ČECH AND STEENROD HOMOTOPY THEORIES
WITH APPLICATIONS TO GEOMETRIC TOPOLOGY

by

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and

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§1. Introduction

Inverse systems of topological spaces occur in many contexts in topology. Some examples are:

a) Geometric Topology. One can associate to any embedding $X \hookrightarrow Y$ the inverse systems $\{U\}$, $\{U \setminus X\}$, $\{(Y \setminus X, U \setminus X)\}$, etc., where $U$ varies over the open neighborhoods of $X$ in $Y$. If $X$ is locally compact, then one can associate to $X$ its end, $E(X) = \{X \setminus C\}$, where $C$ varies over compact subsets of $X$.

b) Algebraic Topology. The Čech construction associates to a topological space an inverse system of CW-complexes. A similar construction in algebraic geometry leads to étale homotopy theory. The Postnikov and pro-finite completion constructions also associate inverse systems of complexes to complexes.


Key words and phrases: closed model category, homotopy category, Pro-H, shape theory, étale functor, Čech cohomology, Vietoris cohomology, completion, open manifold ends, knot theory, embedding theory.
It is often important to have an appropriate category and homotopy category of inverse systems of spaces. In [Gro-1] Grothendieck showed how to associate to any category $C$ another category $pro-C$ whose objects are inverse systems in $C$ indexed by "filtering categories" and whose morphisms are so defined as to make cofinal systems isomorphic. In [Q-1] Quillen introduced the notion of a model category as an axiomatization of homotopy theory on $Top$ and $SS$. A model category is a category $C$ together with three classes of morphisms called cofibrations, fibrations and weak equivalences which satisfy "the usual properties." The homotopy category of $C$, $Ho(C)$, is obtained from $C$ by formally inverting the weak equivalences in $C$. In [A-M] Artin and Mazur developed the algebraic topology of $pro-Ho(C)$. One has a canonical functor $pro-C \to pro-Ho(C)$ and it is natural to consider $pro-Ho(C)$ as the homotopy category of $pro-C$. This point of view goes back to Christie [Chr]. But Christie also realized that for some purposes $pro-Ho(C)$ was too weak and one really wanted a stronger category. It can be shown that $pro-Ho(C)$ is not the homotopy category of a model category structure on $pro-C$. Hastings [Has-1] has shown that $pro-SS$ admits a natural model category structure with homotopy category, $Ho(pro-SS)$, obtained from $pro-SS$ by formally inverting level weak equivalences. $Ho(pro-SS)$ had previously appeared in Porter's work on the stability problem for topological spaces [Por-2]. Grossman [Gros-1] has studied a coarser model category structure on $Towers-SS$.

In the first part of these notes (§§2-5) we develop the algebraic topology of $Ho(pro-C)$ and compare $Ho(pro-C)$ with $pro-Ho(C)$. 
We also give applications to the study of the derived functors of the inverse limit. The second part of these notes (§§6-8) contains applications to proper homotopy theory, group actions on the Hilbert cube, and shape theory.

More precisely, §2 contains background material about pro-categories and model categories. The "Mardesic trick," described in §2.1, says that all inverse systems are pro-equivalent to inverse strongly systems indexed over cofinite, directed sets. Quillen's theory of model categories [Q-1] is reviewed in §2.3.

In §3 we show that for nice closed model categories $C$, $C^J$ (where strongly $J$ is a cofinite, directed set) and pro-$C$ inherit natural closed model structures from $C$, extending [Has-1].

In §4 we show that the natural inclusion $\text{Ho}(C) \rightarrow \text{Ho}(\text{pro-}C)$ has an adjoint $\text{holim}: \text{Ho}(\text{pro-}C) \rightarrow \text{Ho}(C)$ (compare Bousfield and Kan [B-K-1]) and obtain vanishing theorems for $\lim^S$.

The basic algebraic topology of $\text{Ho}(\text{pro-}C)$ is developed in §5. We compare $\text{Ho}(\text{pro-}C)$ with pro-$\text{Ho}(C)$, discuss homotopy and homology pro-groups, and prove various Whitehead theorems. §5.5, Whitehead and stability theorems, includes a survey of work of the first author and R. Geoghegan [E-G-1,4].

In §6 we show that the category of $\sigma$-compact spaces and proper maps may be embedded in a suitable category of towers which is a closed model category. We then apply pro-homotopy theory to proper homotopy theory and weak-proper-homotopy theory (see [Chap-1] and [C-S] for
weak-proper-homotopy theory and its uses in the study of Q-manifolds and shape theory.) Some of our results are announced in [E-H-3].

We apply this theory in §7 to the study of group actions on infinite dimensional manifolds. §7 represents joint work with Jim West [West-1], [E-H-W].

In §8 we discuss strong shape theory and develop generalized Steenrod homology theories using pro-homology. These theories have found recent applications in the Brown-Douglas-Fillmore theory of operator algebras [B-D-F-1-2]; see also Kaminker and Schochet [K-S].

Detailed introductions precede §§3-8.

Acknowledgements. We wish to acknowledge helpful discussions with Tom Chapman, Ross Geoghegan, and Jim West, and correspondence with A. K. (Peter) Bousfield and Jerry Grossman. Some of this material was presented at conferences at Syracuse University (Syracuse, N. Y., December, 1974), Mobile, Alabama (March, 1975), New York (March, 1975), Syracuse University (Syracuse, N. Y., April, 1975), and the University of Georgia (August, 1975), whose hospitality we wish to acknowledge.

The second-named author held a visiting position at the State University of New York at Binghamton during the academic year 1974-75 and wishes to acknowledge their support. He was also partially supported by N.S.F. Institutional Grants at Hofstra University in 1973-74 and 1975-76.

We wish to thank Althea Benjamin for typing this manuscript.
§ 2.1. Pro-Categories.

We need a category of inverse systems such that cofinal subsystems are isomorphic; such a category was first defined by Grothendieck in [Gro-1] and is described in detail in the appendix of [A-M].

(2.1.1) Definitions. A category I is said to be Left Filtering if I is non-empty, and:

(a). Every pair of objects \( i, i'' \in I \) can be embedded in a diagram

\[
\begin{array}{ccc}
& i' & \\
\downarrow & & \downarrow \\
\downarrow & & \\
\downarrow & & \downarrow \\
\end{array}
\]

(b). If \( i' \rightarrow i\) is a pair of maps in I, then there is a map \( i'' \rightarrow i' \in I \) such that the compositions \( i'' \rightarrow i' \) are equal.

If \( C \) and \( I \) are categories, then an \( I \)-Diagram over \( C \) is just a functor \( X:I \rightarrow C \). We will usually use the notion \( \{X_i\} \), suppressing both \( I \) and the bonding morphisms \( X(i) \rightarrow X(i') \).

A pro-object over \( C \) is an \( I \)-diagram over \( C \) where \( I \) is a small left filtering category. The pro-objects over \( C \) form a category pro-\( C \) with maps defined by \( (\text{pro-} C)(\{X_i\}, \{Y_j\}) = \lim_j \text{Colim}_i \{C(X_i, Y_j)\} \).

(Note: the indexing categories are not assumed equal).

We have defined the set of maps in pro-\( C \) from \( X \) to \( I \), but the above definition is somewhat opaque and it's not obvious how to define
the composition of two maps from the above definition. Hence, we shall give an alternative definition. For simplicity, assume that we are given inverse systems \( \{X_i\} \) and \( \{Y_j\} \) in \( C \) which are indexed by directed sets \( I \) and \( J \).

\[ (2.1.2) \quad \text{Definition.} \quad \text{A morphism} \ f : X \longrightarrow Y \ \text{in pro-}C \ \text{is represented by} \ \theta : J \longrightarrow I \ (\text{not necessarily order preserving}) \ \text{and morphisms} \ f_j : X_{\theta(j)} \longrightarrow Y_j \ \text{of} \ C \ \text{for each} \ j \in J, \ \text{subject to the condition that if} \ j \preceq j' \ \text{in} \ J \ \text{then for some} \ i \in I \ \text{such that} \ i \triangleright \theta(j) \ \text{and} \ i \triangleright \theta(j'), \ \text{the diagram}
\]

\[ \begin{array}{c}
X_{i,\theta(j)} \\
\downarrow \\
X_i \\
\downarrow \\
X_{i,\theta(j')} \\
\downarrow \\
X_{\theta(j)} \\
\downarrow \\
f_j \\
\downarrow \\
Y_j \\
\downarrow \\
Y_j' \end{array} \]

\[ \text{commutes} \ (X_{i,\theta(j)} : X_i \longrightarrow X_{i'}, \ \text{etc.}, \ \text{are the bonding maps of the inverse systems). Two pairs} \ (\theta, f_j) \ \text{and} \ (\theta', f'_j) \ \text{represent the same morphism in pro-}C \ \text{if for each} \ j \in J \ \text{there is an} \ i \in I \ \text{such that} \ i \triangleright \theta(j) \ \text{and} \ i \triangleright \theta'(j) \ \text{and} \ f_j X_{i,\theta(j)} = f'_j X_{i,\theta'(j)}. \]

\[ (2.1.3) \quad \text{Remark.} \quad \text{The pro-object} \ \{X_i\} \in \text{pro-}C \ \text{contains much more information about the inverse system than does the inverse limit,} \]

\[ \lim_i \{X_i\} \in C, \ \text{even if the inverse limit exists in} \ C - \text{it might not exist in} \ C. \ \text{The relationship between the pro-object} \ \{X_i\} \ \text{and the inverse limit} \ \lim_i \{X_i\} \ \text{is analogous to that between the germ of a function at a point} \ p \ \text{and the value of} \ f \ \text{at} \ p. \]

The following reindexing results from [A-M] will be needed.
(2.1.4) Proposition. A map \( f : X \rightarrow Y \in \text{pro-} C \) can be represented up to isomorphism (in Maps (pro- C)), by a small left filtering inverse system of maps \( \{ X_i \xrightarrow{f_i} Y_i \} \), i.e., by a pro-object over Maps (C).

More generally, the following holds.

(2.1.5) Proposition. Let \( \Delta \) be a finite diagram with commutation relations, and suppose that \( \Delta \) has no loops, i.e., that the beginning and end of a chain of arrows are always distinct. Let \( D \) be a diagram in \( \text{pro-} C \) of the type of \( \Delta \), i.e., a morphism of \( \Delta \) to \( \text{pro-} C \). There is a left filtering inverse system \( \{ D_i \} \) of diagrams of \( C \) such that the diagram in \( \text{pro-} C \) determined by \( \{ D_i \} \) is isomorphic to \( D \).

Our techniques often require that the indexing categories be

\textbf{cofinite} (each element has finitely many predecessors) \textbf{strongly directed sets} \((a \leq b \text{ and } b \leq a \text{ implies } a = b)\). The following reindexing trick was inspired by Mardescic [Mar-1]. Let \( I \) be a strongly directed set. Define \( M(I) \) to be the set of finite subsets of \( I \) containing a maximum element. \( M(I) \) is a cofinite strongly directed set ordered by inclusion. Associating to a set
A in $\mathcal{M}(I)$ is its maximum element determines a cofinal order preserving map $\mathcal{M}(I) \xrightarrow{M} I$. If $\{X_i\}_{i \in I}$ is an object of pro-$\mathcal{C}$, then

$\{X_{M(A)}\}_{A \in \mathcal{M}(I)}$ is indexed by the cofinite strongly directed set

$\mathcal{M}(I)$ and the natural map $\{X_i\}_{i \in I} \rightarrow \{X_{M(A)}\}_{M(I)}$ is a pro-$\mathcal{C}$ isomorphism. Let pro-$\mathcal{C}$ denote the full subcategory of pro-$\mathcal{C}$ consisting of objects whose indexing category is a strongly directed set. The above observations yield a functor $M: \text{pro-}\mathcal{C} \rightarrow \text{pro-}\mathcal{C}$, naturally equivalent to the identity, such that $M(X)$ has a cofinite indexing set for every $X$ in pro-$\mathcal{C}$. We need a functor

$M: \text{pro-}\mathcal{C} \rightarrow \text{pro-}\mathcal{C}$ with the same properties. Let $I$ be a (small) left filtering category. Let $M'(I)$ be the set of ordered quadruples $(A,B,C,D)$ such that:

a) $A$ is a finite subset of $\text{Ob}(I)$. (= objects of $I$);

b) $B$ is a finite subset of $\text{Mor}(I)$ (=morphisms of $I$) whose domains and ranges lie in $A$;

c) $C$ is an object of $I$;

d) $D$ is a set of maps from $C$ to each object in $A$ (one for each object in $A$) such that if $X,Y \in A$ and $f:X \rightarrow Y$ is in $B$ then the diagram
commutes, where \( g \) and \( h \) are in \( D \).

Say \( (A,B,C,D) < (A',B',C',D') \) if \( A \cup C \subset A' \) and \( B \cup D \subset B' \).

\( M'(I) \) is clearly cofinite. Because \( I \) is filtering, \( M'(I) \) is directed. But we may choose a cofinal strongly directed set in any directed set \( J \) as follows. Partition \( J \) by setting

\[
[j] = \{ j' \in J \mid j \leq j' \text{ and } j' \leq j \}.
\]

Define a strongly directed set \( J' \) by choosing one element in each equivalence class constructed above, together with the partial order induced from \( J \). Applying this construction to \( M'(I) \) yields a strongly directed set \( M(I) \). The association \( (A,B,C,D) \mapsto C \) determines a cofinal functor \( M : M(I) \to I \) and thus (by a strong use of the Axiom of Choice) our desired functor \( M : \text{pro-} C \to \text{pro-} C \). Summarizing, we have the following theorem.

\( (2.1.6) \) \textbf{Theorem.} There exists a functor \( M : \text{pro-} C \to \text{pro-} C \), naturally equivalent to the identity, such that \( M(X) \) is indexed by a cofinite strongly directed set for every \( X \) in \( \text{pro-} C \). \( \square \)
§2.2. Simplicial sets.

In this section we shall sketch some of the basic definitions of the category $SS$ of simplicial sets (the present standard reference is [May]). $SS$ is the prototype for D. Quillen's abstract model category $[0,1]$, a category with sufficient structure to do "homotopy theory". There are two goals for this section: first, to develop enough of the basic theory of $SS$ to make these notes accessible to those not familiar with simplicial techniques; and second, to motivate the concept of abstract model category which is basic to our homotopy theory of pro-spaces.

(2.2.1) Definition. A simplicial set $X$ consists of:

a) A sequence $X_n$, $n \geq 0$, of sets. The elements of $X_n$ are called the $n$-simplicies of $X$.

b) Face maps $d_i^n : X_n \rightarrow X_{n-1}$ for $n > 1$ and $0 \leq i \leq n$.

c) Degeneracy Maps $s_i^n : X_n \rightarrow X_{n+1}$ for $n \geq 0$ and $0 \leq i \leq n$.

The maps are required to satisfy the following identities:

$$d_i d_j = d_{j-1} d_i \quad \text{for} \quad i < j,$$

$$s_i s_j = s_{j+1} s_i \quad \text{for} \quad i < j,$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for} \quad i < j, \\ I_d & \text{for} \quad i = j \text{ or } j+1, \\ s_j d_{i-1} & \text{for} \quad i > j+1. \end{cases}$$

For example, the singular complex $S(X)$ of a topological space $X$ is a simplicial set with typical $n$-simplex a continuous map $f : \Delta^n \rightarrow X$, where $\Delta^n$ is the standard $n$-simplex in $\mathbb{R}^{n+1}$. 
(2.2.2) Definition. A map \( f: X \to Y \) of simplicial sets consists of a sequence of maps \( f_n: X_n \to Y_n, \ n \geq 0 \), which satisfy the identities \( f_{n-1} d_i = d_i f_n \) and \( s_i f_n = f_{n+1} s_j \).

Definitions (2.2.1) and (2.2.2) combine to yield the category of simplicial sets, \( SS \). The following Kan extension condition is crucial to the development of the homotopy theory of \( SS \). Let \( x \) be an \( n \)-simplex of a simplicial set \( X \). The faces of \( x \), \( d_0 x, d_1 x, \ldots, d_n x \) satisfy certain compatibility conditions, for example, \( d_j (d_j x) = d_j (d_{j+1} x) \). A simplicial set \( X \) is said to satisfy the Kan extension condition (or is simply called a Kan complex) if given any \( n \) \((n-1)\)-simplices \( y_0, y_1, \ldots, y_{\hat{1}}, \ldots, y_n \) which satisfy the compatibility conditions to be the faces of an \( n \)-simplex, there exists an \( n \)-simplex \( x \) with \( d_j x = y_j \) for \( j = 0, 1, \ldots, \hat{1}, \ldots, n \). More generally, a map \( p: E \to B \) in \( SS \) is called a Kan fibration if given an \( n \)-simplex \( b \) in \( B \), and \( n \) \((n-1)\)-simplices in \( E, y_0, y_1, \ldots, y_{\hat{1}}, \ldots, y_n \) which satisfy the appropriate compatibility conditions and the requirement \( p(y_j) = d_j b \) for \( j = 0, 1, \ldots, \hat{1}, \ldots, n \), there is an \( n \)-simplex \( x \) in \( E \) with \( d_j x = y_j \) for \( j = 0, 1, \ldots, \hat{1}, \ldots, n \), and \( p(x) = b \).

For example, the singular complex of a space is a Kan complex; also a Hurewicz fibration \( p: E \to B \) of spaces induces a Kan fibration \( S(p): S(E) \to S(B) \).

D. M. Kan[1962] gave a combinatorial description of the homotopy groups of Kan complexes and also described a functorial Postnikov decomposition of Kan complexes. This definition of the homotopy groups is extended to all simplicial sets by defining
\[ \pi_1(X) \cong \pi_1(\text{SRX}) (= \pi_1(\text{RX})), \] where \( R : \text{SS} \to \text{Top} \) denotes Milnor's \[ \text{[Mil-2]} \] geometric realization functor.

Call a map \( f : X \to Y \) of simplicial sets a cofibration if it is an inclusion (i.e., each \( f_n \) is an inclusion), a weak equivalence if \( \pi_0(f) \) is a bijection, and for every choice of basepoints in \( X \), \( \pi_*(f) \) is an isomorphism. The usual homotopy-extension and covering-homotopy properties are combined in the following theorem.

(2.2.3) **Covering Homotopy Extension Theorem.** Given a commutative solid-arrow diagram

\[
\begin{array}{ccc}
A & \longrightarrow & Y \\
\downarrow i & & \downarrow \pi_1 \\
X & \longleftarrow & B
\end{array}
\]

in which \( i \) is a cofibration, \( p \) is a fibration, and either \( i \) or \( p \) is a weak equivalence, then there exists a filler \( f \).

See, e.g. \( \text{[Q-1, \S II.3]} \) for a proof. Theorem (2.2.3) becomes Quillen's Axiom M1 for a model category \( \text{[Q-1]} \); see \( \S 2.3 \).

Note that the usual homotopy extension property only holds for maps into Kan complexes.
§2.3. **MODEL CATEGORIES.**

We shall describe the basic properties of closed model categories and their associated homotopy categories; this theory is due to D. G. Quillen [Q-1, §§I.1 - I.5].

(2.3.1) **Definition.** An ordered pair of maps \((i,p)\) is said to have the **lifting property** if given any solid-arrow commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow{i} & & \downarrow{p} \\
X & \xleftarrow{\phantom{f}} & B
\end{array}
\]

there exists a filler \(f\).

(2.3.2) **Definition.** A **closed model category** consists of a category \(C\) together with three classes of maps in \(C\), called the **fibrations**, **cofibrations**, and **weak equivalences** satisfying the following axioms.

**MO.** \(C\) is closed under finite colimits and limits.

**M1.** If a map \(i\) is a cofibration, a map \(p\) is a fibration, and either \(i\) or \(p\) is a weak equivalence, then the pair \((i,p)\) has the lifting property.

**M2.** Any map \(f\) may be factored as \(f = pi\) where \(i\) is a cofibration and a weak equivalence and \(p\) is a fibration, or \(i\) is a cofibration and \(p\) is a fibration and a weak equivalence.

**M3.** Fibrations (resp. cofibrations) are stable under composition.
and base change (pullbacks) (resp., cobase change (pushouts)).

Any isomorphism is a fibration and a cofibration.

**M4.** The base extension (resp., cobase extension) of a map which is both a fibration (resp., cofibration) and a weak equivalence is a weak equivalence.

**M5.** Let

\[ \begin{CD}
    X @>f>> Y @>g>> Z
\end{CD} \]

be a diagram in \( C \). If any two of the maps \( f, g \), and \( gf \) are weak equivalences, then so is the third.

Any isomorphism is a weak equivalence.

**M6a.** A map \( p \) is a fibration if and only if for all maps \( i \) which are cofibrations and weak equivalences, the pair \((i,p)\) has the lifting property.

**M6b.** A map \( i \) is a cofibration if and only if for all maps \( p \) which are fibrations and weak equivalences, the pair \((i,p)\) has the lifting property.

**M6c.** A map \( f \) is a weak equivalence if and only if \( f = uv \), where for all cofibrations \( i \) and fibrations \( p \), the pairs \((i,u)\) and \((v,p)\) have the lifting property.

Observe that Axioms M5 and M6 imply Axioms M1, M3, and M4; hence to show that a given category is a closed model category it suffices to verify Axioms M0, M2, M5 and M6. We shall want the following technical definitions.
(2.3.3) **Definitions.** A map which is both a fibration (resp., cofibration) and a weak equivalence is called a **trivial fibration** (resp., **trivial cofibration**). The **initial** object of $C$ shall be denoted $\emptyset$; the **terminal** object $\ast$ (these objects exist by Axiom MO). An object $X$ in $C$ is called **fibrant** if the natural map $X \longrightarrow \ast$ is a fibration; **cofibrant** if the natural map $\emptyset \longrightarrow X$ is a cofibration.

