ then we will always assume that $G$ includes the translation group. Our main examples are $V = \mathbb{R}^n, G = SO(n)$ and $V = \mathbb{C}^n, G = U(n)$. In these cases we can also construct continuous invariant valuations $t \in$
Val₁(ℝⁿ), s = Val₂(ℂⁿ) by

\[ t := \int_{Gr_{n-1}} \chi(P \cap \cdot) dP \]
\[ s := \int_{Gr_{n-1}} \chi(Q \cap \cdot) dQ \]

where \(dP, dQ\) are the invariant measures on the given affine Grassmannians. The Alesker products of a valuation \(\phi\) with \(s, t\) are then given by

\[ \phi \cdot t := \int_{Gr_{n-1}} \phi(P \cap \cdot) dP \]
\[ \phi \cdot s := \int_{Gr_{n-1}} \phi(Q \cap \cdot) dQ \]

In particular we find by induction that

\[ t^k := \int_{Gr_{n-k}} \chi(P \cap \cdot) dR \]
\[ s^k := \int_{Gr_{n-k}} \chi(S \cap \cdot) dS \]

where the measures \(dR, dS\) are obtained by pushing forward the product measures \((dP)^k, (dQ)^k\) under the (partially defined) intersection maps

\[ I_k : (Gr_{n-1})^k \to Gr_{n-k}, \]
\[ I_k(P_1, \ldots, P_k) := P_1 \cap P_2 \cap \cdots \cap P_k. \]

Clearly \(dR, dS\) are again invariant.

2. **Poincaré duality**

If \(V\) is a finite dimensional real vector space we denote by \(\text{Val} = \text{Val}(V)\) the vector space of continuous, translation-invariant valuations on \(V\). Put \(\text{Val}_i\) to be the subspace of valuations of degree \(i\), i.e. the space of all \(\phi \in \text{Val}\) such that

\[ \phi(tA) = t^i \phi(A) \]

for all nice \(A \subset V\) and \(t > 0\). P. McMullen showed that

\[ \text{Val} = \bigoplus_{i=0}^{n} \text{Val}_i \]

where \(n = \dim V\). Furthermore \(\text{Val}_0, \text{Val}_n\) are one-dimensional, spanned by \(\chi, \text{vol}\) respectively. If \(\dim V \geq 2\) then \(\dim \text{Val}(V) = \infty\). The Alesker product respects the grading, i.e. \(\text{Val}_i \cdot \text{Val}_j \subset \text{Val}_{i+j}\).

There is a bilinear form on \(\text{Val}(V)\) given by

\[ (\phi, \psi) := (\phi \cdot \psi)_n, \]
the degree $n$ component of the product, where we identify $\text{Val}_n$ with $\mathbb{R}$ by taking $\text{vol} \sim 1$. It is easy to see that
\[(\phi, \psi) = \lim_{R \to \infty} \omega_n^{-1} R^{-n} (\phi \cdot \psi)(B_R)).\]

Fact 2.1. If $\phi \in \text{Val}(V)$ and $K \subset V$ is compact then
\[\sup_{B \subset K} \phi(B) < \infty.\]

We may as well take $V = \mathbb{R}^n$. Let $G \subset SO(n)$ be a closed subgroup, and $\overline{G}$ the subgroup of the euclidean group $SO(n)$ generated by $G$ and translations. Then $\overline{G}$ admits an invariant measure $dg$, normalized so that for every $S \subset \mathbb{R}^n$
\[dg(\{g : go \in S\}) = \text{vol}(S).\]
Here $o$ may be taken to be any arbitrarily chosen point of $\mathbb{R}^n$.

Theorem 2.2 (Alesker-Poincaré duality). The Alesker pairing is perfect, i.e. if $0 \neq \phi \in \text{Val}^G(V)$ then there exists $\psi \in \text{Val}^G(V)$ such that $(\phi, \psi) \neq 0$.

