THE CATEGORY OF CATEGORIES AS A MODEL FOR
THE PLATONIC WORLD OF FORMS

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by

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There have been many fascinating developments recently in mathematics. The early work by Eilenberg and MacLane on category theory has now progressed to the point where it can no longer be said to be "just a language". At the least, it has replaced set theory as a universal language for mathematics. The concept of adjointness, first pointed to by Kan and developed by Freyd, Lawvere, and others, has in the most recent work of Lawvere become the universal principle underlying all of mathematics (or so the writer feels). Following is an attempt to put this work into a historical context and to make it accessible to the non-specialist.

Pre-Greek mathematics was of an essentially empirical character similar to what we today call engineering. For example, in certain surviving fragments of Egyptian mathematics the formulas for the volumes of certain solids were expressed with no distinction being made between exact and approximate formulas. One of the greatest developments in the history of mathematics is the understanding of the concept of proof and formal theory. This first was clearly understood by Thales. He showed that certain statements in mathematics can be derived one from the other by rules of logic. He proved such statements as that the base angles
of an isosceles triangle are equal. This is one of the earliest universal statements to be proven in mathematics, or for that matter in any field. This was the beginning in man's flight from the particular to the general. The Pythagoreans proceeded to develop the formal structure of geometry and number theory. They discovered, "created", through pure thought concepts which had eluded all other civilizations. Two prime examples are the concepts of irrational number and Platonic solid, both of which are beautifully handled by Euclid. One may argue that nothing is eternal, but, though Euclid's Elements is flawed in detail its modus operandi is forever.

Plato, one of the top-ranked mathematicians of his time and of all time, was very impressed by geometry. This is demonstrated by the sayings attributed to him: "God forever geometricizes" and "Let no one enter these halls who is not familiar with geometry". Plato set for himself the program of extending the power of geometry, which only applied to triangles and circles and such, to all of human thought. He failed, but his vision has come to pass.

Plato searched for a universal universe, a universal mathematics, where all of reality would be deducible as are the theorems in Euclid. He never got beyond the conceptual stage and even that was very hazy. There is, however, a counterpart in modern mathematics to the Platonic world of forms. This is known as the meta-category
of categories. In the recent work of Lawvere this has progressed through the conceptual stage to the formal stage. Lawvere has given a set of axioms which uniquely pick out the category of categories. Further yet, he has already made considerable progress in working out their consequences. At the Battelle conference on categorical algebras, Lawvere gave a series of lectures entitled "Hyperdoctrines". These lectures and two papers, "Category-Valued Higher Order Logic" and "Adjointness in Foundations", deserve serious consideration by those interested in the foundations of mathematics and philosophy in general. The natural question to ask is why Lawvere has succeeded where Plato failed. The answer to this question can be understood in the words of Newton, "If I have seen farther than others it is because I have stood on the shoulders of giants". Lawvere's work represents the union of the formalism of Descartes and the conceptualism of Plato. Other philosophers have searched for this union, namely Leibnitz and Whitehead, but the tools which they used were inadequate for the job. Lawvere was in possession of what may become the ultimate abstraction of Descartes' already abstract idea of function, namely the Eilenberg-MacLane theory of categories and functors.

The concept of a function between small finite sets is very clear. Once you leave the realm of small finite sets, and in fact desire to talk about infinite sets and functions
between them, it becomes very necessary to formalize the definition of function. The result is that the definition of function is given implicitly in terms of your axioms of set theory, for example the axiom of choice helps determine what we mean by function. Developments in algebraic topology led Eilenberg and MacLane to formalize the concept of category and functor, which is, in a sense, an abstraction of the concepts of set and function. For example, for the category of finite sets and the standard definition of function we observe certain formal properties of functions, namely: 1. To every set there is a special function, the identity function; 2. Given any function, it uniquely determines two sets called the domain and co-domain; 3. Given any three sets $X, Y, Z$ and two functions $f$ and $g$, such that the domain of $f$ is $X$ and the co-domain of $f = Y = \text{the domain of } g$, and the co-domain of $g = Z$, then there is a uniquely determined function $h$ whose domain is $X$ and whose co-domain is $Z$ called the composition of $f$ with $g$. (The reader will probably recognize this as formalized trivia, but, as is presently the case, from trivia spring monumental ideas.) The same types of structures arise when one considers the category of groups and group homomorphisms, topological spaces and continuous maps, etc. In fact, the meta-category of all categories itself has the structure of a category. This brings up the difference between category theory as foundations of mathematics and
some of the previous attempts at a single, first order theory to serve as a foundation for mathematics. Category theory has the property which I like to describe as stability under transition to the meta-language, namely, both the formal languages and the meta-languages can all use the same notation.

There is another reason why the systematic use of category theory is important. Let me describe certain situations which occur in mathematics. In Euclidian geometry, there are theorems about the plane which cannot be proved without the use of the space axioms. In number theory, theorems such as the prime number theorem require sophisticated tools from analytic number theory and cannot be derived from a set of axioms such as those given by Mendelson in *Introduction to Mathematical Logic*. What I'm referring to, of course, is Godel's incompleteness theorem. What is usually done in mathematics to circumvent this theorem is to imbed the structure in which the problem is originally formulated into a larger super-structure, which is, in a sense, a natural way of adding extra axioms. I feel the recent successes of generalized cohomology theories is an example. When you take a topological space and consider it as an object in the category of topological spaces, and then consider the category of topological spaces as an object in the category of categories, you have put much stronger restrictions on what is meant by a topological
space, and have, in effect, added a great number of new axioms. Thus it is clear why certain theorems are so easy to prove using homology theory and k-theory, but appear to be impossible in any other way.

There is a great amount of work that needs to be done in these fields, especially that of studying the structure of the category of categories and seeing how far the principle of adjointness can be pushed. The subject matter, being so young, and highly philosophical, is very accessible to all, especially the young. The following is a list of articles which will provide a rapid and stimulating introduction to the current research.

Freyd, Professor P. J., "Representation in Abelian Categories".
Lawvere, Professor F. W., "Category of Categories as a Foundation for Mathematics".
Lawvere, Professor F. W., "Adjointness in Foundations" N.Y.P.
Lawvere, Professor F. W., "Category-Valued Higher Order Logic" N.Y.P.
MacLane, Professor S., "Categorical Algebra".

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