(2.3.4) **Definition.** Let $X \in C$. A **cylinder object** for $X$ consists of an object $X \otimes [0,1]$ and a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X \otimes [0,1] \\
\downarrow & & \downarrow \pi_0 = \pi_1 = \text{id}_X \\
X & \longrightarrow & X
\end{array}
$$

where the map $(i_0, i_1)$ is a cofibration, the map $p$ is a weak equivalence, and $\pi_0 = \pi_1 = \text{id}_X$. We shall frequently write $X \otimes 0$ for $i_0(X)$ and $X \otimes 1$ for $i_1(X)$. **Caution:** in general, $X \otimes [0,1]$ is not the tensor product of $X$ with an object $[0,1]$; in fact, in general, $X \otimes [0,1]$ need not depend functorially upon $X$.

We may use cylinder objects to form mapping cylinders with the usual properties. For example, we have the following.

(2.3.5) **Proposition.** Let $f: X \longrightarrow Y$ be a cofibration. Then there exist suitable cylinder objects so that $f$ induces a trivial cofibration

$$
Y \times \emptyset \otimes [0,1] \longrightarrow Y \otimes [0,1],
$$

and a cofibration

$$
(2.3.6) \ Y \otimes [0,1] \otimes [0,1] \longrightarrow Y \otimes [0,1].
$$

If $f$ is a trivial cofibration, the induced map (2.3.6) is also a trivial cofibration.
Proof. Consider the commutative diagram

\[
\begin{align*}
X \otimes 0 & \cup X \otimes 1 \xrightarrow{i_0 + i_1} X \otimes [0,1] \\
Y \otimes 0 & \cup Y \otimes 1 \xrightarrow{j_0 + j_1} Y \otimes 0 \cup X \otimes [0,1] \cup Y \otimes 1 \\
& \quad \Downarrow \, k \\
& \quad \Downarrow \, g \\
\id_Y + \id_Y & \quad \Downarrow \, q \\
& \quad \Downarrow \, f \\
& \quad Y,
\end{align*}
\]

in which the subdiagram

\[
\begin{align*}
X \otimes 0 & \cup X \otimes L \xrightarrow{i_0 + i_1} X \otimes [0,1] \xrightarrow{p} X
\end{align*}
\]

is a cylinder object for \( X \), \( Y \otimes 0 \cup X \otimes [0,1] \cup Y \otimes 1 \) is the pushout of the upper left square, and \( g \) is the map induced by the maps \( fp \) and \( \id_Y + \id_Y \). The map \( j_0 + j_1 \) is the pushout (cobase extension) of the cofibration \( i_0 + i_1 \), hence is itself a cofibration. Factor \( g \) as \( qk \) where \( k \) is a cofibration and \( p \) is a
trivial cofibration (dotted arrows above). We obtain a suitable cylinder object for $Y$, namely

$$Y \otimes 0 \cup Y \otimes 1 \xrightarrow{k_0+k_1} Y \otimes [0,1] \xrightarrow{q} Y,$$

so that $f$ induces cofibrations

$$Y \otimes 0 \cup X \otimes [0,1] \xrightarrow{} Y \otimes [0,1],$$

$$Y \otimes 0 \cup X \otimes [0,1] \cup Y \otimes 1 \xrightarrow{} Y \otimes [0,1].$$

The remaining assertions are easily checked by applying Axiom M5; details are omitted. □

We shall discuss cocylinder objects (dual to cylinder objects) in $C$, and loop and suspension functors as well as the induced cofibration and fibration sequences in §3.4 below.

In these notes we shall always assume that our closed model categories $C$ satisfy the following niceness condition.

**Condition N:**

**N1.** Each cofibration is a pushout of a cofibration of cofibrant objects.

**N2.** Each fibration is a pullback of a fibration of fibrant objects.

**N3.** At least one of the following statements hold:

**N3a.** All objects are cofibrant.

**N3b.** All objects are fibrant.
N4. There exist functorial cylinder objects, denoted by \(-\mathcal{C}[0,1]\) with \(i_0(-) = \mathcal{C} \emptyset\) and \(i_1(-) = \mathcal{C} 1\).

The following closed model categories satisfy Condition N.

SS; (D. M. Kan [Kan -3], [Kan -4], see D. Quillen [Q, §II.3]). Since all objects are cofibrant, Condition N(1) is trivial. N(2) is due to J. C. Moore, see [Q, §II.3]. For N(3), let \(X \otimes [0,1] = X \times \Delta^1\), the usual product.

Top; the category of topological spaces with the following structure: cofibrations and fibrations are defined by the homotopy-extension and covering-homotopy properties, respectively; weak equivalences are ordinary homotopy equivalences. This is due to A. Strøm [Str].

CG; the category of compactly generated spaces, with a similar structure. See N. E. Steenrod [St-3]; also, [Has-3].

Sing; the category of topological spaces with the following singular structure: cofibrations are pushouts of inclusions of sub-complexes of CW complexes, fibrations are Serre fibrations, weak equivalences are weak homotopy equivalences [Q, §II.3].

SSG (SSAG); Simplicial groups (resp., simplicial abelian groups) [Q; §II.3]. Here \(X \otimes [0,1]\) denotes \(F(X \times [0,1]) / F(X \times 0) \sim X \times Q F(X \times 1) \sim X \times 1\), where the products are taken in SS and \(F\) is the free (resp., free abelian) simplicial group functor.
Sp; D. M. Kan's simplicial spectra [Kan-1]; see [Has-Z] for the model structure.

We shall now describe the homotopy theory of a closed model category $\mathcal{C}$ which satisfies condition N.

(2.3.7) **Definition.** The homotopy category of $\mathcal{C}$, $\text{Ho}(\mathcal{C})$, is the quotient category obtained from $\mathcal{C}$ by inverting all weak equivalences.

Quillen proved the following.

(2.3.8) **Proposition** [Q1, Prop. I.5.1]. A map $f$ in $\mathcal{C}$ becomes an equivalence in $\text{Ho}(\mathcal{C})$ if and only if $f$ is a weak equivalence in $\mathcal{C}$.

The following homotopy theory is required for the proof.

(2.3.9) **Definition.** If $X$ is cofibrant and $Y$ is fibrant, maps $f, g: X \rightarrow Y$ will be called homotopic (denoted $f \simeq g$) if there is a map $H: X \times [0,1] \rightarrow Y$ with $H|_{X \times 0} = f$ and $H|_{X \times 1} = g$.

Compare [Q1, §I.1].

(2.3.10) **Definition.** Let $\mathcal{C}_{cf}$ denote the full subcategory of cofibrant, fibrant objects in $\mathcal{C}$.

(2.3.11) **Proposition** [Q1, Lemma I.5.1 and its dual]. A map $f: X \rightarrow Y$ in $\mathcal{C}_{cf}$ is a weak equivalence if and only if there is a map
g: Y → X with gf = id_X and fg = 1_Y.

The proof is analogous to that of [Q, Lemma I.5.1], and is omitted. □

Proof of Proposition (2.3.8). Given a map f: X → Y in C which is invertible in Ho(C), form a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\uparrow & & \uparrow \\
X' & \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow \\
X'' & \xrightarrow{[f'']} & Y''
\end{array}
\]

where X' and Y' are cofibrant, X'' and Y'' are both cofibrant and fibrant, and all vertical maps are weak equivalence. Then use the axioms to realize the map [f''] in Ho(C) by a map f'' in C. Finally, Proposition (2.3.11) implies that f'' is a weak equivalence. See [Q, §II.5] for details.

§2.4. Simplicial Closed Model Categories.

We shall discuss function spaces in SS, and their generalization to the concept of simplicial closed model category given by Quillen [Q, §II.2.2].

The product in SS is given by \( X \times Y = \{(x \times y)_n = x_n \times y_n\} \), together with induced face and degeneracy maps. This product is coadjoint to the "function space" \( \text{HOM}(X,Y) = \{\text{HOM}(X,Y)_n = \)
= SS(X × Δ^n, Y)}, together with the face and degeneracy maps induced from the maps d^i:Δ^{n-1} → Δ^n and s^i:Δ^{n+1} → Δ^n, 0 ≤ i ≤ n. See, e.g., [May], or [Q-1]. In the context of the adjoint pair (x, HOM) we shall write ⊗ for x⊗.

The functors ⊗ and HOM satisfy the following properties.

(i) There is an associative composition

HOM (X,Y) × HOM (Y,Z) → HOM (X,Z), for all X, Y, and Z in C, and a natural isomorphism of functors SS(X,Y) ⊗ HOM (X,Y)_0,

u → u,

such that for u in SS(X,Y), f in HOM (Y,Z)_n, and g in HOM (W,X),

f ◦ (S_0)^n u = HOM (u,Z)_n (f), and

(S_0)^n u ◦ g = HOM (W,u)_n (g)

SS(X,Y) ⊗ HOM (X,Y)_0,

for all Y in C.

(ii) There are natural maps α:Y → HOM (X,X ⊗ Y)

which induce isomorphisms (enriched adjunction [E-K])
\[ \text{HOM} (X \otimes Y, Z) \rightarrow \xrightarrow{\sim} \text{HOM} (Y, \text{HOM} (X, Z)), \]

for all \( X \) and \( Z \) in \( C \).

(iii) For all \( Y \) in \( C \) there are natural maps
\[ \beta: Y \rightarrow \text{HOM} (\text{HOM} (Y, Z), Z) \]
which induce isomorphisms
\[ \text{HOM} (X, \text{HOM} (Y, Z)) \rightarrow \xrightarrow{\sim} \text{HOM} (Y, \text{HOM} (X, Z)), \]

for all \( X \) and \( Z \) in \( C \).

(iv) There are natural isomorphisms
\[ \text{HOM} (*, X) \rightarrow \xrightarrow{\sim} X, \]

for all \( X \) in \( C \).

Consequently,

(v) The composition maps in (i) and \( SS \) are compatible:
\[ \text{i.e., the following diagram commutes for all } \]
\[ X, Y, Z \text{ in } SS: \]
\[ \begin{array}{ccc}
\text{HOM} (X, Y)_0 \times \text{HOM} (Y, Z)_0 & \rightarrow & \text{HOM} (X, Z)_0 \\
\downarrow & & \downarrow \\
\text{SS}(X, Y) \times \text{SS}(Y, Z) & \rightarrow & \text{SS}(X, Z).
\end{array} \]

(vi) \( \otimes \) is coherently associative and commutative, with \( * \)
as coherent unit.
Function spaces and products in CG have similar properties.

For another example, define $\otimes$ on $SS_\ast$ by

$$X \times Y = X \wedge Y = X \times Y / X \vee Y,$$

and HOM as above except that $\Delta^n$ is replaced by $\Delta^n \ast$ obtained from $\Delta^n$ by adjoining a disjoint basepoint.

These ideas have been abstracted in S. Eilenberg and G. M. Kelly's **closed symmetric monoidal category** [E-K].

The singular closed model structure on Top, $Top_{\text{sing}}$ ([Q-1, §II.3], see §2.3) admits a different type of "function space." For $X$ and $Y$ in Top, define $\text{HOM}(X, Y)$ to be the simplicial set with

$$\text{HOM}(X, Y)_n = \text{Top}(X \times R\Delta^n, Y)$$

together with the induced face and degeneracy maps, see above. Here $R$ denotes Milnor's geometric realization functor [Mil-2], [May]. For $X$ in Top and $K$ in $SS$, define $X \otimes K = X \times RK$. Then

$$\text{HOM}(X \otimes K, Y) = \text{HOM}(K, \text{HOM}(X, Y))$$

for $K$ in $SS$, and $X, Y$ in Top. Quillen generalized this concept by introducing **closed simplicial model categories**, described below.

(2.4.1) **Definition** (see [Q-1, Definition II.1.1]) A **simplicial category** is a category $C$ together with the following structure:
(i) A functor $\text{HOM}(-,-)$ from $C \times C$ to $SS$, contravariant in the first variable and covariant in the second.

(ii) For an $X, Y$, and $Z$ in $C$, maps in $SS$

$$\text{HOM}(X,Y) \times \text{HOM}(Y,Z) \rightarrow \text{HOM}(X,Z)$$
called composition.

(iii) An isomorphism of functors

$$C(X,Y) \xrightarrow{\cong} \text{HOM}(X,Y)_0,$$

$$u \mapsto \tilde{u},$$

where $\text{HOM}(X,Y)_0$ consists of the 0-simplices of $\text{HOM}(X,Y)$.

These functors are required to satisfy the following conditions.

(1) Composition is associative.

(2) For $u$ in $C(X,Y)$, $f$ in $\text{HOM}(Y,Z)_n$, and $g$ in $\text{HOM}(W,X)_n$,

$$f \circ (S_0)^n \tilde{u} = \text{HOM}(u,Z)_n(f),$$

and

$$(S_0)^n \tilde{u} \circ g = \text{HOM}(W,u)_n(g).$$

(2.4.2) Definition (see [Q-1, Definition II.1.3]). For $X$ in $C$ and $K$ in $SS$, $X \otimes K$ shall denote an object of $C$ together with a distinguished map $\alpha: K \rightarrow \text{HOM}(X, X \otimes K)$ which
induces a natural isomorphism

\[ \text{HOM} (X \otimes K, Y) \longrightarrow \text{HOM} (K, \text{HOM} (X, Y)). \]

\( \text{HOM} (K, X) \) shall denote an object of \( C \) together with a distinguished map \( \varepsilon : K \longrightarrow \text{HOM} (\text{HOM} (K, X), X) \) which induces a natural isomorphism

\[ \text{HOM} (Y, \text{HOM} (K, X)) \longrightarrow \text{HOM} (K, \text{HOM} (Y, X)). \]

(2.4.3) **Examples.** Clearly \( S S \) with its usual symmetric monoidal structure, is a simplicial category (see [Q-1], [May], [E-K]). \( \text{Top}_{\text{sing}} \) (see §2.3) is also a simplicial category.

(2.4.4) **Definition** [Q-1, Definition II.2.2]. A **closed simplicial model category** consists of a closed model category \( C \) which is also a simplicial category satisfying the following two conditions.

**SMO.** For \( X \) in \( C \) and \( K \) a **finite** simplicial set, then \( X \otimes K \) and \( \text{HOM} (K, X) \) exist.

**SM2.** If \( i : A \longrightarrow X \) is a cofibration in \( C \) and \( p : Y \longrightarrow B \) is a fibration in \( C \), then

\[ \text{HOM} (X, Y) \longrightarrow \text{HOM} (A, Y) \times _{\text{HOM} (A, B)} \text{HOM} (X, B) \]

is a fibration in \( S S \) which is trivial if either \( i \) or \( p \) is trivial.

Recall that for spaces \( X \) and \( Y \), say in \( CG \), and \( \text{HOM} (X, Y) \)
the usual function space,

\[ [X,Y] = \pi_0(\text{HOM} (X,Y)). \]

Similarly, for \( X \) and \( Y \) in \( \text{SS} \) with \( Y \) fibrant (i.e., Kan),
\[ [X,Y] = \pi_0(\text{HOM} (X,Y)). \]  A similar statement holds in an abstract simplicial closed model category \( C \).

(2.4.4) **Proposition** [Q-1, Proposition II.2.5]. If \( X \) is cofibrant in \( C \) and \( Y \) is fibrant in \( C \), then

\[ \text{Ho}(C)(X,Y) = \pi_0(\text{HOM} (X,Y)), \]

the set of path components of \( \text{HOM} (X,Y) \). \( \square \)

(2.4.5) **Remarks.**

(a) Our use of \( \text{HOM} \) in three settings, on \( C \times C \), \( C \times S \), and \( S \times S \), should emphasize the analogy between the functor \( \text{HOM} \) of Definition and the usual "function space" functors.

(b) All of the above results have pointed analogues; replace \( \text{SS} \) by \( \text{SS}_* \) and \( \Delta^n \) by \( \Delta^n_* \), which is obtained from \( \Delta^n \) by adjoining a disjoint basepoint.

(c) In general the function space in \( \text{Top} \) or \( \text{CG} \), \( \text{HOM} (X,Y) \), is not homotopy equivalent to the
realization of the singular "function space"

\[ R(\text{Hom}_{\text{sing}}(X,Y)) \equiv R(\text{Top}(X \times R^n, Y), d_1, s_1). \]

For example, the latter space is always a CW complex.

§2.5. Homotopy theories of pro-spaces.

In this section we shall briefly indicate the need for a
"sophisticated" homotopy theory of pro-spaces. M. Artin and B. Mazur
took pro-Ho(Top) to be the homotopy theory of pro-Top. Unfortunately, this point of view is inadequate for some purposes. For example, Quillen [Q Ch. II, p. 0.3] observed that the category
pro-Ho(Top) was not the homotopy category of a model structure on
pro-Top. One next attempts to define homotopy globally in pro-Top,
that is, to call maps \( f, g : \{X_i\} \rightarrow \{Y_j\} \) homotopic if there is a
homotopy \( H : \{X_i\} \times [0,1] \equiv \{X_i \times [0,1]\} \rightarrow \{Y_j\} \)
from \( f \) to \( g \).

This notion is stronger than the Artin-Mazur notion which would
identify two level maps \( \{X_n \xrightarrow{f_n, g_n}_n \rightarrow Y_n\} \) if there were homotopies
\( H_n : f_n = g_n \) without any coherence criteria among the \( H_n \). For example, let \( D \) denote the inverse system

\[ s^1 \xleftarrow{2} s^1 \xleftarrow{} \ldots, \]

where \( s^1 \xleftarrow{2} s^1 \) is the degree two map \( Z \rightarrow Z^2 \). Then, there is
a unique Artin-Mazur homotopy class of maps from a point to \( D \), but
there are uncountably many global homotopy classes of maps from a point
to $D$ (more precisely, $\lim^1 \{z \leftarrow^2 z \leftarrow^2 \cdots \}$ such classes). The following example shows that the notion of global homotopy is also too naive.

$$\begin{array}{cccccc}
\circ & \downarrow & \circ & \downarrow & \circ & \\
\circ & \downarrow & \circ & \downarrow & \circ & \\
\circ & \downarrow & \circ & \downarrow & \circ & \\
\circ & \downarrow & \circ & \downarrow & \circ & \\
\circ & \downarrow & \circ & \downarrow & \circ & \\
\end{array}$$

$X \xrightarrow{p} Y$

Here $X = \{X_n = (S^1 \vee [0,\infty)) \times \{0,1\} \cup [n,\infty) \times [0,1]\}$, and

$Y = \{Y_n = S^1 \times \{0,1\} \cup [0,1]\}$. The map $p = \{p_n\}: X \rightarrow Y$ is levelly a homotopy equivalence, but there is no homotopy inverse to $p$ in pro-$\text{Top}$. If, however, the bonding maps of the towers $X$ and $Y$ are fibrations, then the notion of global homotopy turns out to be the "right" notion. The right homotopy category, $\text{Ho}(\text{pro-}\text{Top})$, is defined by formally inverting level homotopy
equivalences.

In Chapter 3 we define a natural model structure on \( \text{pro-Top} \) and in Chapter 5 we will compare \( \text{Ho(pro-Top)} \) with \( \text{pro-Ho(Top)} \). The remaining chapters further develop the algebraic topology of \( \text{pro-Top} \) and give geometric applications.
§3. THE MODEL STRUCTURE ON PRO-SPACES

§3.1. Introduction

In this chapter we shall associate to a closed model category $C$ which satisfies condition $N$ (§2.3) a natural closed model structure on $\text{pro}-C$. This chapter is organized as follows.

In §3.2, we discuss the homotopy theory of $C^J$, where $J$ is a cofinite strongly directed set ($a \leq b$ and $b \leq a \Rightarrow a = b$). We shall develop a closed model structure on $C^J$ (Theorem (3.2.2)) which is natural in the following sense (Theorem (3.2.4)). The constant diagram functor $C \rightarrow C^J$ preserves the model structure. The inverse limit functor $\lim: C^J \rightarrow C$ preserves fibrations and trivial fibrations.

In §3.3, we shall extend the closed model structure from the level categories $C^J$ to $\text{pro}-C$ (Theorem (3.3.3)) with the same naturality properties as our closed model structure on $C^J$ (Theorem (3.3.4)). Simplicial structures on $\text{pro}-C$ are discussed in §3.5.

§3.4 is concerned with suspension and loop functors, and cofibration and fibration sequences. D. Quillen [Q-1, §§1.2-3] developed a general theory of suspension and loop functors, and cofibration and
fibration sequences in the homotopy category of an abstract closed model category. We shall sketch this theory in the context of \( \text{Ho(pro-C)} \). We shall show that an inverse system of fibrations over \( C \) is equivalent in \( \text{Ho(pro-C)} \) to a short fibration sequence.

In §3.5 we consider the category \( \text{Maps(pro-C)} \) and a full subcategory \( (C, \text{pro-C}) \) whose objects are maps \( A \rightarrow X \) (in \( \text{pro-C} \)) with \( X \) stable in \( \text{pro-C} \).

We develop useful (see §§6-8) geometric models of \( \text{Ho(Top, tow-Top)} \) and \( \text{Ho(tow-Top)} \) in §3.6.

We shall compare our closed model structure to those of A. K. Bousfield and D. M. Kan [B-K-1] and J. Grossman [Gros-1] in Remarks (3.2.5).

§3.2. The homotopy theory of \( C^J \).

Let \( C \) be a closed model category which satisfies Condition N (§2.3). Let \( J(=\{j\}) \) be a cofinite strongly directed set. We shall show that \( C^J \) inherits a natural closed model structure from \( C \); this will yield the required homotopy category \( \text{Ho}(C^J) \) (see §2.3).

\[ \text{(3.2.1) Definitions. A map } f:X \rightarrow Y \quad (=\{f_j:X_j \rightarrow Y_j\}) \]

in \( C^J \) is a cofibration (resp., weak equivalence) if for all \( j \) in \( J \),
the maps $f_j$ are cofibrations (resp., weak equivalences).

A map $f$ in $C^J$ is a fibration if it has the right-lifting-property with respect to all maps $i$ which are both cofibrations and weak equivalences.

In this section we shall prove the following.
(3.2.2) **Theorem.** $C^J$, together with the above structure, is a closed model category.