Proof. Choose $A \subset V$ so that $\phi(A) \neq 0$. Then
\[(\phi, \mu_A^G) = \lim_{R \to \infty} \omega_n^{-1} R^{-n} (\phi \cdot \mu_A^G(B_R)).\]
\[= \lim_{R \to \infty} \omega_n^{-1} R^{-n} \int_{\overline{G}} \phi(gA \cap B_R) \, dg\]
\[= \lim_{R \to \infty} \omega_n^{-1} R^{-n} \left( \int_{\{gA \subset B_R\}} + \int_{\{gA \cap B_R \neq \emptyset, gA \not\subset B_R\}} \right) \phi(gA \cap B_R) \, dg\]

Since $\phi$ is $G$-invariant, the integrand of the first integral is precisely $\phi(A)$, and for large $R$ the measure of the domain of integration is $\sim \omega_n R^n$. The integrand of the second integral is dominated by $\sup_{B \subset A} \phi(B)$, and the domain of integration is $O(R^{n-1})$. So the value of the pairing is precisely $\phi(A)$. □

3. Numerology

A version of Crofton’s formula states that there is a constant $c$ such that if $\sigma \subset \mathbb{R}^n$ is a curve then $t(\sigma) = c \text{length}(\sigma)$. Similarly, if $\Sigma \subset \mathbb{C}^n$ is a piece of a complex curve then $s(\Sigma) = c' \text{area}(\Sigma)$. We adjust the normalizations of the measures $dP, dQ$ so that
\[t(\sigma) = \frac{2}{\pi} \text{length}(\sigma)\]
\[s(\Sigma) = \frac{1}{\pi} \text{area}(\Sigma).\]
In particular, if $\Sigma$ is a unit disk contained in a complex line $\subset \mathbb{C}^n$ then
\[s(\Sigma) = 1.\]
In this section we establish some values that follow from these choices:

**Proposition 3.1.**
\[e^{\pi t} = \sum \omega_n \mu_n, \text{ i.e. } \frac{\pi n t^n}{n!} = \omega_n \mu_n\]
Proposition 3.2. If $B$ is the unit ball of $\mathbb{C}^n$ then
\[
s^k t^{2n-2k} (B) = \binom{2n-2k}{n-k}.
\]

3.1. Volumes of balls, spheres, and projective spaces. We will need to know the volumes of a few basic objects.

We denote the volume of the unit ball in $\mathbb{R}^n$ by $\omega_n$, and the volume of the $n$-dimensional unit sphere by $\alpha_n$. Evaluating $\omega_{n+1}$ by the shell method we find that
\[
(n + 1)\omega_{n+1} = \alpha_n.
\]

Fact 3.3.
\[
\frac{\omega_{n+2}}{\omega_n} = \frac{2\pi}{n+2}.
\]

From the obvious facts
\[
\omega_0 = 1, \quad \omega_1 = 2
\]
we can now find all the $\omega_n$ by induction. In particular
\[
(7) \quad \omega_{2n} = \frac{\pi^n}{n!}.
\]

We also find by induction from Fact 3.3
\[
(8) \quad \omega_n \omega_{n+1} = 2 \frac{2^n \pi^n}{(n+1)!}.
\]

Next we calculate the volume of $\mathbb{C}P^n$. Recall that this space may be viewed as the quotient of $S^{2n+1} \subset \mathbb{C}^{n+1}$ under the equivalence: $v \sim w$ iff $e^{i\theta}v = w$ for some $\theta$. The metric structure of $\mathbb{C}P^n$ is then given by taking the distance between two points of $\mathbb{C}P^n$ to be the spherical distance between their preimage circles in $S^{2n+1}$. This is something like viewing the real line as the quotient of a cylinder, by collapsing the meridians of the cylinder down to points: the distance between points of the line is precisely the same as the distance along the cylinder between the corresponding meridians. Of course the area of any section of the cylinder is precisely the length of the meridian circle times the length of the corresponding segment of the line. The volumes of $\mathbb{C}P^n$ and $S^{2n+1}$ are related similarly:
\[
\alpha_{2n+1} = 2\pi \text{vol}(\mathbb{C}P^n).
\]

Since
\[
\alpha_{2n+1} = 2(n+1)\omega_{2n+2} = 2 \frac{\pi^{n+1}}{n!}
\]
we compute
\[
(9) \quad \text{vol}(\mathbb{C}P^n) = \frac{\pi^n}{n!} = \omega_{2n}.
\]