A map $f:X \to Y$ in $C^J$ is a fibration if for each $j$ in $J$, the induced map $q_j$ in the diagram

\[
\begin{array}{cccccc}
X_j & \longrightarrow & Y_j & \longrightarrow & \lim_k < j X_k \\
\downarrow_{q_j} & & \downarrow & & \downarrow \\
\lim_k < j Y_k & \longrightarrow & \lim_k < j X_k & \longrightarrow & \lim_k < j Y_k \\
\end{array}
\]

($P_j$ is the pullback) is a fibration.

This structure is natural in the following sense.

(3.2.4) **Theorem.** The constant diagram functor $C \to C^J$ preserves cofibrations, fibrations, and weak equivalences. The inverse limit functor $\lim: C^J \to C$ preserves fibrations, and trivial fibrations (maps which are both fibrations and weak equivalences).
Proof. Immediate from Definitions (3.2.1). □

(3.2.5) Remarks. Bousfield and Kan [B-K, p. 314] defined a different closed model structure on $C^J$ by defining fibrations and weak equivalences degree-wise, and defining cofibrations by the appropriate lifting property. The Bousfield-Kan structure has the disadvantage that most of Theorem (4.4) is false: only fibrations and weak equivalences are preserved by the constant diagram functor; none of the model structure is preserved by the inverse limit functor. Consequently, our homotopy inverse limit functor (§4.2) is simpler than theirs; this simplicity makes evident the applications to homological algebra in §§4.5-4.6; see also below.

J. Grossman [Gros-1] also introduced a closed model structure on the category of towers of simplicial sets. His structure is weaker than ours; essentially, he inverts $\eta$-isomorphisms in the sense of [A-M].

Our definition of fibration was motivated by the definition of a cofibration of pairs (for the inclusion $(X,A) \rightarrow (Y,B)$ to be a cofibration one usually asks that the induced map $X \cup_A B \rightarrow Y$ be a cofibration), and the analogous definition of a cofibration of CW spectra (see [Vogt-2]). Also:

(a) Our definition of fibration is consistent with the definition of a flasque pro-group (see §4.);

(b) The associated definition of cofibration means that a proper cofibration $X \rightarrow Y$ induces a cofibration of the ends $\epsilon(X) \rightarrow \epsilon(Y)$ (see §6).
The proof of Theorem (3.2.2) is contained in Propositions (3.2.6), (3.2.7), (3.2.24), (3.2.27) and (3.2.28), below.

(3.2.6) **Proposition.** (Verification of Axiom MO). $C^J$ admits finite limits and colimits.

**Proof.** Let $D$ be a finite diagram in $C^J$. The induced diagrams $D_j$ over $C$ have colimits $\text{colim } D_j$ and limits $\text{lim } D_j$ in $C$ by Axiom MO for $C$. These yield objects \{colim $D_j$\} and \{lim $D_j$\} in $C^J$ which are easily seen to be the colimit and limit of $D$, respectively. $\square$

In fact, if $C$ admits more general colimits or limits, so does $C^J$.

In order to verify Axioms M2, M5 and M6 for $C^J$ we shall give explicit descriptions of fibrations (Proposition (3.2.7)) and trivial fibrations (Proposition (3.2.17)) in $C^J$. Our descriptions will involve diagram (3.2.3).

(3.2.7) **Proposition.** A map $p:Y \rightarrow B$ in $C^J$ is a fibration if and only if for each $j \in J$ the induced map $q_j$ in the diagram
(P_\text{\text{j}} \text{ is the pullback}) \text{ is a fibration in } \mathcal{C}.

\textbf{Proof.} First, let \text{p} be a fibration in C^J, that is, assume that \text{p} has the right-lifting-property with respect to the class of trivial cofibrations in C^J. We shall show that each induced map q_\text{j}:Y_\text{j} \to P_\text{j} has the same right-lifting-property by constructing suitable "test maps" K \to L in C^J which are trivial cofibrations.

Consider a solid-arrow commutative diagram

\begin{equation}
\begin{array}{ccc}
A & \longrightarrow & Y_\text{j} \\
\downarrow i & & \downarrow q_\text{j} \\
X & \longrightarrow & P_\text{j}
\end{array}
\end{equation}

in \mathcal{C} in which \text{i} is a trivial cofibration. Define objects
K = K_k and L = L_k in C^j as follows:

\[
K_k = \begin{cases} 
X & \text{for } k < j, \\
A & \text{for } k = j, \\
\phi & \text{otherwise;}
\end{cases}
\]

\[
L_k = \begin{cases} 
X & \text{for } k \leq j, \\
\phi & \text{otherwise}.
\end{cases}
\]

The required bonding maps are induced by \(i\) and \(id_X\). Then there is an induced trivial cofibration \(i'^*: K \to L\) in \(C^j\). Diagram (3.2.8) induces a solid-arrow commutative diagram

\[
\begin{array}{ccc}
K & \to & Y \\
\downarrow & & \downarrow \phi \\
L & \to & B
\end{array}
\]

in \(C^j\) (the maps \(K_k \to Y_k\) are induced from the map \(A \to Y_j\) for \(j = k\) and the composite maps \(X \to P_j \to Y_k\) for \(k < j\) (see diagram (3.2.8)), the other maps in diagram (3.2.9) are defined similarly).

The right-lifting-property of \(p\) yields a filler \(g\) in diagram (3.2.9). Because diagram (3.2.8) is the \(j\)th level of diagram (3.2.9), the map \(g_j: L_j = X \to Y_j\) is the required filler in diagram
(3.2.8). Hence $q_j$ is a fibration, as required.

Conversely, let $p: Y \to B$ be a map in $C^J$ with the property that the induced maps $q_j: Y_j \to F_j$ (see diagram (3.2.3)) are fibrations. Consider a solid-arrow commutative diagram in $C^J$

$$
\begin{array}{ccc}
A & \xrightarrow{H} & Y \\
\downarrow i & & \downarrow p \\
X & \xleftarrow{h} & B \\
\end{array}
$$

(3.2.10)

in which the map $i$ is a trivial cofibration. We shall obtain the required filler

$$f = \{f_j: X_j \to Y_j\}$$

by induction on $j$.

Consider a fixed index $j$. Suppose that for all $k < j$, there exist maps $f_k: X_k \to Y_k$ with the following properties:

$$
\begin{align*}
(f_k i_k &= H_k \\
p_k f_k &= h_k
\end{align*}
$$

(3.2.11)

(3.2.12) $f_k \circ \operatorname{bond} = \operatorname{bond} \circ f_k$ for $k < k$ (if $j$ has no predecessors these restrictions are vacuous). Formula (3.2.12) yields
a composite map

\[(3.2.13) \quad X_j \longrightarrow \lim_k < j \quad Y_k \longrightarrow \lim_k < j B_j ;\]

by formulas (3.11) this map is equal to the composite map

\[(3.2.14) \quad X_j \longrightarrow B_j \longrightarrow \lim_k < j B_k .\]

In fact, formulas (3.2.10) - (3.2.14) yield a solid-arrow commutative diagram

\[(3.2.15) \quad A_j \xrightarrow{H_j} Y_j \]
\[\downarrow i_j \quad \downarrow f_j \quad \downarrow q_j \]
\[X_j \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad P_j \]

in C. (This uses the definition of \( P_j \) in diagram (3.2.3).) Because \( i_j \) is a trivial cofibration and \( g_j \) is a fibration in C, there exists a filler \( f_j \) in diagram (3.2.15). Further, diagram (3.2.15) yields formulas (3.2.11) and (3.2.12) for \( f_j \).

By continuing inductively, we obtain the required filler \( f = \{ f_j \} \) in diagram (3.2.10). \( \Box \)

\[(3.2.16) \quad \text{Proposition. Let } p: X \longrightarrow B \text{ be a fibration in } C^J. \text{ Then at each level, the map } p_j: Y_j \longrightarrow B_j \text{ is a fibration in } C. \]
Proof. For a given \( j \) in \( J \), consider the commutative diagram

\[
\begin{array}{c}
\text{\( Y_j \)} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{\( q_j \)} & \quad \rightarrow & \quad \text{\( p_j \)} & \quad \rightarrow & \quad \text{\( \lim_k < j Y_k \)} \\
\downarrow & & \downarrow & & \\
\text{\( p'_j \)} & \rightarrow & \text{\( b_j \)} & \rightarrow & \text{\( \lim_k < j B_k \)} \\
\end{array}
\tag{3.2.3}
\]

By introducing the indexing category

\[ K = \{ k | k < j \} \subset J, \]

we see that the map \( \lim_k < j P_k \) is a fibration in \( C \) (apply Theorem (3.2.4)). Hence \( p'_j \), and thus the composite map \( p_j = p'_j q_j \) are fibrations in \( C \). \( \square \)

(3.2.17) Remarks. The above proof illustrates the usefulness of strongly directed cofinite indexing sets: maps in \( C^J \) may be constructed inductively.

Trivial fibrations in \( C^J \) admit a similar characterization.
(3.2.18) **Proposition.** A map \( p:Y \to B \) in \( C^J \) is a trivial fibration if and only if the induced maps \( q_j:Y_j \to P_j \) (see Proposition (3.2.6)) are trivial fibrations in \( C \).

**Proof.** First, let \( p:Y \to B \) be a trivial fibration. If \( j \) is an initial element in \( J \), the maps \( q_j:Y_j \to P_j \) and \( p_j:Y_j \to B_j \) are equal. But \( p_j \) is a fibration by Proposition (3.2.7) and a weak equivalence by hypothesis. Hence \( q_j(=p_j) \) is a trivial fibration, as required.

Now suppose that for a fixed (non-initial) \( j \) in \( J \), and for all \( k < j \), the induced maps \( q_k \) are trivial fibrations. We shall show that \( q_j \) is a trivial fibration.

Consider the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{q_j} & P_j \\
\downarrow{f_j} & & \downarrow{\lim_{k < j} Y_k} \\
B_k & \rightarrow & \lim_{k < j} B_k
\end{array}
\]

in which \( P_j \) is a pullback. If \( \lim_{k < j} P_k \) were a trivial fibration, the induced map \( P_j \to B_k \) would also be a trivial fibration (by Axiom M4 for \( C \)) hence a weak equivalence. But then \( q_j \) would
be a weak equivalence (since $f_j$ is a weak equivalence by hypothesis, this follows from Axiom M5 for C). But $f_j$ is a fibration by Proposition (3.2.7), so that $f_j$ would be a trivial fibration, as required. It therefore suffices to show that the maps

\[(3.2.19) \quad \lim_{k < j} p_k : \lim_{k < j} Y_k \to \lim_{k < j} B_k\]

are trivial fibrations.

To do this, we consider a commutative solid-arrow diagram of the form

\[
\begin{array}{ccc}
K & \longrightarrow & \lim_{k < j} Y_k \\
\downarrow i & & \downarrow \lim_{k < j} p_k \\
L & \longrightarrow & \lim_{k < j} B_k
\end{array}
\]

in C in which $i$ is a cofibration. We may define a filler $f$ in diagram (3.2.19) by defining maps

\[f_k : L \longrightarrow Y_k\]

for $k < j$ which make the diagrams
(3.2.21) i

and

(3.2.22)

commute. This requires a second induction; this time on $k$. Suppose that for a fixed $k$, and all $\ell < k$ there exist the required fillers $f_{\ell}$ (if $k$ has no predecessors, this condition is vacuous).

We obtain a map

$$K \longrightarrow Y_k$$

(see diagram (3.2.3)) for which the solid-arrow diagram
commutes. But the map $q_k$ is a trivial fibration by our inductive assumption. Hence there exists a filler $f_k$ for the upper left corner of diagram (3.2.23). Diagram (3.2.23) immediately implies that the required diagrams (3.2.21) and (3.2.22) commute. Continuing inductively yields the required maps $f_k$ and filler

$$f = \lim_k \downarrow f_k$$

in diagram (3.2.20). Hence, the maps (3.2.19) are trivial fibrations, as required.

The proof of the converse is similar to the proof of the "if" part of Proposition (3.2.6) and is omitted.

We may now verify that $C$ satisfies Axioms M2, M5, and M6 for a
closed model category.

(3.2.24) Proposition. (Verification of Axiom M2) Any map 
\[ f: X \to Y \] 
in \( \mathcal{C}^J \) may be factored as 
\[ X \xrightarrow{i} Z \xrightarrow{p} Y \]

where \( i \) is a cofibration, \( p \) is a fibration, and either \( i \) or \( p \) is a weak equivalence.

Proof. We shall factor \( f \) as \( pi \) with \( i \) a weak equivalence. The proof of the other case is similar and omitted. To factor \( f \), we shall factor the maps \( f_j: X_j \to Y_j \) as 
\[ X_j \xrightarrow{i_j} Z_j \xrightarrow{p_j} Y_j \]

where:

(a) for \( k < j \), \( i_j \) and \( p_j \) cover \( i_k \) and \( p_k \), respectively;

(b) the maps \( i_j \) are trivial cofibrations; and

(c) the induced maps \( q_j: Z_j \to P_j \) associated with

the maps \( p_j: Z_j \to Y_j \) (see diagram (3.2.3))

are fibrations.

Suppose for a given \( j \) and for all \( k < j \), the maps \( f_k \)
have been so factored. If \( j \) has no predecessors, this condition is vacuous. We may then form the commutative diagram

\[
\begin{array}{ccc}
X_j & \longrightarrow & \lim_{k < j} X_k \\
\downarrow & & \downarrow \\
Q_j & \longrightarrow & \lim_{k < j} Z_k \\
\downarrow & & \downarrow \\
Y_j & \longrightarrow & \lim_{k < j} Y_k \\
\end{array}
\]

(3.2.26)

where \( Q_j \) is a pullback, and the map \( g_j \) is induced from the map \( \lim_{k < j} P_k \). As in the proof of Proposition (3.2.17), see diagram (3.2.20), the map \( \lim_{k < j} P_k \) is a fibration. Hence the induced map \( g_j \) (see diagram (3.2.26)) is a fibration (by Axiom M3 for \( C \)).

Now, factor the map \( X_j \rightarrow Q_j \) (in diagram (3.2.26)) as the composite

\[
\begin{array}{ccc}
X_j & \xrightarrow{i_j} & Z_j & \xrightarrow{q_j'} & Q_j \\
\end{array}
\]

(using Axiom M2 for \( C \)). Finally, let

\[
p_j = g_j q_j' : Z_j \longrightarrow Q_j \longrightarrow Y_j .
\]
Then the factorization

\[ X_j \xrightarrow{i_j} Z_j \xrightarrow{p_j} Y_j \]

(see diagram (3.2.25)) has the required properties. The conclusion follows. □

(3.2.27) **Proposition.** (Verification of Axiom M5). If two of the maps in the diagram

\[ X \xrightarrow{f} Y \xrightarrow{g} Z \]

are weak equivalence, then so is the third.

**Proof.** This follows from Axiom M5 for \( C \) since weak equivalences in \( C^J \) are defined degreewise. □

(3.2.28) **Proposition.** (Verification of Axiom M6).

(a) A map is a fibration if and only if it has the right-lifting-property with respect to the class of trivial cofibrations.

(b) A map is a cofibration if and only if it has the left-lifting-property with respect to the class of trivial
fibrations.

(c) A map \( f \) is a weak equivalence if and only if 
\[ f = uv \]  
where \( v \) has the left-lifting-property 
with respect to the class of fibrations and \( u \) 
has the right-lifting-property with respect to 
the class of cofibrations.

**Proof.** Part (a) is contained in Definitions (3.2.1).

The proof of Part (b) is similar to that of Proposition (3.2.7).
The main step is to associate to an element \( j \) of \( J \) and a fibration 
\[ Y \to B \text{ in } C, \]  
the objects \( E = \{E_k\} \) and \( A = \{A_k\} \) in \( C^J \) defined 
by setting

\[
E_k = \begin{cases} 
Y & \text{for } k \geq j, \\
* & \text{otherwise}, 
\end{cases}
\]

\[
A_k = \begin{cases} 
B & \text{for } k \geq j, \\
* & \text{otherwise}, 
\end{cases}
\]

with the evident bonding maps and also the fibration \( E \to B \text{ in } C^J \).

One may then proceed as in the discussion following diagram (3.2.8).

Remaining details are omitted.

For Part (c), first let \( f \) be a weak equivalence. Factor \( f \) 
as \( uv \) where \( u \) is a trivial fibration and \( v \) is a fibration. By
Axiom M5 for $C^J$ (Proposition (3.2.27)) and Propositions (3.2.7) and (3.2.17), $v$ is a trivial cofibration. The required lifting properties follow easily.

Conversely, arguments similar to the proof of Proposition (3.2.18) show that maps $u$ and $v$ with the given lifting properties are weak equivalences. Hence $f = uv$ is a weak equivalence by Axiom M5 for $C^J$. Details are omitted. □

This completes the proof of Theorem (3.2.2).

§3.3. The homotopy theory of $\text{pro-}C$.

Let $C$ be a closed model category which satisfies Condition N (§2.3). We shall show that $\text{pro-}C$ inherits a natural closed model structure from the closed model categories $C^J$ ($J$ is a cofinite strongly directed set); this will yield the required homotopy category $\text{Ho}(\text{pro-}C)$ (see §2.3). One of our main tools is the Mardešić trick (Theorem (2.1.6) which states that any inverse system is isomorphic to an inverse system indexed by a cofinite strongly directed set).

(3.3.1) Definitions. A map $f$ in $\text{pro-}C$ is called a strongly cofibration if $f$ is the image in $\text{pro-}C$ of a (level) cofibration $\{f_j\}$ in some $C^J$, where $J$ is a cofinite strongly directed set. Strong fibrations, strongly trivial cofibrations, and strongly trivial fibrations are defined similarly.
A map in pro-\(C\) is a cofibration if it is the retract in Maps (pro-\(C\)) of a strong cofibration. Fibrations, trivial cofibrations, and trivial fibrations in pro-\(C\) are defined similarly. A map \(f\) in pro-\(C\) is a weak equivalence if \(f = p_1\) where \(p\) is a trivial fibration and \(i\) is a trivial cofibration.

(3.3.2) Remarks. Clearly, trivial cofibrations (resp., trivial fibrations) in pro-SS are both cofibrations (resp., fibrations) and equivalences. Corollary (3.3.13), below, shows that the converse assertions hold.

We shall also need to show that the class of weak equivalences is closed under retracts. Then Definitions (3.3.1) yield an apparently larger class of weak equivalences than the class of retracts of the (level) weak equivalences in the categories \(C^J\) (Definitions (3.2.1)). Definitions (3.3.1) are essentially forced by the requirement that the composition of two weak equivalences yields a weak equivalence (Axiom M5). We do not know if every weak equivalence in pro-\(C\) is a retract of a (level) weak equivalence in some \(C^J\).

In this section we shall prove the following.

(3.3.3) Theorem. pro-\(C\), together with the above structure, is a closed model category.

(3.3.4) Theorem. The constant diagram functor \(C \to \text{pro-}C\) preserves cofibrations, fibrations, and weak equivalences. The inverse limit functor \(\lim:C^J \to C\) preserves fibrations and trivial fibrations.
Proof. Immediate from the Definitions (3.1.1) and Theorem (3.3.3).

The proof involves the following main steps:

Verification of Axiom M0 (Proposition (3.3.5));
Verification of Axiom M2 (Proposition (3.3.8));
Verification of Axiom M6 (Propositions (3.3.9), (3.3.15), and (3.3.17));
Verification of Axiom M5:
Special cases (Propositions (3.3.18), and (3.3.26));
General case (Proposition (3.3.35)).

(3.3.5) Proposition (Verification of Axiom M0). Pro-C admits finite colimits and limits.

Proof. Let \( \Delta \) be a finite diagram in Pro-C. We shall show that \( \Delta \) has a colimit; the construction of a limit for \( \Delta \) is similar and omitted. By inserting identity maps if necessary, we may assume that \( \Delta \) has no loops (the colimits of the original and new diagrams will be isomorphic). Applying the Artin-Mazur reindexing (see Proposition (2.1.5)) to \( \Delta \) yields an inverse system \( \{ \Delta_i \} \) of diagrams of C which determines a diagram in Pro-C isomorphic to \( \Delta \). Applying the Mardešić construction (Theorem (2.1.6)) to \( \{ \Delta_i \} \) yields a diagram \( \Delta' \) over some level category \( C^J \) indexed by a cofinite directed set \( J \); further \( \Delta' \) and \( \Delta \) are isomorphic diagrams over Pro-C. Now let \( X' = \text{colim} \Delta' \) in \( C^J \) (\( X' \) is defined levelwise by Proposition (3.2.6)).
We shall now check that \( X' = \text{colim} \Delta \) in \( C^J \). Clearly there exists a coherent family of maps from the objects of \( \Delta \) (\( \cong \Delta' \)) to \( X' \). To check the universality of \( X' \), suppose that there exists a coherent family of maps from \( \Delta \) to an object \( Y \) in \( \text{Pro-} C \). Assume, without loss of generality, that the indexing set of \( Y \) has a least element \( 0 \), and that \( Y_0 = \ast \). Let \( \tilde{\Delta} \) be the diagram consisting of \( \Delta, Y \), and the maps from \( \Delta \) to \( Y \) in \( \text{Pro-} C \). As above, we may define a diagram \( \tilde{\Delta}'' \) in some level category \( C^K \) indexed by a cofinite directed set \( K \) with \( \tilde{\Delta}'' \cong \tilde{\Delta} \) in \( \text{Pro-} C \). Because the indexing set of \( Y \) has a least element there exists a cofinal functor \( T : F \to J \) (this is easy but tedious). Now let \( \Delta'' \) and \( Y'' \) be the appropriate restrictions of \( \tilde{\Delta}'' \). Let \( X'' \) be the colimit of \( \Delta'' \) in \( C^K \). Then the maps from \( \Delta'' \) to \( Y'' \) factor uniquely through \( X'' \). Because \( \Delta'' \cong T^x \Delta' \cong \Delta \) and \( X'' \cong T^x X' \cong X' \) in \( \text{Pro-} C \), \( X' \) has the required universal property.

Artin and Mazur give a non-constructive proof of the following more general result.

(3.3.6) Proposition.\footnote{\textit{A-M}, Propositions A.4.3 and A.4.4]. Let \( U \) be a universe such that \( SS \) is \( U \)-small. Then \( \text{Pro-} SS \) admits \( U \)-small colimits and limits.}

(3.3.7) Remarks. Let \( \Delta \) be an infinite diagram over some level category \( C^J \), let \( X \) be its colimit in \( C^J \) (\( X \) is defined levelwise as in Proposition (3.2.6)). It is easy to see that in general \( X \) is not the colimit of \( \Delta \) in \( \text{Pro-} C \).
The statement of the following proposition is a technical reformulation of Axiom M2; see Remarks (3.3.2).

(3.3.8) **Proposition.** (Verification of Axiom M2). Any map \( f \) in \( \mathbf{Pro} - \mathbf{C} \) may be factored as \( f = p \iota \) where \( \iota \) is a trivial cofibration and \( p \) is a fibration, or \( \iota \) is a cofibration and \( p \) is a trivial fibration.

**Proof.** By Propositions (2.1.5) and (2.1.6) we may factor \( f \) as the composite

\[
\begin{array}{ccc}
X & \xrightarrow{\approx} & X' \\
\downarrow{=} & & \downarrow{=} \\
Y' & \xrightarrow{f'} & Y \\
\end{array}
\]

where \( f' \) is a level map indexed by a cofinite directed set \( J \). Consider \( f' \) as a map in \( \mathbf{C}^J \). By Axiom M2 for \( \mathbf{C}^J \) (Proposition (3.2.24)), we may factor \( f \) as \( p'\iota' \) in \( \mathbf{C}^J \) where \( p' \) and \( i' \) have the required properties. To complete the proof, let \( \iota \) and \( p \) be the respective composite mappings

\[
\begin{array}{ccc}
X & \xrightarrow{\approx} & X' \\
\downarrow{=} & & \downarrow{=} \\
Z' & \xrightarrow{p'} & Y' \\
\end{array}
\]

We shall now begin the verification of Axiom M6 for \( \mathbf{Pro} - \mathbf{C} \). The following proposition is a special case of Axiom M1 for \( \mathbf{Pro} - \mathbf{C} \).

(3.3.9) **Proposition.** Given any commutative solid-arrow diagram

(3.3.10)

\[
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow{i} & & \downarrow{=} \\
X & \xrightarrow{=} & B \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
\downarrow{=} & & \\
X & \xrightarrow{=} & B \\
\end{array}
\]
in which either \( i \) is a trivial cofibration and \( p \) is a fibration, or \( i \) is a cofibration, there exists a filler \( h \).

**Proof.** We shall only discuss the case in which \( i \) is a trivial cofibration. The proof of the other case is similar and omitted.

Since the lifting property described in Diagram (3.3.10) is preserved under the formation of retracts, we may assume that \( i \) is a strongly strongly trivial cofibration indexed by a cofinite directed set \( J \) and that \( p \) is a strongly strongly fibration indexed by a cofinite directed set \( K \).

We shall now replace diagram (3.3.10) by a suitable level diagram. We cannot merely apply the reindexing techniques of §2.1 since it appears unlikely that a cofinal functor \( T: L \to K \) maps fibrations in \( C^K \) into fibrations in \( C^L \) (fibrations are not defined levelwise).

Since \( J \) is cofinite, we may inductively define an order preserving function \( J \to K \) \((j \mapsto k(j)) \) and commutative solid-arrow diagrams

\[
\begin{array}{ccc}
A_k(j) & \xrightarrow{i_k(j)} & Y_j \\
\downarrow{h_j} & & \downarrow{p_j} \\
X_k(j) & \xrightarrow{i_k(j)} & B_j \\
\end{array} \tag{3.3.11}
\]

which represent Diagram (3.3.10) (that is, there are maps of Diagram
(3.3.10) to Diagrams (3.3.11) over \( \text{Pro-} \mathcal{C} \) such that if \( j' > j \) there is a commutative diagram of diagrams

\[
\begin{array}{ccc}
(3.3.10) & \rightarrow & (3.3.11), \text{ level } j' \\
& \downarrow & \\
& (3.3.11), \text{ level } j.
\end{array}
\]

We thus obtain a commutative solid-arrow diagram

\[
\begin{array}{ccc}
\{A_k\} & \rightarrow & \{A_k(j)\} \rightarrow \{\phi_j\} \rightarrow \{y_j\} \\
\downarrow & & \downarrow \\
\{i_k\} & \rightarrow & \{i_k(j)\} \\
\downarrow & & \downarrow \\
\{x_k\} & \rightarrow & \{x_k(j)\} \\
\downarrow & & \downarrow \\
\{b_j\} & & \{p_j\}
\end{array}
\]

with the following properties: the composites along the top and bottom rows are \( f \) and \( g \) respectively, and the right-hand square is a level diagram indexed by \( J \) (that is, a diagram in \( \mathcal{C}^J \)). Note: in general the function \( J \rightarrow K \) need not be cofinal, hence \( \{A_k\} \triangleright \{A_k(j)\} \), etc. Further, since trivial cofibrations in \( \mathcal{C}^K \) and \( \mathcal{C}^J \) are defined levelwise, \( \{i_k(j)\} \) is a trivial cofibration in \( \mathcal{C}^J \).
Since, by hypothesis, \( \{p_j\} \) is a fibration in \( C^J \), by Axiom M6 for \( C^J \) (Proposition (3.2.28)), there exists a filler \( \{h'_j\} \) in Diagram (3.3.12). It is clear that the composite mapping

\[
\begin{array}{ccc}
\{X_k\} & \longrightarrow & \{X_{k(j)}\} \\
\downarrow & & \downarrow \\
\{Y_j\} & \longrightarrow & \{Y_j\}
\end{array}
\]

is the required filler in Diagram (3.3.10).

(3.3.13) **Corollary.** A map is a trivial cofibration if and only if it is both a cofibration and a weak equivalence. A similar description holds for trivial fibrations.

**Proof.** The "only if" assertions hold by definition. Conversely, let \( f: X \rightarrow Y \) be both a cofibration and a weak equivalence. Using Definitions (3.3.1), write \( f = \pi i \) where \( i \) is a trivial cofibration and \( p \) is a trivial fibration. We shall see that \( f \) is a retract of \( i \), and hence a trivial cofibration.

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Z \\
\downarrow^f & & \downarrow^p \\
Y & \xrightarrow{\text{id.}} & Y
\end{array}
\]

(3.3.14)

Proposition (3.3.9) yields a filler \( g \) in Diagram (3.3.14). Rewriting Diagram (3.3.14) as shown
where \( pg = id_y \), shows that \( f \) is a retract of \( i \).

Verification of the assertion about fibrations is similar and omitted. \( \square \)

This answers one of the questions raised in Remarks (3.3.2).

(3.3.15) Proposition. (Verification of Axioms M6a and M6b).

(a) A map \( p \) is a fibration if and only if for all maps \( i \) which are cofibrations and equivalences, the pair \( (i, p) \) has the lifting property.

(b) A map \( i \) is a cofibration if and only if for all maps \( p \) which are fibrations and equivalences, the pair \( (i, p) \) has the lifting property.

Proof. (a) The "only if" part follows from Corollary (3.3.13) (which shows that \( i \) is a trivial cofibration) and Proposition (3.3.9). Conversely, let \( p \) be a map with the lifting property of hypothesis (a). Use Axiom M2 (Proposition 3.3.8) to write \( f = uv \), where \( u \) is a fibration and \( v \) is a trivial cofibration. As in the proof of Corollary (3.3.13), it follows that \( p \) is a retract of \( u \), and hence a fibration.

(b) The proof is similar to the proof of (a) and is
omitted. □

Similar arguments yield the following.

(3.3.16) Proposition. A map \( i \) is a trivial cofibration (cofibration and equivalence) if and only if for all fibrations \( p \), the pair \( (i,p) \) has the lifting property.

A map \( p \) is a trivial fibration (fibration and equivalence) if for all cofibrations \( i \), the pair \( (i,p) \) has the lifting property. □

(3.3.17) Proposition. (Verification of Axiom M5c). A map \( f \) is a weak equivalence if and only if \( f = uv \) where for all cofibrations \( i \) and fibrations \( p \), the pairs \( (i,u) \) and \( (v,p) \) have the lifting property.

Proof. By Proposition (3.3.16), the above characterization of weak equivalences is equivalent to that of Definitions (3.3.1). □

We have completed the verification of Axiom M5 for \( {\text{pro-}}C \), and shall now begin the verification of Axiom M5: weak equivalence is a congruence. This relatively lengthy process consists of first using the lifting properties developed above to verify Axiom M5 under the further assumption that all maps are cofibrations or all maps are fibrations. Secondly, we use the factorizations given by Axiom M2 to verify the general case of Axiom M5 for \( {\text{pro-}}C \).

(3.3.18) Proposition. Suppose that the maps \( f:X \to Y \) and \( g:Y \to Z \) are cofibrations. If any two of the maps \( f, g, \) and \( gf \)
are weak equivalences, then so is the third map.

Proof. There are three cases. For Cases I we use Corollary (3.3.13) and Proposition (3.3.9) to characterize maps which are both cofibrations and weak equivalences by their lifting properties.

**Case I:** Let \( f \) and \( g \) be weak equivalences. Then for all fibrations \( p \), the pairs \( (f,p) \) and \( (g,p) \) have the lifting property. Consequently, their composite has the same lifting property; thus \( gf \) is a weak equivalence, as required.

**Case II:** Let \( f \) and \( gf \) be weak equivalences. We shall show that \( g \) is a weak equivalence by verifying that for all fibrations \( p \), the pair \( (g,p) \) has the lifting property. Consider a commutative solid-arrow diagram of the form

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{f} & \searrow{hf} & \downarrow{h} \\
Y & \xleftarrow{K} & E \\
\downarrow{g} & \nearrow{k} & \downarrow{p} \\
Z & \xrightarrow{K} & B \\
\end{array}
\]

(3.3.19)

in which \( p \) is a fibration. We shall show that \( g \) is a trivial cofibration, as required, by constructing a filler \( K \), above.
Because the composite map \( gf \) is a trivial cofibration, there is a map \( K':Z \to E \) such that \( K'gf = hf \) and \( pK' = k \). Caution: in general \( K'g \neq h \). We shall deform \( K' \) into the required filler \( K:Z \to E \).

Because \( f \) is a retract of a trivial cofibration \( f':X' \to Y' \) \( \text{strongly} \) in some level category \( C^J \), where \( J \) is a cofinite directed set, Proposition (2.3.5) implies that \( f' \) induces a trivial cofibration

\[
i':Y' \times 0 \cup X' \times [0,1] \cup Y' \times 1 \to Y' \times [0,1]
\]

in \( C^J \). Hence \( f \) induces a trivial cofibration

\[
i:Y \times 0 \cup X \times [0,1] \cup Y \times 1 \to Y \times [0,1]
\]

in \( \text{pro-}C \). Form the solid-arrow commutative diagram

\[
\begin{array}{ccc}
Y \times 0 \cup X \times [0,1] \cup Y \times 1 & \xrightarrow{K'g \cup hg \cdot \text{proj} \cup h} & E \\
\downarrow i & & \downarrow p \\
Y \times [0,1] & \xrightarrow{kg \cdot \text{proj}} & B
\end{array}
\]

By Axiom M6 (see Proposition (3.3.9)), there exists a filler \( K^{(2)} \) above.
As above, Proposition (3.2.29) implies that the cofibration
\[ g: Y \to Z \]
duces a trivial cofibration
\[ i': Z \times 0 \cup Y \times [0,1] \to Z \times [0,1]. \]

Now form the commutative solid-arrow diagram

\[
\begin{array}{ccc}
Z \times 0 \cup Y \times [0,1] & \xrightarrow{K' \cup K^{(2)}} & E \\
\downarrow i' & & \downarrow p \\
Z \times [0,1] & \xrightarrow{K \circ \text{proj}} & B
\end{array}
\]

Again, as above, there exists a filler \( K^{(3)} \).

Finally, let \( K \) be the composite mapping
\[ Z = Z \times 1 \to Z \times I \xrightarrow{K^{(3)}} E. \]

Then \( K \) is the required filler in Diagram (3.3.19) (easy check), so \( g \)
is an equivalence, as required.

**Case III:** Let \( g \) and \( gf \) be weak equivalences. Assume, for now, the following lemma.

(3.3.20) **Lemma.** Suppose that a map \( i \) in \( \text{pro-} C \) has the
left-lifting-property with respect to all fibrations of fibrant objects.
Then \( i \) is a trivial cofibration.
Consider a solid-arrow commutative diagram

(3.3.21)

in which p is a fibration, and B is fibrant (that is, q is a fibration). Because g and gf are weak equivalences we may successively construct the fillers h and kg above. Because kg is a filler in the top square of Diagram (3.3.21) above, f has the required lifting property to be a trivial cofibration (lemma (3.3.20)). Hence, f is a weak equivalence, as required. □

(3.3.22) Proof of Lemma 3.3.20. Let L denote the class of all p for which the pair (i,p) has the lifting property. It is easy to check that L has the following three properties.

(a) A pullback of a map in L is in L.

(b) Let \( \{E(j)\} \) be an inverse system of objects in \( \text{pro-}C \) indexed by a cofinite directed set \( J \) with a least element 0. If all the induced maps
E(j) \rightarrow \lim_{k < j} \{E(k)\}, \ j \in J

are in \ L, \ then \ the \ induced \ map

\lim_j \{E(j)\} \rightarrow E(0)

is in \ L.

(c) A retract of a map in \ L is in \ L.

To show that \ i is a trivial cofibration, we shall show that \ L contains all fibrations in pro-\ C. By Theorem (3.3.4), \ L contains all fibrations of fibrant objects in \ C; by property N2 of \ C (see §2.3), \ L contains all fibrations in \ C.

Now let \ p:E \rightarrow B \ be a fibration in \ C^J, \ where \ J is a strongly cofinite directed set. Let \ J^* = J \cup \{0\}, \ where \ 0 < j \ for all \ j \ in \ J. \ Define \ an \ inverse \ system \ \{E(j) | j \in J^*\} \ over \ pro-\ C \ as follows. \ Set \ E(0) = B. \ For \ j \in J, \ define \ E(j) \ by \ the \ pull-back \ diagram

\[
\begin{array}{ccc}
E(j) & \rightarrow & E_j \\
\downarrow & & \downarrow \\
B & \rightarrow & B_j
\end{array}
\]

(3.2.23)

in \ pro-\ C. \ That is, for \ k < j, \ E(j)_k = E_k, \ otherwise, \ E(j)_k
is a suitable pullback. Because $J$ is a cofinite directed set, $\lim_{\Delta} \{E(j)\} \cong E$. We shall see that $\{E(j)\}$ satisfies the hypothesis of property (b) above.

For now, consider a fixed $j$ in $J$. To show that the map

$$E(j) \longrightarrow \lim_{k < j} \{E(k)\}$$

is in $L$, we shall show that given a commutative solid-arrow diagram of the form

$$
\begin{array}{ccc}
A & \xrightarrow{h_1} & E(j) \\
\downarrow{h_2} & & \downarrow{H} \\
X & \xrightarrow{h_2} & \lim_{k < j} E(k)
\end{array}
$$

in pro-$C$, there exists a filler $H$. To do this, form the pullback diagram in pro-$C$

$$
\begin{array}{ccc}
\lim_{k < j} E(k) & \longrightarrow & \lim_{k < j} E_k \\
\downarrow & & \downarrow \\
B & \longrightarrow & \lim_{k < j} \{B_k\}.
\end{array}
$$

Thus $h_2$ corresponds to maps
\[ h_2': X \longrightarrow \lim_{k < j} \{E_k\}, \quad \text{and} \]
\[ h_2'': X \longrightarrow B \]

whose images in \( \lim_{k < j} \{E_k\} \) agree. Also, because \( E(j) \) is defined by the pullback diagram (3.2.23) in \( \text{pro-}C \), \( h_1 \) corresponds to the pair of maps

\[ h_1': A \longrightarrow E_j, \quad \text{and} \]
\[ h_1'': A \longrightarrow B, \]

whose images in \( E_j \) are equal. Now consider the resulting commutative solid-arrow diagram

\[ \begin{array}{c}
A \\
\downarrow h_4 \downarrow \downarrow h_3 \downarrow \downarrow h_2' \downarrow \\
X \\
\downarrow h_2 \downarrow \\
B \\
\downarrow \\
E_j \\
\downarrow p_j \\
\lim_{k < j} \{E_k\} \\
\end{array} \]

(3.3.25)

Here \( P_j \) is a pullback, and the composite mapping \( A \longrightarrow X \longrightarrow B \) is \( h_1'' \). By the universal property of pullbacks, there is a unique filler \( h_3 \). By Proposition (3.2.7), \( q_j \) is a fibration in \( C \),
hence \( q_j \in L \), so that there exists a filler \( h_4 \) above. Diagrams (3.2.23) and (3.2.25) show that the maps \( h_2 \): \( X \rightarrow B \) and \( h_4 : X \rightarrow E_j \) induce a unique map \( H : X \rightarrow E(j) \), which is the required filler in diagram (3.2.24). We have shown that the inverse system \( E(j) \) satisfies the hypothesis of condition (b) above. Hence the induced map

\[
p : E = \lim \{ E(j) \} \rightarrow E(0) = B
\]

is in \( L \). Therefore \( L \) contains all strong fibrations in \( \text{pro-}C \).

By property (c) above, \( L \) contains all fibrations in \( \text{pro-}C \). Therefore \( i \) is a trivial cofibration, as required. \( \square \)

Similar techniques yield the following.

(3.3.26) **Proposition.** Let \( f : X \rightarrow Y \), and \( g : Y \rightarrow Z \) be fibrations. If any two of the maps \( f, g, \) and \( gf \) are weak equivalences, so is the third.

The proof is analogous to the proof of Proposition (3.3.18) and requires a lemma "dual" to Lemma (3.3.20).

(3.3.27) **Lemma.** Suppose that a map \( p \) in \( \text{pro-}C \) has the right-lifting-property with respect to all cofibrations of cofibrant objects. Then \( p \) is a trivial fibration.
The proof is similar to the proof (3.3.22), and somewhat simpler because cofibrations in $C^J$ are defined levelwise. Details of the proofs of the above Proposition and Lemma are omitted.

We now begin the proof of the general case of Axiom M5 with four preliminary lemmas.

(3.3.28) **Lemma.** Let $f : X \to Y$ be a weak equivalence. Suppose that $f = p\iota$, where $\iota$ is a trivial cofibration and $p$ is a fibration. Then $p$ is a trivial fibration.

**Proof.** By Definitions (3.3.1), we may write $f = p'\iota'$, where $\iota'$ is a trivial cofibration and $p'$ is a trivial fibration. Form the commutative solid-arrow diagram

(3.3.29)

By Axiom M6a (Proposition (3.3.15)), there exist maps $f$ and $f'$ (as shown above) such that Diagram (3.3.29) together with either dotted arrow, commutes. Hence $f'f\iota = \iota$ and $pf'f = p$. Thus there exists a commutative solid-arrow diagram (see Proposition (2.3.5))
As the proof of Case II of Proposition (3.3.18), \( j \) is a trivial cofibration. Hence there exists a filler \( H \), that is, a homotopy \( H : \text{id}_Z = f'f \) relative to \( X \) which covers \( \text{id}_Y \). A similar construction yields a homotopy \( H' : \text{id}_Z = ff' \). Therefore the fibrations \( p \) and \( p' \) are fibre-homotopy-equivalent. Hence they have similar lifting properties (use deformations analogous to those in the proof of Case II of Proposition (3.3.18)), so that \( p \) is a trivial fibration by Proposition (3.3.16). □

(3.3.30) Lemma. Let \( f : X \to Y \) be a weak equivalence. Suppose that \( f = pi \), where \( p \) is a trivial fibration and \( i \) is a cofibration. Then \( i \) is a trivial cofibration.

The proof is similar to that of Lemma (3.3.28), and is omitted.

At this point we recall Condition N3 for \( C \): at least one of the following statements holds.
N3a. All objects of $C$ are cofibrant.

N3b. All objects of $C$ are fibrant.

We shall assume N3a for the remainder of this section, unless otherwise specified. If instead N3b holds, replace "fibration" by "cofibration" in Lemma (3.3.31), below, "dualize" the proof of Lemma (3.3.32), below, and make some other obvious changes. Details are omitted.

(3.3.31) Lemma. Let $p:E \to B$ be a trivial fibration. Then there exists a section $s:B \to E$ (that is, $ps = id_B$); further, any section is a trivial cofibration.

Proof. By Assumption N3, $B$ is cofibrant, so Axiom M6 (see Proposition (3.3.9)) yields a filler $s$ in the commutative solid-arrow diagram

```
\phi \quad E \\
\downarrow \quad s \quad \downarrow p \\
B \quad \downarrow \quad B
```

Now, let $s':B \to E$ be a section to $p$. As in the proof of Lemma (3.3.28), there exists a homotopy $H:E \times [0,1] \to E$ over $id_B$ from
sp to id_E. As in the proof of Proposition (3.3.18), Case II, or the
proof of [Q-1, Lemma I.5.1, §4], we may use H to show that s' is a
trivial cofibration. □

(3.3.32) Lemma. Let f:X → Y be a trivial fibration and
g:Y → Z be a trivial cofibration. Then the composite gf is a
weak equivalence.