3.2. Powers of $t$. In principle one could carry out the calculations of Propositions 3.1 and 3.2 by carrying out the integrals explicitly, but it turns out that we may avoid such unpleasantness by means of a very useful trick. The idea is to switch to a compact space, where the calculation is much easier. This is called the “transfer principle.” We carry this out first for Proposition 3.1, using the unit sphere $S^n$ as a stand-in for $\mathbb{R}^n$. 
Recall that we have defined a valuation $\phi$ on the $\mathbb{S}^n$ as

$$(10) \quad \phi = \int_{\text{Gr}_{n-1}} \chi(\cdot \cap H) \, dH$$

where $\text{Gr}_{n-1}$ here denotes the space of great hyperspheres in $\mathbb{S}^n$ (which may be identified with $\mathbb{RP}^n$) with its $SO(n+1)$-invariant measure $dH$. Crofton's formula tells us that if $\sigma \subset \mathbb{S}^n$ is a curve then $\phi(\sigma)$ is some multiple of the length of $\sigma$, depending on the normalization of $dH$. To make this compatible with (4), let us adjust this normalization so that the coefficient is $\frac{2}{\pi}$:

$$(11) \quad \phi(\sigma) = \frac{2}{\pi} \text{length}(\sigma).$$

Taking $\sigma$ to be a great circle $S^1$ we find that

$$4 = \frac{2}{\pi} \cdot 2\pi \quad = \phi(S^1) \quad = \int_{\text{Gr}_{n-1}} \chi(H \cap S^1) \, dH \quad = 2 \, dH(\text{Gr}_{n-1}),$$

i.e. $dH(\text{Gr}_{n-1}) = 2$.

By the definition of the Alesker product it follows that

$$(12) \quad \phi^k = \int_{\text{Gr}_{n-k}} \chi(\cdot \cap L) \, dL$$

where $\text{Gr}_{n-k}$ denotes the space of great sub hyperspheres of codimension $k$ and $dL$ is the $SO(n+1)$-invariant measure of total mass $2^k$. Taking $k = n$, and recalling that $\text{Gr}_0$ is the space of all pairs of antipodal points of $\mathbb{S}^n$, we compute

$$(13) \quad \phi^n(S^n) = \int_{\text{Gr}_0} \chi(S^n \cap K) \, dK = \int_{\text{Gr}_0} \chi(K) \, dK = \int_{\text{Gr}_0} 2 \, dK = 2^{n+1}$$

Since $\phi^n$ is a multiple of the $n$-dimensional volume measure on $\mathbb{S}^n$, we conclude that

$$(14) \quad \phi^n = \frac{2^{n+1}}{\alpha_n} \, \text{vol}_n.$$

In order to relate this conclusion to $\mathbb{R}^n$, we observe that $\mathbb{S}^n$ looks like $\mathbb{R}^n$ at very small scales. Moreover, the great hyperspheres look like affine hyperplanes, and the invariant measures also correspond. In view of (12) we conclude that $\phi$ and $t$ coincide at vanishingly small scales, and therefore

$$(15) \quad t^n = \frac{2^{n+1}}{\alpha_n} \, \text{vol}_n.$$

But

$$\frac{2^{n+1}}{\alpha_n} = \frac{2^{n+1}}{(n+1)\omega_{n+1}} \quad = \frac{2^{n+1}}{(n+1)} \cdot \frac{\omega_n(n+1)!}{2(2\pi)^n} \quad \text{(by (3.3))} \quad = \frac{\omega_n n!}{\pi^n}.$$
3.3. Values of monomials on the unit ball. To prove Proposition 3.2 we use the transfer principle again, letting $\mathbb{C}P^n$ stand in for $\mathbb{C}^n$. The idea is that $\mathbb{C}P^n$ looks like $\mathbb{C}^n$ at small scales.