Proof. By Axiom M2 (Proposition (3.3.8)) we may factor gf as
pi, where i is a cofibration and p is a trivial fibration. Form
the commutative solid-arrow diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & W \\
\uparrow f & & \uparrow t \\
y & \xrightarrow{s} & p & \xrightarrow{t} & Z
\end{array}
\]

Lemma (3.3.31) yields a section s to f. Because pis = gfs = g,
there exists a commutative solid-arrow diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{is} & W \\
\uparrow g & & \uparrow p \\
z & \xrightarrow{id} & Z
\end{array}
\]

Axiom M6 (see Proposition (3.3.9)) yields a filler t in diagram
(3.3.34). The map t is a section to p and satisfies tg = is
(see diagram (3.3.33)). Axiom M5 for cofibrations (Proposition
(3.3.18)) implies that \( t_g (= i_s) \) and \( i \) are trivial cofibrations.
Hence \( g f (= p_i) \) is a weak equivalence (Definitions (3.3.1)), as
required. \( \square \)

(3.3.35) Proposition. (Verification of Axiom M5). Let

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{g} \\
\end{array}
\]

be a diagram in which two of the maps \( f, g, \) and \( g f \) are weak equi-
valences. Then so is the third map.

Proof. There are three cases.

Case I: Let \( f \) and \( g \) be weak equivalences. As in Defini-
tions (3.3.1), write \( f = p_i \) and \( g = q_j \), where \( p \) and \( q \) are
trivial fibrations and \( i \) and \( j \) are trivial cofibrations. By
Lemma (3.3.32) the composite mapping \( j p \) is a weak equivalence write
\( j p = r k \), where \( r \) is a trivial fibration and \( k \) is a trivial
cofibration. By Propositions (3.3.26) and (3.3.18) respectively, the
composite mapping \( q r \) is a trivial fibration and the composite mapping
\( k i \) is a trivial cofibration. Hence \( g f (= q r k i) \) is a weak equi-
valence, as required.

Case II: Let \( f \) and \( g f \) be weak equivalences. Write
\( f = p_i \) and \( g = q_j \), where \( p \) is a trivial fibration, \( i \) and \( j \)
are trivial cofibrations, and \( q \) is a fibration. Then \( g f = q j p i \).
By Lemma (3.3.32), \( j p \) is a weak equivalence. We may therefore write
\( j p = r k \), where \( r \) is a trivial fibration and \( k \) is a trivial
cofibration. We have thus factored the weak equivalence $gf$ as $(qr) (ki)$, where $ki$ is a trivial fibration (use Proposition (3.3.18)) and $qr$ is a fibration (by Axiom M6a (Proposition 3.3.15) which implies that the class of fibrations is closed under composition). By Lemma (3.3.28), $qr$ is a trivial fibration. Finally, Proposition (3.3.26) implies that $q$ is a trivial fibration, so that $g(=qj)$ is a weak equivalence as required.

**Case III.** Let $g$ and $gf$ be weak equivalences. As in Case II, factor $g$ and $f$ so that $gf = qjpi$, where $j$ is a trivial cofibration, $p$ and $q$ are trivial fibrations, and $i$ is a cofibration. Proceed as in Case II, using Lemmas (3.3.32) and (3.3.30). Details are omitted. □

This completes the proof that pro-$C$ is a closed model category.

We shall conclude this section by describing cofibrations and trivial cofibrations up to isomorphism. This extends J. Grossman's [Gros-1,54] characterization of cofibrations in his closed model structure on tow-$SS$.

(3.3.36) Proposition. A map $f$ in pro-$C$ is a cofibration (respectively, trivial cofibration) if and only if $f$ is isomorphic to a strong cofibration (respectively, strong trivial cofibration) (in some $C^J$).

Proof. The "only if" part is obvious. For the "if" part, first reindex $f$ if necessary (Proposition (3.1.4) and Theorem (2.1.6))
so that \( f = \{f_j\}: \{X_j\} \to \{Y_j\}, \quad j \in J, \) with \( J \) cofinite. By Definitions (3.3.1), \( \{f_j\} \) is a retract of a strong cofibration (respectively, strong trivial cofibration) \( \{f'_k\}: \{X'_k\} \to \{Y'_k\}, \quad k \in K, \) with \( K \) cofinite. Form the following commutative diagram in pro-\( C \):

\[
\begin{array}{ccc}
{X_j} & \xrightarrow{id} & {X_j} \\
\downarrow & & \downarrow \\
{X'_k} & \xrightarrow{id} & {Y'_k} \\
\downarrow & & \downarrow \\
{Y_j} & \xrightarrow{id} & {Y_j} \\
\end{array}
\]

(3.3.37)

\[
\begin{array}{ccc}
\{f_j\} & \xrightarrow{f_j} & \{f_j\} \\
\downarrow & & \downarrow \\
\{f'_k\} & \xrightarrow{f'_k} & \{f'_k\} \\
\end{array}
\]

We shall use diagram (3.3.37) to construct a level cofibration (respectively, level trivial cofibration) \( f'' \) isomorphic to \( f \). First consider the left front square of diagram (3.3.37). We shall say that

\[
(3.3.38) \quad (f'_k: X'_k \to Y'_k) < (f_m: X_m \to Y_m)
\]

if the square (shown in perspective)
is a left front square of Diagram (3.3.37). Similarly, use the right front square of Diagram (3.3.37) to define relations of the form

\[(3.3.40) \quad (f'_m : X'_m \rightarrow Y'_m) \triangleleft (f'_n : X'_n \rightarrow Y'_n).\]

By diagram (3.3.37), relations (3.3.38) and (3.3.40) and their composites yield an inverse system of maps

\[\tilde{\mathcal{F}} = \{ \tilde{f}_k : \tilde{X}_k \rightarrow \tilde{Y}_k \}, \quad k \in K\]

indexed by a (cofinite) directed set \(K\).

Further, \(\{ f'_j : X'_j \rightarrow Y'_j \}, \quad j \in J\), together with the bonding maps induced by Diagrams (3.3.37) - (3.3.40) is a cofibration (resp., trivial cofibration) and is cofinal in \(\mathcal{F}\), hence isomorphic to \(\tilde{\mathcal{F}}\). Also, \(\{ f_i : X_i \rightarrow Y_i \}\) admits a cofinal subsystem which is cofinal in \(\tilde{\mathcal{F}}\) (the bonding maps agree by Diagrams (3.3.37) - (3.3.40)).

The conclusion follows. \[\square\]
(3.3.41) **Remarks.** The above proof used the fact that cofibrations in \( C^J \) are defined levelwise. We do not know whether the analogue of Proposition (3.3.36) for fibrations holds.

§3.4. **Suspension and loop functors, cofibration and fibration sequences**

D. Quillen [Q-1, §§L2-3] developed a general theory of suspension and loop functors and cofibration and fibration sequences in \( \text{Ho}(C) \), where \( C \) is a closed model category. We shall sketch this theory within the context of our closed model structures on \( \text{pro}-C \).

(3.4.1) **Morphisms in \( \text{Ho}(\text{pro}-C) \).** Let \( X \) be a cofibrant object and \( Y \) be a fibrant object in \( \text{pro}-C \). Then

\[
\text{Ho}(\text{pro}-C)(X,Y) = [X,Y],
\]

where \( [X,Y] \) denotes the set of homotopy classes of maps from \( X \) to \( Y \) (in \( \text{pro}-C \)) with respect to the cylinder \( X \otimes [0,1] \).

There is another dual description of \( \text{Ho}(\text{pro}-C)(X,Y) \). Factoring the diagonal map \( Y \to Y \times Y \) as the composite of a trivial cofibration followed by a fibration

\[
Y \to Y^I \to Y \times Y
\]

yields the cocylinder \( Y^I \). **Caution:** in general \( Y^I \) need not depend functorially upon \( Y \) (compare with Definition (2.3.4)).

One can easily show that maps \( f,g:X \to Y \) are homotopic if and only if there exists a commutative diagram
here \( p_0 \) and \( p_1 \) denote the projections onto the first \((Y^0)\) and second \((Y^1)\) factors in \( Y \times Y \), respectively.

Now, let \( C_* \) be a pointed closed model category, that is, a closed model category which is also a pointed category. Then \( \text{pro-} C_* \) becomes a pointed closed model category (the point \( * \) of \( C \) is also the point of \( \text{pro-} C_* \)). We shall follow the "usual" conventions and write \( V \) for the sum (coproduct) in \( \text{pro-} C_* \).

(3.4.2) **Definitions.** If \( f: X \rightarrow Y \) is a cofibration in \( \text{pro-} C_* \) we shall write \( * \vee_X Y \) for the cofibre of \( f \); defined by the pushout diagram

\[
\begin{array}{ccc}
X & \rightarrow & * \\
\downarrow f & & \downarrow \text{cofibre} \\
Y & \rightarrow & * \vee_X Y
\end{array}
\]
If \( f: X \to Y \) is a fibration in \( \text{pro-} C_* \), we shall write \( \ast_{x_Y} X \) for the fibre of \( f \) defined by the pullback diagram

\[
\begin{array}{ccc}
\ast_{x_Y} X & \to & X \\
\downarrow & & \downarrow \\
\ast & \to & Y 
\end{array}
\]

Note that the cofibre of a strong (level) cofibration is just the levelwise cofibre; similarly the fibre of a strong (level) fibration is the levelwise fibre.

(3.4.3) Suspensions and loop spaces. Let \( X \in \text{pro-} C_* \) be cofibrant. Choose a cylinder object \( X \times [0,1] \), and let \( \Sigma X \) be the cofibre of the map \( i_0 + i_1: X \vee X \to X \times [0,1] \). We shall call \( \Sigma X \) a suspension of \( X \). Loop spaces are defined dually. Let \( Y \in \text{pro-} C_* \) be fibrant. Choose a cocylinder object \( Y^{[0,1]} \), and let \( \Omega Y \) be the fibre of the map \( (p_0, p_1): Y^{[0,1]} \to Y \times Y \). We shall call \( \Omega Y \) a loop-space of \( Y \). Note that \( \Sigma X \) above is cofibrant and \( \Omega Y \) is fibrant. Caution: In general \( \Sigma \) and \( \Omega \) need not be functors on \( \text{pro-} C_* \) (see Definition (2.3.4) and paragraph (3.4.1)).

On the other hand, Quillen proves the following theorem for
arbitrary closed pointed model categories.

(3.4.4) **Theorem** [Q-1, §§I.2]. We may extend $\Sigma$ and $\Omega$ to an adjoint pair of functors on all of $\text{Ho}(\text{pro-}C_\times^\ast)$:

$$\text{Ho}(\text{pro-}C_\times^\ast)(\Sigma X, Y) = \text{Ho}(\text{pro-}C_\times^\ast)(X, \Omega Y).$$

If $X$ is cofibrant and $Y$ is fibrant,

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

$\Sigma$ and $\Omega$ are defined up to canonical isomorphism. Also,

$$\text{Ho}(\text{pro-}C_\times^\ast)(\Sigma -, -) = \text{Ho}(\text{pro-}C_\times^\ast(-, \Omega -))$$

are functors from $\text{pro-}C_\times^\ast$ to $\text{pro-}C_\times^\ast$ to groups.

The proof is similar to the proof for the category of pointed spaces, except that somewhat more care is needed because of the choices involved in defining $\Sigma$ and $\Omega$. The group structure on $\text{Ho}(\text{pro-}C_\times^\ast)(\Sigma X, Y)$ comes from a **co-H** structure $\Sigma X \longrightarrow \Sigma X \vee \Sigma X$ which makes $\Sigma X$ a cogroup object in $\text{Ho}(\text{pro-}C_\times^\ast)$; the corresponding group structure on $\text{Ho}(\text{pro-}C_\times^\ast)(X, \Omega Y)$ comes from an **H**-structure $\Omega Y \times \Omega Y \longrightarrow \Omega Y$ which makes $Y$ a group object in $\text{Ho}(\text{pro-}C_\times^\ast)$.

(3.4.5) **Short cofibration sequences.** Let $f: A \rightarrow X$ be a cofibration in $\text{Top}_\times^\ast$. Let
\[ M_f = A \times [0,1] \cup A \times \ast \times [0,1] \]

be the mapping cylinder of \( f \), and consider the induced cofibration

\[
\begin{array}{c}
A \quad \xrightarrow{i_0} \quad M_f \\
I \\
\end{array}
\]

If \( C_{i_0} \) denotes the cofibre of \( i_0 \), \( CA \) the reduced cone on \( A \), and \( \Sigma A \) the reduced suspension of \( A \), we may form the sequence

\[
\begin{array}{c}
A \quad \xrightarrow{i_0} \quad M_f \xrightarrow{} \quad C_{i_0} \\
I \\
\end{array}
\]

and induced diagram

\[
\begin{array}{c}
C_{i_0} \xleftarrow{=} \quad CA \cup_A M_f \xrightarrow{} \quad \Sigma A \vee C_{i_0} \\
\end{array}
\]

Further, in \( \text{Ho}(\text{Top}_\ast) \) the composite mapping \( n : C_{i_0} \rightarrow \Sigma A \vee C_{i_0} \) is a coaction of the cogroup object \((\co-H \text{ space with a co-inverse})\)

\( \Sigma A \) on \( C_{i_0} \).

We shall extend the above observations to arbitrary cofibrations.
in \((\text{pro-}C_\ast)_C\), the full subcategory of cofibrant objects in \(\text{pro-}C_\ast\). (The "cofibrant" condition means that the map \(* \to X\) is a cofibration.) The following notation is needed. If \(A \to X\) is a map in \(\text{pro-}C_\ast\) we shall write \(A \otimes [0,1] \vee_A X\) for the cofibre sum (pushout)

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
A \otimes [0,1] & \longrightarrow & A \otimes [0,1] \vee_A X.
\end{array}
\]

More generally, we shall write \(A \otimes [0,1] \vee_A -\) to denote the cofibre sum with \(i_1: A \to A \otimes [0,1]\), and \(- \vee_A A \otimes [0,1]\) to denote the cofibre sum with \(i_0: A \to A \otimes [0,1]\).

Now let \(A \to X\) be a cofibration in \((\text{pro-}C_\ast)_C\), that is, \(A\) is cofibrant in \(\text{pro-}C_\ast\). Let \(Z\) be the cofibre of \(A\). We shall define a coaction of the cogroup \(ZA\) on \(Z\). To do this, form the commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i_1 & & \downarrow \phi \\
A \otimes [0,1] & \longrightarrow & A \otimes [0,1] \vee_A X.
\end{array}
\]
in which the cylinder object $X \otimes [0,1]$ is obtained by factoring the natural map $X \vee_A A \otimes [0,1] \vee_A X \to X$ as a cofibration followed by a trivial fibration

$X \vee_A A \otimes [0,1] \vee_A X \to X \otimes [0,1] \to X$

(see Proposition (2.3.5)). By construction both squares in Diagram (3.4.6) are pushout squares. Further, $i$ (see Proposition (2.3.5)) and $i_0$ are trivial cofibrations, hence so are their respective pushouts $i'$ and $i'_0$. Diagram (3.4.6) induces the composite mapping $c: Z \to \Sigma A \vee Z$ in $\text{Ho(pro-} C_\star\text{)}$ defined below.
\[ Z \xrightarrow{[i_0]} Z \vee X \times [0,1] \]

\[ \rightarrow * \vee A \otimes [0,1] \vee A X \]

\[ \rightarrow * \vee A \otimes [0,1] \vee A \vee Z \]

\[ = \Sigma A \vee Z \]

It is easy to check that, in \( \text{Ho}(\text{pro}-C_k) \), \( c \) is a coaction of the cogroup \( A \) on \( Z \) and that \( c \) is independent of any choices made above (see [Q-1, Proposition I.3.1] and the following remarks). We therefore define a short cofibration sequence in \( \text{Ho}(\text{pro}-C_k) \) to be a diagram in \( \text{Ho}(\text{pro}-C_k) \):

\[ (3.4.7) \quad A' \rightarrow X' \rightarrow Z', \quad Z' \rightarrow \Sigma A' \vee Z', \]

which for some cofibration \( A \rightarrow X \) in \( (\text{pro}-C_k)_c \) is isomorphic to the diagram

\[ (3.4.8) \quad A \rightarrow X \rightarrow Z, \quad Z \rightarrow \Sigma A \vee Z \]

constructed above.

\[ (3.4.9) \quad \text{Proposition.} \quad \text{A short cofibration sequence} \]

\[ A \rightarrow X \rightarrow Z, \quad Z \rightarrow \Sigma A \vee Z \]

induces a short cofibration sequence

\[ X \rightarrow Z \rightarrow \Sigma A, \quad \Sigma A \rightarrow \Sigma X \vee \Sigma A, \]
the "connecting map" \( Z \to \Sigma A \) is the composite

\[
Z \to \Sigma A \vee Z \to \Sigma A,
\]

and the coaction \( \Sigma A \to \Sigma X \vee \Sigma A \) is the composite

\[
\Sigma A \to \Sigma A \vee \Sigma A \xrightarrow{-id \vee id} \Sigma A \vee \Sigma A \to \Sigma X \vee \Sigma A,
\]

where \(-id: \Sigma A \to \Sigma A\) is the inverse in the cogroup \( \Sigma A \).

The proof is similar to the usual proof of the corresponding assertion in \( \text{Ho}(\text{Top}_{\ast}) \), and dual to the proof of \( R-1\), Proposition I.3.3]. Details are omitted.

(3.4.10) **Long cofibration sequences.** A short cofibration sequence \( A \to X \to Z, \ Z \to \Sigma A \vee Z \) in \( \text{Ho}(\text{pro-}C_{\ast}) \) induces a long cofibration sequence (Barratt-Puppe sequence)

\[
A \xrightarrow{\varphi} X \to Z \to \Sigma A \to \ldots \to \Sigma^n A \to \Sigma^n X \to \Sigma^n Z \to \Sigma^{n+1} A \to \ldots.
\]

Also, for any object \( Y \) in \( \text{pro-}C_{\ast} \), the sequence

(3.4.11) \( \ldots \to \text{Ho}(\text{pro-}C_{\ast})(\Sigma^n Z,Y) \to \text{Ho}(\text{pro-}C_{\ast})(\Sigma^n X,Y) \)

\[
\to \text{Ho}(\text{pro-}C_{\ast})(\Sigma^n A,Y) \to \ldots
\]

\[
\to \text{Ho}(\text{pro-}C_{\ast})(\Sigma A,Y) \to \text{Ho}(\text{pro-}C_{\ast})(Z,Y)
\]

\[
\to \text{Ho}(\text{pro-}C_{\ast})(X,Y) \to \text{Ho}(\text{pro-}C_{\ast})(A,Y)
\]

has the usual exactness properties:
(a) Sequence (3.4.11) is exact as a sequence of maps of pointed sets;

(b) Sequence (3.4.11) is exact to the left of $\text{Ho}(\text{pro-} C_\kappa)(\Sigma X, Y)$ as a sequence of group homomorphisms;

(c) Two maps in $\text{Ho}(\text{pro-} C_\kappa)(Z, Y)$ have the same image in $\text{Ho}(\text{pro-} C_\kappa)(X, Y)$ if and only if they differ by the action of the group $\text{Ho}(\text{pro-} C_\kappa)(\Sigma A, Y)$ on $\text{Ho}(\text{pro-} C_\kappa)(Z Y)$.

(d) Two maps $g_1, g_2 \in \text{Ho}(\text{pro-} C_\kappa)(\Sigma A, Y)$ have the same image in $\text{Ho}(\text{pro-} C_\kappa)(Z, Y)$ if and only if $g_2 = (\Sigma f^\kappa) h \circ g_1$ for some map $h$ in $\text{Ho}(\text{pro-} C_\kappa)(\Sigma X, Y)$.

As above, compare the usual exactness properties of Barratt-Puppe sequences and [Q-1, Proposition I.3.4'] for the proof.

We now summarize the properties of cofibration sequences.

(3.4.12) Proposition. (dual to [Q-1, Proposition I.3.5]). The class of short cofibration sequences in $\text{Ho}(\text{pro-} C_\kappa)$ has the following properties:
(a) Any map \( f : X \rightarrow Y \) in \( \text{Ho}(\text{pro-} C_\ast) \) may be embedded in a cofibration sequence

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
Z & \rightarrow & Z, \quad Z \rightarrow \Sigma X \vee Z.
\end{array}
\]

(b) Given a commutative solid-arrow diagram

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
Z & \rightarrow & \Sigma X \vee Z
\end{array}
\quad
\begin{array}{ccc}
X \rightarrow Y \rightarrow Z & & Z \rightarrow \Sigma X \vee Z \\
\downarrow & & \downarrow \\
X' \rightarrow Y' \rightarrow Z' & & Z' \rightarrow \Sigma X' \vee Z'
\end{array}
\]

(3.4.13)

in which the rows are short cofibration sequences, the filler \( h \) exists.

(c) If the maps \( f \) and \( g \) in diagram (3.4.13) are weak equivalences, so is the filler \( h \).

(d) Proposition (3.4.9) holds.

We omit the proof.

The following straightforward proposition yields many cofibration sequences in \( \text{Ho}(\text{pro-} C_\ast) \).

(3.4.13) \textbf{Proposition.} Let \( \{ A_j \rightarrow X_j \rightarrow Z_j \} \) be an inverse system of cofibrations of cofibrant objects \( A_j \rightarrow X_j \), with cofibres \( Z_j \), over \( C_\ast \) indexed by a cofinite directed set \( J \). Then there is
a induced cofibration sequence

\[ \{A_j\} \to \{X_j\} \to \{Z_j\}, \quad \{Z_j\} \to \Sigma \{X_j\} \vee \{Z_j\} \]

in \( \text{Ho(pro-}C_\infty) \). \[\square\]

All of the above theory may be dualized to obtain short and long fibration sequences. Quillen \([Q^{-1}, \S 1.3]\) discusses fibrations explicitly. We shall summarize this discussion below.

A fibration \( p:Y \to B \) in \( \text{pro-}C_\infty \) \( \) (that is, \( B \) is fibrant in \( \text{Pro-}C_\infty \)) induces a short fibration sequence in \( \text{Ho(pro-}C_\infty) \),

\[ (3.4.14) \quad F \to Y \to B, \quad \Omega B \times F \to F, \]

in which \( m \) is a well-defined action of the group object \( \Omega B \) on \( F \) in \( \text{Ho(pro-}C_\infty) \). There is also an induced short fibration sequence

\[ (3.4.15) \quad \Omega B \to F \to E, \quad \Omega E \times \Omega B \to \Omega B, \]

where \( \partial \) is the composite map

\[ \Omega B \xrightarrow{id \times \ast} \Omega B \times F \to F, \]

and \( n \) is the composite map

\[ \Omega E \times \Omega B \to \Omega B \times \Omega B \xrightarrow{-id \times id} \Omega B \times \Omega B \to \Omega B. \]
This is \([Q - 1, \text{Proposition I.3.3}], \text{compare Proposition (3.4.9)}. \) Hence, there is an induced long fibration sequence

\[
\cdots \longrightarrow \Omega^{n+1}_B \longrightarrow \Omega^n_F \longrightarrow \Omega^n_E \longrightarrow \Omega^n_B \longrightarrow \cdots
\]

\[
\longrightarrow \Omega_B \longrightarrow F \longrightarrow E \longrightarrow B
\]

with exactness properties analogous to properties (a) - (d) of long cofibration sequences (see Paragraph (3.4.10) above)\([Q - 1, \text{Proposition I.3.4}].\)

We shall need the following dual of Proposition (3.4.13).

\[
(3.4.17) \text{Proposition. } \text{Let } \{F_j \longrightarrow E_j \longrightarrow B_j\} \text{ be an inverse system of fibrations of fibrant objects } E_j \text{ and } B_j, \text{ with fibres } F_j \text{ over } * \in B_j, \text{ indexed by a cofinite strongly directed set } J. \text{ Then there is an induced fibration sequence}
\]

\[
\{F_j\} \longrightarrow \{E_j\} \longrightarrow \{B_j\}, \ \Omega(B_j) \times \{F_j\} \longrightarrow \{F_j\}
\]

\text{in } Ho(pro - C_*).

\text{Proof. We first replace } \{B_j\} \text{ by a fibrant object } \{B'_j\}, \text{ and next replace the map } \{E_j\} \longrightarrow \{B_j\} \text{ by a fibration}

\[
\{E'_j\} \longrightarrow \{B'_j\} \text{ in } C^J_*. \text{ (Recall that the natural functor}
\]

\[
C^J_* \longrightarrow pro - C_* \text{ preserves the closed model structures.)} \text{ Caution:}
recall that fibrations in $C_*^J$ are not defined levelwise, but the fibre of a strong fibration in $\text{pro-}C_*$, that is, a fibration in $C_*^J$ is just the levelwise fibre. Use the proof of Proposition (3.2.24) (Axiom M2 for $C_*^J$) first to factor the map $\{B_j\} \rightarrow \ast$ as a level trivial cofibration followed by a fibration

$$\{B_j\} \xrightarrow{i_j} \{B'_j\} \rightarrow \ast$$

(this makes $B'_j$ fibrant) and then to factor the composite map

$$\{E_j\} \rightarrow \{B_j\} \rightarrow \{B'_j\}$$

as a level trivial cofibration followed by a fibration

$$\{E_j\} \xrightarrow{i'_j} \{E'_j\} \xrightarrow{p'_j} \{B'_j\}.$$ 

Let $\{F'_j\}$ be the (levelwise) fibre of $\{p_j\}$. We obtain the following commutative diagram in $C_*^J$.

$$\begin{array}{c}
\{F_j\} \rightarrow \{E_j\} \xrightarrow{p_j} \{B_j\} \\
\downarrow \downarrow \downarrow \downarrow \\
\{F'_j\} \rightarrow \{E'_j\} \xrightarrow{p'_j} \{B'_j\} \\
(3.4.18) \quad \{i''_j\} \rightarrow \{i'_j\} \rightarrow \{i_j\} \\
\downarrow \downarrow \downarrow \\
\{F'_j\} \rightarrow \{E'_j\} \xrightarrow{p'_j} \{B'_j\}
\end{array}$$

In diagram (3.4.18) $\{i''_j\}$ is the restriction of $\{i'_j\}$. Regard
Diagram (3.4.18) as an inverse system made up of the diagrams

\[
\begin{array}{ccc}
F_j & \longrightarrow & E_j & \longrightarrow & B_j \\
\downarrow{\scriptstyle i''_j} & & \downarrow{\scriptstyle i'_j} & & \downarrow{\scriptstyle i_j} \\
F'_j & \longrightarrow & E'_j & \longrightarrow & B'_j \\
\end{array}
\]

(3.4.19)

in \( C_\ast \). Because \( p_j \) and \( p'_j \) are fibrations (a fibration in \( C_\ast \)) is a levelwise fibration, see Proposition (3.2.16), although the converse assertion need not hold), and \( i_j \) and \( i'_j \) are weak equivalences, the maps \( i''_j \) are weak equivalences. Hence the diagrams

\[
\{F_j\} \longrightarrow \{E_j\} \longrightarrow \{B_j\},
\]

\[
\{F'_j\} \longrightarrow \{E'_j\} \longrightarrow \{B'_j\}
\]

are isomorphic over \( \text{Ho}(\text{pro-}C_\ast) \). The conclusion follows. \( \square \)

(3.4.20) \textbf{Remarks.} Diagrams (3.4.19) may be extended to maps of short fibration sequences in \( \text{Ho}(C_\ast) \)
§3.5. **Simplicial Model Structures.**

In this section we shall prove that a simplicial closed model structure (satisfying condition N of §2.3) on $\mathbf{C}$ induces such a structure on pro-$\mathbf{C}$. These results can be readily extended to pointed categories; details are similar and omitted.

For a finite simplicial set $K$ and $X, Y$, in $\mathbf{C}$, let $X \otimes K$ and $\text{HOM}(X,Y)$ denote the "tensor product" and "function space" constructions in $\mathbf{C}$ (recall that $\text{HOM}(X,Y)$ is a simplicial set), and let $\text{HOM}(K,X)$ denote the "function space" connecting $\mathbf{SS}$ and $\mathbf{C}$; see §2.4. Let $\{X_j\}, \{Y_k\} \in \mathbf{C}$.

(3.5.1) **Definition.** Let

$$\{X_j\} \otimes K = \{X_j \otimes K\}, \quad \text{and}$$

$$\text{HOM}(K,\{X_j\}) = \{\text{HOM}(K,X_j)\}$$

together with the induced bonding maps, and let

$$\text{HOM}(\{X_j\}, \{Y_k\}) = \lim_k \colim_j \{\text{HOM}(X_j,Y_k)\}.$$ 

These constructions extend to functors

$$\otimes: \text{pro-}\mathbf{C} \times \mathbf{SS} \to \text{pro-}\mathbf{C}, \quad \text{HOM}: (\text{finite simplicial sets})^{\text{op}} \times \text{pro-}\mathbf{C} \to \text{pro-}\mathbf{C},$$

and $\text{HOM}: (\text{pro-}\mathbf{C})^{\text{op}} \times \text{pro-}\mathbf{C} \to \mathbf{SS}$, respectively. Axiom SMO, and the following propositions are immediate consequences.
(3.5.2) Proposition. For \( \{X_j\} \) and \( \{Y_k\} \) in \( \text{pro-} C \), the set of \( 0 \)-simplices of \( \text{HOM}(\{X_j\}, \{Y_k\}) \),

\[
\text{HOM}(\{X_j\}, \{Y_k\})_0 \cong \text{pro-} C(\{X_j\}, \{Y_k\}),
\]

naturally in \( \{X_j\} \) and \( \{Y_k\} \).

(3.5.3) Theorem (Exponential Law). Let \( K \) be a finite simplicial set and let \( \{X_j\}, \{Y_k\} \in \text{pro-} C \). Then

\[
\text{HOM}(\{X_j \otimes K\}, \{Y_k\}) \cong \text{HOM}(K, \text{HOM}(\{X_j\}, \{Y_k\})),
\]

naturally in all variables. (\( \text{HOM} \) is used to denote the "function space" construction in both \( \text{pro-} C \) and \( \text{SS} \).)

Proof. Because

\[
\text{HOM}(\{X_j \otimes K\}, \{Y_k\}) = \text{HOM}(\{X_j \otimes K\}, \{Y_k\})
\]

\[
= \lim_k \{\text{colim}_j \{\text{HOM}(X_j \otimes K, Y_k)\}\}
\]

\[
= \lim_k \{\text{HOM}(\{X_j\} \otimes K, Y_k)\},
\]

and \( \text{HOM}(K, ?) : \text{SS} \to \text{SS} \) preserves limits, we may reduce the general case to the case where \( \{Y_k\} \) is an object \( Y \) of \( C \).
In this case,

\[ \text{HOM} \left( \{x_j\} \otimes K, Y \right) = \text{colim}_j \{\text{HOM} (x_j \otimes K, Y)\} \]

\[= \text{colim}_j \{\text{HOM} (K, \text{HOM} (x_j, Y))\} \]

Because \( K \) has finitely many non-degenerate simplices,

\[ \text{colim}_j \{\text{HOM} (K, \text{HOM} (x_j, Y))\} = \text{HOM} (K, \text{colim}_j \{\text{HOM} (x_j, Y)\}) \]

\[= \text{HOM} (K, \text{HOM} (\{x_j\}, Y)) \]

as required. Naturality follows easily. \( \square \)

(3.5.4) Corollary. With \( K, \{x_j\}, \) and \( \{y_k\} \) as above,

\[ \text{pro-} C(\{x_j\} \otimes K, \{y_k\}) = \text{SS}(K, \text{HOM} (\{x_j\}, \{y_k\})) \]

naturally in all variables. \( \square \)

(3.5.5) Remarks.

(a) The corresponding assertion for \( \text{HOM} (K, K) \) (Definition (2.4.2)) is proven similarly. Details are omitted.

(b) The above results fail for infinite \( K \); to construct counter-examples, use the fact that

\( \text{HOM} (K, -) : \text{SS} \to \text{SS} \) does not commute with colimits for infinite \( K \). Similarly the function space

\( \text{MAP} (K, -) : \text{Top} \to \text{Top} \) does not commute with colimits.
for non-compact \( K \).

(3.5.6) **Theorem** (Verification of Axiom SM7). Let

\[ i: \{ A_j \} \to \{ X_k \} \]
be a cofibration in pro-C, and let

\[ p: \{ Y_\ell \} \to \{ B_m \} \]
be a fibration in pro-C. Then:

(a) The induced map

(3.5.5)

\[
\text{HOM} (\{ X_k \}, \{ Y_\ell \}) \to \text{HOM} (\{ A_j \}, \{ Y_\ell \}) \times \text{HOM}(\{ A_j \}, \{ B_m \} \text{HOM} (\{ X_k \}, \{ B_m \})
\]

is a fibration in SS (i.e., a Kan fibration);

(b) If either \( i \) or \( p \) is also a weak equivalence,

then the map \( q \) above is also a weak equivalence.

**Proof.**

(a) Consider a solid-arrow commutative diagram in SS

of the form

(3.5.6)

\[
\Delta^n \xrightarrow{g} \text{HOM} (\{ A_j \}, \{ Y_\ell \}) \times \text{HOM} (\{ A_j \}, \{ B_m \} \text{HOM} (\{ X_k \}, \{ B_m \})
\]

\[ f \]
\[ \text{HOM} (\{ X_k \}, \{ Y_\ell \}) \]
in which $v^n,k$ is obtained from $\partial \Delta^n$, the boundary of $\Delta^n$, by deleting the $k$th face. The maps $g$ and $q$ correspond respectively to pairs of maps

$$
(3.5.7) \quad g'^*: \Delta^n \to \text{HOM} (\{A_j\}, \{Y_k\}),
$$

$$
\quad g''*: \Delta^n \to \text{HOM} (\{X_k\}, \{B_m\});
$$

$$
q'^*: \text{HOM} (\{X_k\}, \{Y_j\}) \to \text{HOM} (\{A_j\}, \{Y_k\}),
$$

$$
\quad q''*: \text{HOM} (\{X_k\}, \{Y_j\}) \to \text{HOM} (\{X_k\}, \{B_m\});
$$

such that the appropriate composite maps into $\text{HOM} (\{A_j\}, \{B_m\})$ are equal. Applying the exponential law (Theorem (3.5.3)) to $f, g', g'', q', q''$, and assembling the induced maps with the above coherence data (Diagram (3.5.6)), we obtain a commutative solid-arrow diagram

$$
(3.5.8)
$$

\begin{align*}
\{A_j\} \otimes \Delta^n &\to \{A_j\} \otimes v^n,k \{X_k\} \otimes v^n,k \to \{Y_k\} \\
\{X_k\} \otimes \Delta^n &\to \{X_k\} \to \{B_m\}.
\end{align*}
By Definitions (3.3.1), the map \( \iota: \{ A_j \} \to \{ X_k \} \) is a retract of a levelwise cofibration \( \{ \iota' \}: \{ A'_{\ell} \} \to \{ X'_{\ell} \} \). By Axiom SM7 for C, (see §2.4) the induced maps

\[(3.5.9)\]

\[A'_{\ell} \otimes \Delta^n \cup_{A'_{\ell} \otimes \Delta^n} X'_{\ell} \otimes \Delta^n \to X'_{\ell} \otimes \Delta^n\]

are trivial cofibrations; hence the induced map

\[(3.5.10)\]

\[\{ \iota' \}_\ast: \{ A'_{\ell} \} \otimes \Delta^n \cup_{\{ A'_{\ell} \} \otimes \Delta^n} \{ X'_{\ell} \} \otimes \Delta^n \to \{ X'_{\ell} \} \otimes \Delta^n\]

is a trivial cofibration. Because \( \iota' \) is a retract of \( \{ \iota' \}_\ast \) by construction, \( \iota' \ast \) is also a trivial cofibration; hence Axiom M1 for pro-C yields the filler \( h' \) in diagram (3.5.8). Applying the exponential law (Theorem 3.5.3)) to \( h' \) yields a map \( h: \Delta^n \to \text{HOM} (\{ X_k \}, \{ Y_g \}) \). By construction ((3.5.7) - (3.5.10)), the map \( h \) makes diagram (3.5.6) commute. Thus \( q \) is a (Kan) fibration, as required.

(b) If the map \( \iota \) is a trivial cofibration, then \( \iota \)

induces a trivial cofibration
(3.5.11)

$$i_\#: \{A_j\} \otimes \Delta^n \cup \{A_j\} \otimes \Delta^n \to \{X_k\} \otimes \Delta^n$$

by analogues of (3.5.9) - (3.5.10). This yields a filler in the analogue of Diagram (3.5.8) with $\nabla^{n,k}$ replaced by $\vartriangle^n$; hence a filler the analogue of Diagram (3.5.6) with the same replacement.

Thus $q$ is a trivial (Kan) fibration, as required.

Finally, suppose that $p$ is a trivial fibration. The cofibration $i_\#$ (3.5.11) induced by $i$ has the left-lifting-property with respect to $p$, so the required fillers in suitable analogues of Diagrams (3.5.8) and (3.5.6) (see above) exist. Thus $q$ is a trivial (Kan) fibration, as required. 

Theorems (3.5.3) and (3.5.6), together with the earlier proof that $pro-C$ is a closed model category (§3.2 - 3.3) imply that $pro-C$ is a simplicial closed model category.
§3.6. Pairs.

We shall use the Bousfield-Kan \([B,K\wedge]\) model structure on Maps (pro-C). More precisely, Maps (pro-C) is the category whose objects are maps

\[ A \rightarrow X \]

in pro-C, and whose morphisms are commutative squares. A map

\[ \begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \quad \downarrow \\
X \xleftarrow{g} Y
\end{array} \]

in Maps (pro-C) is a weak equivalence if \(g\) and \(f\) are weak equivalences, a fibration if \(g\) and \(f\) are fibrations, and a cofibration if the appropriate lifting property is satisfied. Explicitly, \((g,f)\) is a cofibration if the induced map

\[ X \amalg_A B \rightarrow Y \]

are cofibrations in pro-C \((X \amalg_A B\) is the pushout obtained from most of diagram(3.6.1)).

(3.6.2) Definition. Let \((C,\text{pro-C})\) be the full subcategory of Maps(pro-C) consisting of maps \(A \rightarrow X\) with \(X\) in \(C\).

It is easy to prove as in §3.2:

(3.6.3) Theorem. The closed model structure on pro-C induces natural closed model structures on Maps (pro-C), \((C,\text{pro-C})\), and \((C,\text{tow-C})\). \(\square\)

It is convenient to represent \((C,\text{tow-C})\) as follows. Objects are towers \(X = (X_0 \leftarrow X_1 \leftarrow \ldots)\); maps are maps \(f:X \rightarrow Y\) in tow-C together with maps \(f_0:X_0 \rightarrow Y_0\) in C.
Alternatively, a map \( f: \{x_n\} \rightarrow \{y_n\} \) consists of a cofinal subtower \( \{x_m\} \subseteq \{x_n\} \) with \( x_{m_0} = x_0 \) and a level map \( \{x_m\} \rightarrow \{y_n\} \).

§3.7, Geometric Models.

We shall develop geometric models of \( \text{Ho}(\text{Top}, \text{tow-Top}) \) and \( \text{Ho}(\text{tow-Top}) \) via filtered spaces and a telescope construction. The geometric model of \( \text{Ho}(\text{Top}, \text{tow-Top}) \) will be used in proper homotopy theory in §6. The model of \( \text{Ho}(\text{tow-Top}) \) was used in shape theory by the first author and R. Geoghegan (unpublished).

(3.7.1) Definitions. A filtered space \( X \) consists of an underlying space \( X \) together with a sequence of closed subspaces \( X = x_0 \supset x_1 \supset x_2 \supset \ldots \), with each \( x_n \subseteq \text{int} x_{n+1} \). A filtered map \( f: X \rightarrow Y \) (of filtered spaces) is a continuous map such that for each number \( n \geq 0 \) there is a number \( m \geq 0 \) with \( f(x_m) \subseteq Y_n \). A filtered space \( X \) induces a natural filtration on its cylinder \( X \times [0,1] \); this yields a natural notion of filtered homotopy of filtered maps. There results a filtered category \( \text{Filt} \) and its associated filtered homotopy category (quotient under the relation of filtered homotopy) \( \text{Ho}(\text{Filt}) \). Caution: \( \text{Filt} \) is not a model category.

(3.7.2) Definition. The telescope of a tower \( X = \{x_n\} \) is the space

\[
\text{Tel}(X) = x_0 \times [0,1] \cup \text{bond} x_1 \times [0,1] \cup \text{bond} x_2 \times [1,2] \cup \text{bond} \ldots
\]

\( \text{Tel}(X) \) is filtered by setting

\[
\text{Tel}(X)_n = x_n \times \text{bond} x_{n+1} \times [n,n+1] \cup \text{bond} \ldots
\]

This construction extends to a functor

\( \text{Tel}: \text{Top}^\mathbb{N} \rightarrow \text{Filt} \).
Further, Tel takes a " $x[0,1]$ - homotopy" in $\text{Top}^N$ into a filtered homotopy.

(3.7.3) **Definition.** The category Tel of telescopes is the full subcategory of Filt consisting of telescopes.

This should cause no confusion. Observe that the functor Tel factors through the telescope category Tel; also that $\text{Tel}(X_n)$ is the full subcategory of $\text{Top}^N$ so that cylinders may be formed within Tel. We therefore let $\text{Ho}(\text{Tel})$ be the full subcategory of telescopes in $\text{Ho}(\text{Filt})$.

(3.7.4) **Proposition.** Let $\{f_n\}:\{X_n\} \rightarrow \{Y_n\}$ be a weak equivalence in $\text{Top}^N$. Then $\text{Tel}(f_n)$ is invertible in $\text{Ho}(\text{Tel})$.

**Proof.** Observe that $\text{Tel}(X_n)$ is a strong deformation retract in Tel of the mapping cylinder

$$\text{Map}(\text{Tel}(f_n)) = \text{Tel}\{\text{Map}(f_n)\},$$

$$= \text{Tel}(X_n \times [0,1] \cup Y_n / (x,1) \sim f(x)).$$

We thus assume that $\{f_n\}$ is also a cofibration in $\text{Top}^N$.

For each $n$ choose a retraction $r_n:Y_n \rightarrow X_n$, and a homotopy $h_n:Y_n \times I \rightarrow Y_n$ with $h_n|_0 = \text{id}$ and $h_n|_1 = f_n r_n$.

**Caution:** in general

$\text{bond} r_{n+1} \neq r_n \circ \text{bond}$, and

$\text{bond} h_{n+1} \neq h_n \circ \text{bond}$.

However,

(3.7.5) $\text{bond} r_{n+1} \circ f_{n+1} = \text{bond}$

$= r_n \circ f_n \circ \text{bond}$

$= r_n \circ \text{bond} \circ f_{n+1}$, and

(3.7.6) $\text{bond} h_{n+1} \circ (f_{n+1} \times \text{id})$

$= h_n \circ \text{bond} \circ (f_{n+1} \times \text{id})$

$x_{n+1} \times [0,1] \rightarrow Y_n.$
We shall now use (3.6.5) to define a filtered map
g: Tel\{Y_n\} \to Tel\{X_n\}, which will be shown to be a filtered-homotopy-inverse to Tel\{f_n\}. Because the map \(f_{n+1}\) is a trivial cofibration, \(Y_{n+1} \times [0, 1] \cup X_{n+1} \times 1 \) is a strong deformation retract of \(Y_{n+1} \times [0, 1]\) for \(n = 0\). Let \(\rho_{n+1}\) be the retraction. The composite map, to be denoted \(\nu_{n+1}\)

\[
(3.7.7) \quad Y_{n+1} \times [0, 1] \xrightarrow{\rho_{n+1}} Y_{n+1} \cup X_n \times [0, 1] \cup Y_{n+1} \times 1
\]

(see (3.5)) yields a homotopy from \(\rho_{n} \circ \text{bond} \circ Y_{n+1}\). Now define a map \(g: Tel\{Y_n\} \to Tel\{X_n\}\) as follows:
g maps \(Y_{n+1} \times [n + 1/2, n+1]\) to \(X_{n+1} \times [n, n+1]\) according to the formula

\[
(3.7.8) \quad g(y, t) = (\rho_{n+1}(y), 2t-n-1);
\]
g maps \(Y_{n+1} \times [n, n+1/2]\) to \(X_n \times n\) according to the formula

\[
(3.7.9) \quad g(y, t) = (K_{n+1}(y, 2t-2n), n);
\]
and \(g\) maps \(Y_n \times n\) to \(X_n \times n\) according to the formula

\[
(3.7.10) \quad g(y, t) = (\rho_n(y), n).
\]
Then \(g\) is a filtered map. Schematically,
**Claim 1:** The maps \( gf \) and \( \text{id}_{\text{Tel}(X_n)} \) are filtered-homotopic. To check this, first observe that

\[
K_{n+1} \overset{X_{n+1} \times [0,1]}{\longrightarrow} X_{n+1} \times [0,1]
\]

is the projection onto \( X_{n+1} \) by (3.6.5). Thus

\[
(3.7.11) \quad gf(x, t) = \begin{cases} (x, 2t - n - 1), & n + 1/2 \leq t \leq n + 1, \\ (x, n), & n \leq t \leq n + 1/2 \end{cases}
\]

by (3.6.8)-(3.6.10). Claim 1 follows easily.