The analogue for $\mathbb{C}P^n$ of the valuation $s$ on $\mathbb{C}^n$ is

$$
\sigma := \int_{\text{Gr}_{n-1}^C} \chi(\cdot \cap H) \, dH
$$

where $\text{Gr}_{n-1}^C$ now denotes the space of complex projective hyperplanes (copies of $\mathbb{C}P^{n-1}$) in $\mathbb{C}P^n$. This space is naturally identified with $\mathbb{C}P^n$ itself. Again, $dH$ is the $U(n+1)$-invariant measure on $\text{Gr}_{n-1}^C$; we wish to normalize this measure to make it compatible with normalization of $s$ given in (6). In this case Bézout’s theorem tells us that if $\Sigma \subset \text{Gr}_{n-1}^C$ is a complex curve then $\sigma(\Sigma)$ is a multiple of the area of $\Sigma$. By (6), we want the coefficient to be $\frac{2}{\pi}$. Taking $\Sigma = \mathbb{C}P^1$, and using the case $n = 1$ of (9), we get

$$
1 = \frac{1}{\pi} \text{area}(\mathbb{C} P^1)
= \sigma(\mathbb{C} P^1)
= \int_{\text{Gr}_{n-1}^C} \chi(\mathbb{C} P^1 \cap H) \, dH
= dH(\text{Gr}_{n-1}^C)
$$

since $\mathbb{C} P^1 \cap H$ is a single point if $H$ is in general position. In other words, we take $dH$ to be the invariant probability measure. It follows that

$$
\sigma^k = \int_{\text{Gr}_{n-k}^C} \chi(\cdot \cap L) \, dL
$$

where $dL$ is the invariant probability measure on the codimension $k$ complex Grassmannian $\text{Gr}_{n-k}^C$.

We know that $\sigma^k t^{2n-2k}$ is some multiple of the volume. Since (as in the case of $\mathbb{R}^n$ and $S^n$) $s$ and $\sigma$ coincide at vanishingly small scales, we conclude that $\sigma^k t^{2n-2k}$ and $s^k t^{2n-2k}$ are both the same multiple of the volume. Therefore by (9)

$$
\sigma^k t^{2n-2k}(\mathbb{C}P^n) = s^k t^{2n-2k}(B)
$$

and we wish to show that this common value is $\binom{2n-2k}{n-k}$.

Taking $k = 0$ we compute from Proposition 3.1

$$
t^{2n}(\mathbb{C}P^n) = \frac{\omega_{2n}(2n)!}{\pi^{2n}} \text{vol}_{2n}(\mathbb{C}P^n) = \frac{(2n)!}{\pi^{2n}} \left( \frac{\pi^n}{n!} \right)^2 = \binom{2n}{n}.
$$

Therefore, again invoking the definition of the Alesker product,

$$
\sigma^k t^{2n-2k}(\mathbb{C}P^n) = \int_{\text{Gr}_{n-k}^C} t^{2n-2k}(\mathbb{C}P^n \cap L) \, dL
= t^{2n-2k}(\mathbb{C}P^{n-k})
= \binom{2n-2k}{n-k}
$$

after replacing $n$ by $n - k$ in (18).
3.4. Swept under the rug.

(1) I haven’t discussed here what $t$ means as a valuation on $\mathbb{C}P^n$, relying only on the assertion that each power $t^{2n-2k}$ acts as a multiple of the volume for $(2n - 2k)$-dimensional objects, with coefficient given by Proposition 3.1.

(2) These notes give a complete (maybe slightly informal) justification for our algebraic/numeric procedure for checking whether a given polynomial in $s,t$ vanishes as an element of $\text{Val}^{U(n)}(\mathbb{C}^n) = V_0^n$. As I never tire of saying, this serves to reduce many questions about the integral geometry of $\mathbb{C}^n$ to pure algebra.

I have also described a checking procedure for the algebra $V_1^n$ of invariant valuations on $\mathbb{C}P^n$, giving an algebraic/numeric criterion for the vanishing of the invariant valuation corresponding to a given polynomial in $\sigma, t$. Following the calculations above, this amounts to evaluating $t^i(\mathbb{C}P^{n-k})$ for values of $i$ other than $2n - 2k$, or more generally the values of $t^i$ at other algebraic subvarieties $W \subset \mathbb{C}P^n$. Obviously this turns on item (1) above.

The geometric side of the answer (due essentially to A. Gray) is most naturally expressed in terms of the Chern classes of $W$. As far as I know, this is the only place where characteristic classes explicitly enter the integral geometry picture (even though the two subjects display many other uncanny similarities).