**Claim 2:** The maps \( fg \) and \( \text{id}_{\text{Tel}(Y_n)} \) are filtered-homotopic. We shall use (3.6.6) to imitate the construction of \( g \) and obtain the required homotopy. As above,

\[
X_{n+1} \times [0,1] \times [0,1] \cup Y_{n+1} \times \text{bd}([0,1] \times [0,1])
\]

is a strong deformation retract of \( Y_{n+1} \times [0,1] \times [0,1] \). (\([0,1] \) is just a second unit interval.) We may therefore define a homotopy

\[
\Gamma_{n+1} \text{ from } K_{n+1} \text{ to bond rel } X_{n+1} \times [0,1] \times [0,1] \cup Y_{n+1} \times \text{bd}([0,1] \times [0,1])
\]

Schematically,
Gluing these maps together yields a filtered homotopy

\[ \Gamma : \text{Tel}\{Y_n\} \times [0,1] \rightarrow \text{Tel}\{Y_n \times [0,1]\} \rightarrow \text{Tel}\{Y_n\} \]

from \(fg\) to a map \(h : \text{Tel}\{Y_n\} \rightarrow \text{Tel}\{Y_n\}\) which changes only vertical coordinates, and moves those coordinates at most 1/2 unit as in (3.6.11). Details are analogous to those in Claim 1 and are omitted. Claim 2 follows. \(\square\)

(3.7.12) Corollary. The functor \(\text{Tel} : \text{Top}^N \rightarrow \text{Ho}(\text{Tel})\)
factors through \(\text{Ho}(\text{Top}^N)\) to induce

\(\text{Tel} : \text{Ho}(\text{Top}^N) \rightarrow \text{Ho}(\text{Tel})\).

Proof. By the model structure, each map in \(\text{Ho}(\text{Top}^N)\) may be represented by a diagram

\[ \{X_n\} \rightarrow \{Y_n\} \]

\[ \{i_n\} \]

\[ \{Y'_n\} \]
where \( \{ Y_n \} \) is fibrant in \( \text{Top}^N \) (i.e., a tower of fibrations) and \( \{ j_n \} \) is a (levelwise) weak equivalence. (Note: all objects in \( \text{Top}^N \) are cofibrant). Any two such maps are homotopic with respect to the cylinder \(-X[0,1]\). Further, a weak equivalence between fibrant objects in \( \text{Top}^N \) admits a homotopy inverse with respect to \(-X[0,1]\). The conclusion follows by proposition (3.6.4). \( \square \)

We shall now extend \( \text{Tel} \) to \( \text{Ho}(\text{Top, tow-Top}) \).

(3.7.13) **Proposition.** Suppose that \( \{ x_{n_k} \} \) is a cofinal subtower of \( \{ x_n \} \) with \( x_{n_0} = x_0 \). Then there is a natural equivalence \( \text{Tel}(x_n) \to \text{Tel}(x_{n_k}) \) in \( \text{Ho}(\text{Tel}) \).

**Proof.** The required map \( \text{Tel}(x_n) \to \text{Tel}(x_{n_k}) \) is defined by mapping \( x_{n_k} \times x \) to \( x_{n_k} \times x \) under the identity and extending to \( \text{Tel}(x_n) \) as illustrated.

This map is easily seen to be a filtered homotopy equivalence. \( \square \)
(3.7.14) **Proposition.** (a) The functor $\text{Tel}: \text{Top}^N \rightarrow \text{Ho(Tel)}$ factors through $(\text{Top}, \text{tow}-\text{Top})$.

(b) This functor factors through $\text{Ho(Top}, \text{tow}-\text{Top})$, to induce

$$\text{Tel}: \text{Ho(Top}, \text{tow}-\text{Top}) \rightarrow \text{Ho(Tel)}.$$

**Proof.** Part (a) follows from Proposition (3.6.13) because each map in $(\text{Top}, \text{tow}-\text{Top})$ may be represented by a diagram

$$\{x_n\} \Rightarrow \{x^n_k\} \rightarrow \{y^n_k\}$$

where $\{x^n_k\}$ is cofinal in $\{x_n\}$, $x^n_0 = x^0$, and $\{f^n_k\}$ is a level map (i.e. a map in $\text{Top}^N$).

Part (b) follows from Part (a) and the observation that $\{y^n\}$ is fibrant in $(\text{Top}, \text{tow}-\text{Top})$ if $\{y^n\}$ is a tower of fibrations by using the **Proof** of Corollary (3.6.12). □

We are ready to show that $\text{Tel}: \text{Ho(Top}, \text{tow}-\text{Top}) \rightarrow \text{Ho(Tel)}$ is an equivalence.

(3.7.15) **Definition.** The end of a filtered space $X$ is the tower $E(X) = \{x^n\}$ in $(\text{Top}, \text{tow}-\text{Top})$.

Then $E$ extends to a functor

$$E: \text{Tel} \rightarrow (\text{Top}, \text{tow}-\text{Top}).$$

The definition of $\text{Ho(Tel)}$ implies the following.

(3.6.16) **Proposition.** $E$ induces a functor

$$E: \text{Ho(Tel)} \rightarrow \text{Ho(Top}, \text{tow}-\text{Top}).$$

(3.7.17) **Definitions.** Let $X = \{x^n\} \in \text{Top}^N$. Let $p^*: E^* \text{Tel}(X) \rightarrow X$

be the map on $\text{Top}^N$ given by letting $p^n|_{x^n_k}^{x^n_{k+l},k}$ be the composite
\[ x_k \xrightarrow{[x-1,k]} x_k \xrightarrow{\text{bond}} \cdots \xrightarrow{\text{bond}} x_n \]

for \( k > n \).

Let

\[ q = \text{Tel}(p) \circ (\text{Tel} \circ E)(\text{Tel}(X)) \xrightarrow{} \text{Tel}(X). \]

(3.7.18) **Proposition.** (a) The maps \( p \) are natural weak equivalences in \( \text{Ho(Top,tow-Top)} \).

(b) The maps \( q \) are natural weak equivalences in \( \text{Ho(Tel)} \).

**Proof.** Part (a) follows immediately from the definitions of \( \text{Tel}, E, \) and \( p \) (note that each \( p_n \) is an equivalence in \( \text{Ho(Top)} \)). Part (b) follows from Part (a), the definition of the telescope category and proposition (3.6.14b).

Propositions (3.6.14b), (3.6.16), and (3.6.18) imply the following.

(3.7.19) **Theorem.** The categories \( \text{Ho(Top,tow-Top)} \) and \( \text{Ho(Tel)} \) are naturally equivalent. \( \square \)

A geometric model for \( \text{Ho(tow-Top)} \) may be obtained as follows. We may assume that all towers \( X = \{X_n\} \) satisfy \( X_0 = \ast \). This embeds \( \text{tow-Top} \) in \( \text{(Top,tow-Top)} \) (regard \( X \) as \( (\ast, X) \)), and gives rise to a full subcategory of \( \text{Tel, ContTel} \) (contractible telescopes) consisting of those telescopes \( \text{Tel}((\ast = X_0 \leftarrow X_1 \leftarrow \ldots)) \).

(3.7.20) **Corollary.** The categories \( \text{Ho(tow-Top)} \) and \( \text{Ho(ContTel)} \) are naturally equivalent.
(3.7.21) Remarks. In R. Vogt's [Vogt] approach to the homotopy theory of categories of diagrams, $\text{Tel} \{ X_n \}$ is the homotopy colimit of the diagram $\{ X_0 \leftarrow X_1 \leftarrow \ldots \}$. In this setting our homotopy category $\text{Ho}(\text{Tel})$ represents "coherent pro-homotopy", that is, a version of pro-homotopy where maps and homotopies are required to satisfy various coherency conditions. The development of coherent pro-homotopy theory (Vogt only works with level maps) is an interesting problem whose solution should have applications to proper homotopy theory (see §6), and shape theory, especially in alternative proofs of the Chapman [Chap-1] complement theorem (see §8).
§4. THE HOMOTOPY INVERSE LIMIT AND
ITS APPLICATIONS TO HOMOLOGICAL ALGEBRA

§4.1. Introduction.

In this chapter we shall define a homotopy inverse limit functor
\[ \text{holim}: \text{Ho}(\text{pro-}C) \longrightarrow \text{Ho}(C) \]
adjoint to the inclusion functor
\[ \text{Ho}(C) \longrightarrow \text{Ho}(\text{pro-}C) \]
for suitable closed model categories \( C \) (§4.2) and obtain applications to the study of the derived functors \( \lim^s \) of the inverse limit. Bousfield and Kan [B-K; Chapter XI] gave a less intuitive construction of a (somewhat different) homotopy inverse limit on \( \text{pro-SS} \) (see §§4.2, 4.9), and suggested the study of homotopy inverse limits on other \( \text{pro-} \) categories. §4.3 is an appendix to §4.2.

In §4.4 we survey some of the results of other authors, notably Z. Z. Yeh, J.-E. Roos, J.-L. Verdier, C. Y. Jensen, and B. Osofsky, on \( \lim^s \) as background for our own work. Our results are stated in §4.5 and proved in the following sections: §4.6, Algebraic description of \( \lim^s \); §4.7, Topological description of \( \lim^s \); §4.8, Vanishing theorems for \( \lim^s \).

In this chapter, \( C \) shall denote a closed model category which satisfies Condition \( N \) (§2.3) and admits arbitrary (inverse) limits. In particular, we shall discuss \( C = \text{SS}, \text{SSG} \) (simplicial groups) and \( \text{SSAG} \) (simplicial abelian groups).
§4.2 The homotopy inverse limit.

In this section we shall define a homotopy inverse limit functor

$$\text{holim}: \text{Ho}(\text{pro-}C) \longrightarrow \text{Ho}(C)$$

adjoint to the inclusion $\text{Ho}(C) \longrightarrow \text{Ho}(\text{pro-}C)$. That is, for

$X$ in $C$ and $\{Y_j\}$ in $\text{pro-}C$,

$$(4.2.1) \quad \text{Ho}(\text{pro-}C)(X, \{Y_j\}) \cong \text{Ho}(C) (X, \text{holim} \{Y_j\}).$$

We begin by considering formula (4.2.1) in the case that $X$ is cofibrant in $C$, and hence in $\text{pro-}C$, and $\{Y_j\}$ is fibrant in $\text{pro-}C$.

$$(4.2.2) \quad \text{Proposition. The inverse limit functor}$$

$$\text{lim} : (\text{pro-}C)_{\text{cf}} \longrightarrow C \quad \text{induces a functor on the homotopy categories}$$

$$\text{lim} : \text{Ho((pro-}C)_{\text{cf}}) \longrightarrow \text{Ho}(C).$$

$\text{Proof.}$ To define $\text{lim}$ on maps in $\text{Ho((pro-}C)_{\text{cf}})$, recall that for $\{X_j\}, \{Y_k\} \in (\text{pro-}C)_{\text{cf}},$

$$(4.2.3) \quad \text{Ho((pro-}C)_{\text{cf}})(\{X_j\},\{Y_k\}) = [[X_j],\{Y_k\}],$$

the set of homotopy classes of maps from $\{X_j\}$ to $\{Y_k\}$ with respect to a cylinder $\{X_j\} \otimes [0,1]$. Let $f,g: \{X_j\} \longrightarrow \{Y_k\}$ be homotopic maps in $\text{pro-}C$, and let $H: \{X_j\} \otimes [0,1] \longrightarrow Y$ be a homotopy with $H_0 = f$ and $H_1 = g$. Applying the inverse limit to the
Now observe that the maps \( i_0 \) and \( i_1 \) in Diagram (4.2.4) are sections to the trivial fibration \( p: \{X_j\} \otimes [0,1] \to \{X_j\} \). Hence, \( \lim i_0 \) and \( \lim i_1 \) are sections to the induced (\( \lim \) preserves trivial fibrations, see Theorem (3.3.4)) trivial fibration \( \lim p: \lim \{X_j\} \otimes [0,1] \to \lim \{X_j\} \). This implies that the maps \( \lim i_0 \) and \( \lim i_1 \) are weak equivalences. By diagram (4.2.5), the maps
\[ \lim f = \lim H \circ \lim i \]
\[ \lim g = \lim H \circ \lim i \]
become equivalent in \( Ho(C) \). \( \square \)

(4.25) **Proposition.** The inverse limit functor
\[ \lim: (pro-C) \rightarrow C \] induces a functor on the homotopy categories.
\[ \lim: Ho((pro-C)_{\bar{f}}) \rightarrow Ho(C). \]

**Proof.** Let \( \{ X_j \} \) be \( pro-C \). Factor the natural map
\[ \phi \rightarrow \{ X_j \} \] as \( \phi \rightarrow \{ x^j_k \} \rightarrow \{ X_j \} \), a cofibration followed by a trivial fibration. Then \( Ho(pro-C)(\{ X_j \},-) = Ho(pro-C)(\{ x^j_k \},-) \);
also, because \( \lim \) preserves trivial fibrations, \( Ho(C)(\lim \{ X_j \},-) \]
\[ = Ho(C)(\lim \{ x^j_k \},-) \]. The conclusion now follows from Proposition (4.22). \( \square \)
(4.2.6) **Proposition.** For $X$ cofibrant in $C$ and $\{Y_j\}$ fibrant in $\text{pro-}C$,

(4.2.7) $\text{Ho}(\text{pro-}C)(X, \{Y_j\}) \cong \text{Ho}(C)(X, \lim \{Y_j\})$.

**Proof.** Formula (4.2.7) follows easily from adjointness of $\lim$ on $\text{pro-}C$,

$$\text{pro-}C(X, \{Y_j\}) = C(X, \lim \{Y_j\})$$

and Proposition (4.2.2). \qed

We are now ready to define the homotopy inverse limit. By Quillen's theory of model categories (see §2.3), the homotopy theory of fibrant objects in $\text{pro-}C$, $\text{Ho}((\text{pro-}C)_f)$ is equivalent to the homotopy theory of $\text{pro-}C$, $\text{Ho}(\text{pro-}C)$. Following Quillen, for each object $X$ in $\text{pro-}C$, factor the map $X \to \text{pt}$ as a trivial cofibration $i_X$ followed by a fibration

(4.2.8) $X \xrightarrow{i_X} \text{Ex}^\infty X \to \ast$
(If $X$ is fibrant, choose $\text{Ex}^\infty X = X$). This construction induces a functor

\[(4.2.9) \quad \text{Ex}^\infty : \text{Ho}(\text{pro-} C) \to \text{Ho}((\text{pro-} C)_f)\]

$\text{Ex}^\infty$ is defined on morphisms by applying Axiom M6 to obtain fillers in the diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow^{i_X} & & \downarrow^{i_Y} \\
\text{Ex}^\infty X & \xrightarrow{\text{Ex}^\infty f} & \text{Ex}^\infty Y.
\end{array}
$$

The homotopy class of $\text{Ex}^\infty f$ depends only on the homotopy class of $f$ because

$$
[\text{Ex}^\infty f] = [i_Y][f][i_X]^{-1}.
$$

We therefore define the homotopy inverse limit $\text{holim}$, to be the composite functor

\[(4.2.10) \quad \text{holim} = \lim \circ \text{Ex}^\infty : \text{Ho}(\text{pro-} C) \to \text{Ho}((\text{pro-} C)_f) \to \text{Ho}(C).\]

\[(4.2.11) \quad \textbf{Remarks}. \quad \text{If } X \in \text{pro-} C \text{ is fibrant, we may take } \text{Ex}^\infty X = X \text{ in (4.2.8). Hence, } \text{holim } X = \lim X \text{ on } (\text{pro-} C)_f.\]

Propositions (4.2.5) and (4.2.6) immediately yield the following.
(4.2.12) **Theorem.** The functor $\text{holim} : \text{Ho}(\text{pro-}C) \to \text{Ho}(C)$ is adjoint to the inclusion $\text{Ho}(C) \to \text{Ho}(\text{pro-}C)$.

(4.2.13) **Remarks.** Bousfield and Kan [B-K, Chapter XI] defined a different "homotopy inverse limit" functor $\text{holim}_{B-K} : \text{Ho}(\text{SS}^J) \to \text{Ho}(\text{SS})$. They observed that for a tower of fibrations $X$, $\text{holim}_{B-K} X \simeq \lim X$. But, for a tower of fibrations $X$, we may take $\text{Ex}^\infty X = X$; hence their definition is equivalent with ours on such systems. In general our definitions differ except on fibrant objects. For example, Bousfield and Kan only obtain the following weak analogue of Theorem (4.2.11): $R\text{holim}_{B-K}$ is adjoint to the inclusion $\text{Ho}(\text{SS}) \to \text{Ho}(\text{SS}^J)$, where $R\text{holim}_{B-K}$ is Quillen's total right derived functor [Q-1,§1.4] of $\text{holim}_{B-K}$. In fact, for our holim, $\text{holim} = R\text{holim} = R\text{lim}$, where $\lim$ is the ordinary inverse limit.

In §4.3 we shall describe an explicit $\text{Ex}^\infty$ functor from $\text{Ho}(\text{pro-SS})$ to $\text{Ho}(\text{(pro-SS)}_f)$.

§4.3. **$\text{Ex}^\infty$ on pro-SS.**

In this section we shall describe an explicit $\text{Ex}^\infty$ functor from $\text{Ho}(\text{pro-SS})$ to $\text{Ho}(\text{pro-SS})_f$, together with natural (in $\text{Ho}(\text{pro-SS})$) trivial cofibrations $X \to \text{Ex}^\infty X$ in pro-SS.
Compare (4.2.8) - (4.2.9).

\[ (4.3.1) \quad \operatorname{Ex}^{\infty}_{\text{on objects. Let } X \in \text{pro-SS. First apply}} \]

the Mardešić construction of $\S 2.1$ which replaces $X$ functorially by $\text{strongly}$ directed set $J = \{j\}$. We thus need only define $\{Z_j\} = \operatorname{Ex}^{\infty}\{X_j\}$ where $J = \{j\}$ is a cofinite $\text{strongly}$ directed set. We shall proceed inductively.

First, suppose that $j$ is an initial object of $J$. Define a fibrant simplicial sets $Z_j$ and a trivial cofibration $X_j \to Z_j$ by functorially factoring the maps $X_j \to \star$ as in [Q, $\S$II.3].

Next, assume inductively that for a given $j$ in $J$, and all $k < j$, that simplicial sets $Z_k$, bonding maps $Z_k \to Z_j$ for $\ell < k$, and trivial cofibrations $X_k \to Z_k$ have been defined so that:

(a) for $\ell < k$ the diagrams

\[
\begin{array}{ccc}
X_k & \longrightarrow & Y_k \\
\downarrow & & \downarrow \\
X_\ell & \longrightarrow & Y_\ell
\end{array}
\]

commute;
(b) for \( m < j < k \) the diagrams

\[
\begin{array}{c}
Z_k \\
\downarrow \\
Z_j \\
\downarrow \\
Z_m
\end{array}
\]

commute; and

(c) The maps \( Z_k \to \lim_{k < j} \{Z_k\} \) induced by (b) are fibrations.

Apply the Quillen factorization to the composite mapping

\[
X_j \to \lim_{k < j} \{X_k\} \to \lim_{k < j} \{Z_k\}
\]

to obtain a diagram

\[
X_j \to Z_j \to \lim_{k < j} \{Z_k\}
\]

consisting of a trivial cofibration followed by a fibration. The map \( Z_j \to \lim_{k < j} \{Z_k\} \) induces bonding maps \( Z_j \to Z_k \) for \( k < j \). It is easy to see that these maps satisfy conditions (a) and (b) above.

Condition (c) above is satisfied by construction.

Continuing inductively yields a fibrant pro-(simplical set)

\[ \{Z_j\} = \Ex^\infty \{X_j\} \]
In order to define $\text{Ex}^\infty$ on morphisms in pro-$SS$, we need the following cofinality lemma.

(4.3.3) Lemma. Let $T: J \to K$ be a cofinal functor on strongly cofinite directed sets. Then for any $\{X_k\}$, $T$ induces an isomorphism $\text{Ex}^\infty X_k \to \text{Ex}^\infty X_{T(j)}$ in $\text{Ho}(\text{pro-SS})$, in other words, the diagram

\[
\begin{array}{ccc}
SS^K & \xrightarrow{T^*} & SS^J \\
\downarrow{\text{Ex}^\infty} & & \downarrow{\text{Ex}^\infty} \\
SS^K & \xrightarrow{\pi} & SS^J \\
\downarrow{\pi} & & \downarrow{\pi} \\
\text{Ho}(\text{pro-SS}) & & \text{Ho}(\text{pro-SS})
\end{array}
\]

commutes up to natural equivalence of functors.

Proof. Let $\{Y_j\}$ denote $\text{Ex}^\infty X_{T(j)}$ and $\{Z_k\}$ denote $\text{Ex}^\infty X_k$. We may assume that $J$ and $K$ have initial elements $j_0$ and $k_0$ with $T(j_0) = k_0$, and also that $X_{k_0} = *$. This yields a natural map $Z_{T(j_0)} \to Y_{k_0}$. Now fix $j'$ in $J$, and assume that the natural maps $Z_{T(j)} \to Y_j$ have been defined for $j < j'$ such that the diagrams
(4.3.4)

\[ Z_{T(j)} \rightarrow Y_j \rightarrow Z_{T(j)} \quad \text{and} \quad X_{T(j)} \rightarrow Z_{T(j)} \]

\[ Z_{T(j'')} \rightarrow Y_{j''} \]

commute. Consider the induced commutative diagram

(4.3.5)

\[ X_{T(j')} \rightarrow \lim_{k < T(j')} \{X_k\} \rightarrow \lim_{k < T(j')} \{Z_k\} \]

\[ \lim_{T(j) < T(j')} \{X_{T(j)}\} \rightarrow \lim_{T(j) < T(j')} \{Z_{T(j)}\} \]

\[ X_{T(j')} \rightarrow \lim_{j < j'} \{X_{T(j)}\} \rightarrow \lim_{j < j'} \{Y_j\} \]

Since \( Z_{T(j')} \) is defined by applying the Quillen factorization to the composite along the top row of diagram (4.3.5), namely the map

\[ X_{T(j')} \rightarrow \lim_{k < T(j')} \{Z_k\}, \quad \text{and} \quad Y_j, \]

is defined by applying the Quillen factorization to the composite along the bottom row of (4.3.5),

\[ X_{T(j')} \rightarrow \lim_{j < j'} \{Y_j\}, \quad \text{there is an induced map} \]

(4.3.6)

\[ Z_{T(j')} \rightarrow Y_{j'}. \]
By Diagram (4.3.5), the map (4.3.6) satisfies the conditions of Diagrams (4.3.4). Continue inductively to define maps \( Z_{T(j)} \to Y_j \) for all \( j \) in \( J \). Diagrams (4.3.4) then yield a commutative diagram in \( SS^J \)

\[
\begin{align*}
\{X_{T(j)}\} & \longrightarrow \{Z_{T(j)}\} = T^* \text{Ex}^\infty \{X_k\} \\
& \downarrow \tau \\
\{Y_j\} = \text{Ex}^\infty T^* \{X_k\} & .
\end{align*}
\]

Since for each \( j \), \( X_{T(j)} = Z_{T(j)} \) and \( X_{T(j)} = Y_j \), the map \( \tau \) in (4.3.7) is a level weak equivalence, hence an isomorphism in \( \text{Ho(pro-SS)} \).

Since the constructions which led to Diagrams (4.3.7) were natural, we have defined the required natural isomorphism

\[
\pi \circ \text{Ex}^\infty \simeq \pi \circ T^* \circ \text{Ex}^\infty \\
\pi \circ \text{Ex}^\infty \circ T^* .
\]

We have thus proven that the construction \( X \to MX \to \text{Ex}^\infty MX \), where \( M \) denotes the Mardešić construction, extends to a functor from \( \text{pro-SS} \) to \( \text{Ho}(\text{pro-SS}_f) \) and that there are trivial cofibrations \( X \to MX \to \text{Ex}^\infty MX \), which are natural in \( \text{Ho(pro-SS)} \). By construction, \( \text{Ex}^\infty M \) factors through \( \text{Ho(pro-SS)} \), as required.

(4.3.4) \textbf{Remarks.} The only special property of \( SS \) which we needed is existence of functorial factorizations in Axiom M2 (5.2.3).
4.4. The derived functors of the inverse limit: background.

In this section we shall briefly summarize vanishing theorems and cofinality theorems for right-derived functors \( \lim^S \) of the inverse limit

\[
\lim: (AG)^J \to AG
\]

(where \( AG \) is the category of abelian groups). Because \( \lim \) is left-exact, but not, in general, right exact, vanishing theorems measure the exactness of \( \lim^S \). Cofinality theorems are used to extend \( \lim^S \) to \( \text{pro-} AG \).

There is a close relationship between these results and the homological dimension of modules which we shall not discuss here; see B. L. Osofsky [Osof] for a good survey. We refer the reader to [Mit-2], for example, for the basic theory of derived functors due to H. Cartan and S. Eilenberg [C-E].


(4.4.1) Definition (Yeh, Roos). An inverse system \( \{G_i\} \) indexed by a directed set \( I \) is called flasque (or star-epimorphic) if for each ordered (not necessarily directed) set \( J \) contained in \( I \), the natural map \( \lim_{i \in I} \{G_i\} \to \lim_{i \in J} \{G_i\} \) is surjective.
(4.4.2) **Proposition** (Yeh, Roos). If \( \{G_i\} \) is flasque, then
\[
\lim^s \{G_i\} = 0 \quad \text{for } s \geq 1.
\]

Roos obtains this result by showing that \( \lim^s \{G_i\} \) is the homology of a complex obtained from \( \{G_i\} \). Compare J. Milnor's [Mil-3] use of \( \lim \) and \( \lim^1 \) in axiomatizing the cohomology of infinite CW complexes. In fact, Roos only requires that \( I \) be ordered, not necessarily directed.

(4.4.3) **Definition** (Jensen). An inverse system \( \{G_i\} \) indexed by a directed set \( I \) is called weakly flasque if for each directed set \( J \) contained in \( I \), the natural map \( \lim_{i \in I} \{G_i\} \to \lim_{i \in J} \{G_i\} \) is surjective.

(4.4.4) **Proposition** (Jensen). If \( \{G_i\} \) is weakly flasque, then \( \lim^s \{G_i\} = 0 \quad \text{for } s \geq 1. \)

Jensen first proves that \( \lim^s \) applied to a weakly flasque system is 0 by showing that \( \lim \) is right-exact on such systems. Vanishing of \( \lim^s \) then follows by a suitable iteration.

(4.4.5) **Remarks.**

(a) Clearly flasque implies weakly flasque. Jensen observed that the converse is false.
(b) Roos and Jensen actually worked in categories of the form $A^I$, where $A$ is an abelian category which satisfies suitable exactness axioms of A. Grothendieck [Gro-2].

One may extend $\lim$ and $\lim^s$ from $(AG)^J$ to $pro-AG$ as follows. First, a cofinal functor $T: J \to K$ induces an isomorphism $T^*: \{G_k\} \to \{G_{T(j)}\}$ in $pro-AG$, hence we need the following result of Roos (1961), Jensen (1972) and B. Mitchell [Mit-2] (1973).

(4.4.6) **Theorem** (Mitchell). A cofinal functor $T: J \to K$ between filtering categories induces commutative diagrams

$$
\begin{array}{ccc}
(AG)^K & \xrightarrow{\lim^s} & \lim^s \\
| & \downarrow{T^*} & \\
(AG)^J & \xrightarrow{\lim^s} & \lim^s \\
\end{array}
$$

Secondly, although Artin and Mazur [A-M, A.3] (see 2.1 above) gave a natural representation of a map $\{G_j \mid j \in J\} \to \{H_k \mid k \in K\}$ in $pro-AG$ by a level map $\{G_j(\ell) \to H_k(\ell)\}$ in some $(AG)_L$, a given map in $pro-AG$ may have many level representatives. Therefore we need to show that commutative diagram
in pro-\(\text{AG}\), where \(L = \{\ell\}\), and \(M = \{m\}\) are filtering categories, induces commutative diagrams

\[
\begin{align*}
\lim_L^s \{G_j(\ell)\} & \longrightarrow \lim_L^s \{H_k(\ell)\} \\
\Downarrow \quad \cong \quad \Downarrow \\
\lim_M^s \{G_j(m)\} & \longrightarrow \lim_M^s \{H_k(m)\}.
\end{align*}
\]

J.-L. Verdier [Ver] announced this result in 1965, and hence extended the inverse limit functor and its derived functors to pro-\(\text{AG}\).

In particular, the diagrams

\[
\begin{align*}
\text{(AG)}^J & \longrightarrow \lim_j^s \longrightarrow \text{AG} \\
\Downarrow & \\
\text{pro-\(\text{AG}\)} & \longrightarrow \lim_{\text{pro}}^s
\end{align*}
\]
commute, where $\lim^s_{\text{pro}}$ is the $s$th right derived functor of $\lim_{\text{pro}}$. We shall include independent proofs of these results, see §4.5.

(4.4.7) \textbf{lim and HOM functors.} Let $R$ be a commutative ring, let $R\text{-Mod}$ be the category of $R$-modules. Let $\text{HOM}$ denote the internal mapping functor on $R\text{-Mod}$, that is, $\text{HOM}(X,Y)$ is the natural $R$-module with $R\text{-Mod}(X,Y)$ as underlying set. In 1973 B. L. Osofsky [Osof] gave the following representation of the functors $\lim^s:(R\text{-Mod})^J \longrightarrow R\text{-Mod}$. She defined a "tensor product" ring $R \otimes J$ such that the categories $(R\text{-Mod})^J$ and $(R \otimes J)\text{-Mod}$ are isomorphic. This isomorphism induces natural equivalences of functors

\[
\lim: (R\text{-Mod})^J \longrightarrow R\text{-Mod}
\]

\[(\equiv \text{HOM}_{(R\text{-Mod})^J}(R, \lim(-)): (R\text{-Mod})^J \longrightarrow R\text{-Mod})
\]

\[\equiv \text{HOM}_{(R \otimes J\text{-Mod})^J}(R \otimes I, -): R \otimes G\text{-Mod} \longrightarrow R\text{-Mod},
\]

hence also

\[\lim^s: (R\text{-Mod})^J \longrightarrow R\text{-Mod}
\]

\[\equiv \text{Ext}_{(R \otimes J)\text{-Mod}}^s(R \otimes J, -): (R \otimes J\text{-Mod}) \longrightarrow R\text{-Mod}.
\]
Later in this chapter we shall use pro-categories to give more natural results. See §4.5.

§4.5. Results on derived functors of the inverse limit.

We shall prove the following three theorems in §4.6.

**Theorem A.** Let $J$ be a cofinite strongly directed set, and let $\text{HOM}_J^\text{pro}, \text{Ext}_J^\text{pro}$, $\text{Ext}_J$ and $\text{Ext}_J^\text{pro}$ be the appropriate $\text{HOM}$ and $\text{Ext}$ functors. Then

(a) $\lim_J = \text{HOM}_J^J(Z,-) : \mathcal{A}G^J \to \mathcal{A}G$.

(b) $\lim_{\text{pro}} = \text{HOM}_J^\text{pro}(Z,-) : \mathcal{P}ro- \mathcal{A}G \to \mathcal{A}G$.

(c) $\lim_J^s = \text{Ext}_J^s(Z,-) : \mathcal{A}G^J \to \mathcal{A}G$.

(d) $\lim_{\text{pro}}^s = \text{Ext}_J^\text{pro}(Z,-) : \mathcal{P}ro- \mathcal{A}G \to \mathcal{A}G$.

**Theorem B.** Let $J$ be a cofinite strongly directed set. Then

the diagrams

\[
\begin{array}{ccc}
\mathcal{A}G^J & \xrightarrow{\pi} & \mathcal{P}ro- \mathcal{A}G \\
\downarrow{\lim_J^s} & & \downarrow{\lim_{\text{pro}}^s} \\
\mathcal{A}G & & \mathcal{A}G
\end{array}
\]

commute up to natural equivalence.
Theorem C. Let \( \{G_j\} \) be a stable pro-group. Then

\[
\lim_s^J \{G_j\} = \lim_{\pro}^s \{G_j\} = \lim_{\pro}^s (\lim \{G_j\}) = \begin{cases} 
\lim \{G_j\}, & s = 0, \\
0, & s \neq 0
\end{cases}
\]

In §4.7 we shall use the relationship between the topological and algebraic structures on \( \pro-\text{SSAG} \) to prove the following.

Theorem D. Let \( \{G_j\} \) be an inverse system of free abelian groups and \( \{H_k\} \in \pro-\text{AG} \). Then

\[
\Ext^S(\{G_j\}, \{H_k\}) = \Ho(\pro-\text{SSAG})(\{G_j\}, \{W^S H_k\})
\]

where \( W \) is the \( W \)-construction of Eilenberg and MacLane (see (4.7.9)).

Theorems A and D imply the following.

**Theorem E.** \( \lim_s^J \{G_j\} = \Ho(\pro-\text{SSAG})(Z, \{W^S G_j\}) = \pi_0(W^S G_j) \) on \( \pro-\text{AG} \). An analogous formula holds on \( \pro-G \) for \( s = 0 \) and \( 1 \).

**Theorem F.** \( \lim_s^J \{G_j\} = \Ho(\pro-\text{Sp})(S, \{K_{G_j}\}) \) where \( \text{Sp} \) denotes the category of simplicial spectra and \( K_{G_j} \) the simplicial Eilenberg-MacLane spectrum with

\[
\pi_n(K_{G_j}) = \begin{cases} 
G_j, & n = 0, \\
0, & n \neq 0
\end{cases}
\]

In §4.8 we shall define strongly-Mittag-Leffler pro-groups and prove the following.

**Theorem G.** Let \( \{G_j\} \) be a strongly-Mittag-Leffler pro-group.

Then,

(a) \( \lim^1 \{G_j\} = 0 \);

(b) If \( \{G_j\} \) is abelian, then \( \lim^s \{G_j\} = 0 \) for \( s > 0 \);

(c) If \( \lim \{G_j\} = 0 \), then \( \{G_j\} \cong 0 \) in \( \text{pro-}G \).

§4.6. **Algebraic description of \( \lim^s \).**

In this section we shall interpret the inverse limit functors

\[
\lim^J : \text{AG}^J \longrightarrow \text{AG} \quad \text{and} \quad \lim^\text{pro} : \text{pro-} \text{AG} \longrightarrow \text{AG}
\]

as suitable HOM functors. This will show that \( \lim^\text{pro} \) extends \( \lim^J \), and also identify the derived functors \( \lim^s \) as the derived functors \( \text{Ext}^S \) of HOM. Compare the results of B. Osofsky described in (4.4.7). Our results can be extended easily to categories of modules over a commutative ring with identity. We shall need the following structure.

(4.6.1) **Theorem.**
(a) \( AG^J \) is an abelian category.

(b) pro-\( AG \) is an abelian category.

Part (a) is well-known. For part (b), see [A-M, A.4]. Some of the abelian structure is described below.

(4.6.2) Definition. A sequence \( 0 \to \{A_j\} \to \{B_j\} \to \{C_j\} \to 0 \) in \( AG^J \) is exact if for each \( j \) in \( J \), the sequence
\[
0 \to A_j \to B_j \to C_j \to 0
\]
is exact.

(4.6.3) Proposition. A sequence
\[
0 \to \{A_j\} \to \{B_k\} \to \{C_k\} \to 0
\]
in pro-\( AG \) is exact if and only if there exists a commutative diagram
\[
\begin{array}{ccc}
0 & \to & \{A_j\} \\
\downarrow & & \downarrow \\
0 & \to & \{B_k\}
\end{array}
\]
\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
0 & \to & \{C_k\}
\end{array}
\]

in which the bottom row is a short exact sequence in the appropriate level category \( AG^M \), \( M = \{m\} \).

Proof. The "if" part is clear. For the "only if" part, first reindex the given short exact sequence to obtain a diagram.
\[ 0 \rightarrow \{ A_j \} \rightarrow \{ B_k \} \rightarrow \{ C_\ell \} \rightarrow 0 \]

\[(4.6.5) \]

\[ 0 \rightarrow \{ A_{j(m)} \} \rightarrow \{ B_{k(m)} \} \rightarrow \{ C_{\ell(m)} \} \rightarrow 0 \]

in which the rows are exact, and the bottom row consists of level maps.

Let \( M = \{ m \} \), and let \( A'_m \) be the kernel of the map \( B_{k(m)} \rightarrow C_{\ell(m)} \). Then the map \( \{ A'_m \} \rightarrow \{ B_{k(m)} \} \) is a kernel of the map \( \{ B_{k(m)} \} \rightarrow \{ C_{\ell(m)} \} \) in \( \text{pro-} \mathcal{A}_G \), hence we may replace the bottom row of Diagram (4.6.5) by the isomorphic short exact sequence

\[(4.6.6) \quad 0 \rightarrow \{ A'_m \} \rightarrow \{ B_{k(m)} \} \rightarrow \{ C_{\ell(m)} \} \rightarrow 0 . \]

Similarly, let \( C'_m \) be the cokernel of the map \( A'_m \rightarrow B_{k(m)} \).

There results a short exact sequence in \( \mathcal{A}_G^M \),

\[ 0 \rightarrow \{ A'_m \} \rightarrow \{ B_{k(m)} \} \rightarrow \{ C'_m \} \rightarrow 0 , \]

which is isomorphic to (4.6.6), via reindexing in the middle term, as required. \( \square \)

(4.6.7) Remarks. Proposition (4.6.3) can easily be extended to finite diagrams of short exact sequences without loops; see §2.1.
(4.6.8) **Definitions.** Direct sums in $A^J$ are defined degree-wise. We may define direct sums in pro-$A$ as follows. Given
\{X_j\} and \{Y_k\} indexed by $J$ and $K$ respectively; let
\{X_j\} + \{Y_k\} = \{X_j + Y_k\}, indexed over the product category
$J \times K$. If $J$ and $K$ are directed sets, $(j,k) \leq (j',k')$ if
$j \leq j'$ and $k \leq k'$.

The internal mapping functor $\text{HOM}$ on $A$ also extends to
$A^J$ and pro-$A$.

(4.6.9) **Definitions.** Given \{X_j\} and \{Y_j\} in $A^J$,
declare $\text{HOM}_j(\{X_j\},\{Y_j\}) = A^J(\{X_j\},\{Y_j\})$, with group operations
induced from \{Y_j\}. Similarly, given \{X_j\} and \{Y_k\} in pro-$A$,
let $\text{HOM}_\text{pro}(\{X_j\},\{Y_k\}) = \text{pro-$A$}(\{X_j\},\{Y_k\})$, with group operations
induced from \{Y_k\}.

Then $\text{HOM}_j$ and $\text{HOM}_\text{pro}$ may be extended to functors.

Because the inverse limit $\lim_j : A^J \to A$ (respectively,
$\lim_{\text{pro}} : \text{pro-$A$} \to A$) is adjoint to the inclusion $A \to A^J$
(respectively, $A \to \text{pro-$A$}$), and $\text{HOM}(Z,X) = X$ for $X$ in $A$,
we have the following.

(4.6.10) **Theorem.**

(a) $\lim_j = \text{HOM}_j(Z,-) : A^J \to A$. 
(b) \( \lim_{\text{pro}} = \text{HOM}_{\text{pro}}(Z, -) : \text{pro} - \text{AG} \to \text{AG} \).

(4.6.11) **Corollary.** \( \lim_{J} = \lim_{\text{pro}} \circ \pi : \text{AG}^{J} \to \text{AG} \).

(4.6.12) **Corollary.**

(a) \( \lim_{J}^{S} = \text{Ext}_{J}^{S}(Z, -) : \text{AG}^{J} \to \text{AG} \).

(b) \( \lim_{\text{pro}}^{S} = \text{Ext}_{\text{pro}}^{S}(Z, -) : \text{pro} - \text{AG} \to \text{AG} \).

**Proof.** In each case, \( \lim^{S} \) and \( \text{Ext}^{S} \) are the \( S \)th derived functors of isomorphic functors. Now apply D. Buchsbaum's characterization of derived functors [Buch] [see also, [Mit-1, p. 193]. The conclusion follows.

The above theorem and corollaries form Theorem A. We shall now define a natural transformation of connected sequences of functors

\( \tau : \{\text{Ext}_{J}^{S}\} \to \{\text{Ext}_{\text{pro}}^{S}\} \), and show that if \( J \) is a cofinite directed set, then \( \tau \) induces an isomorphism of connected sequences of functors

\( \tau : \{\text{Ext}_{J}^{S}(Z, -)\} \to \{\text{Ext}_{\text{pro}}^{S}(Z, -)\} \).

(4.6.13). **Construction of \( \tau \).** Because the natural quotient functors \( \pi : \text{AG}^{J} \to \text{pro} - \text{AG} \) preserve abelian structures, they map an extension (long exact sequence) in \( \text{AG}^{J} \) into an extension in \( \text{pro} - \text{AG} \), and send a map of extensions in \( \text{AG}^{J} \) into a map of extensions in
pro-AG. Therefore, \( \pi \) induces the required natural transformation of connected sequences of functors \( \tau: \{\text{Ext}_j^S\} \to \{\text{Ext}_{\text{pro}}^S\} \).

\[\text{(4.6.14) Theorem.} \quad \text{Let } J \text{ be a cofinite directed set. Then the diagrams} \]

\[
\begin{array}{ccc}
AG^J & \xrightarrow{\pi} & \text{pro-AG} \\
\downarrow & & \downarrow \\
\text{Ext}_j^S(Z,-) & \xrightarrow{\text{strongly}} & \text{Ext}_{\text{pro}}^S(Z,-) \\
AG & & \\
\end{array}
\]

\text{commute up to natural equivalence.}

\textbf{Proof.} For } s = 0, \text{ this follows from (4.6.10) - (4.6.13). The conclusion now follows by Buchsbaum's characterization of derived functors [Buch], or by the following alternative direct proof for } s > 0.

\text{We shall first show that } \tau: \text{Ext}_j^1(Z, \{G_j\}) \to \text{Ext}_{\text{pro}}^1(Z, \{G_j\}) \text{ is an epimorphism for each cofinite directed set } J \text{ and } \{G_j\} \text{ in } AG^J.

Consider an extension in pro-AG

\[\text{(4.6.15)} \quad 0 \to \{G_j\} \to \{H_k\} \to Z \to 0.\]

By the Mardešić construction, §2.1, we may assume that \( K \) as well as \( J \) is a cofinite directed set. We shall reindex (4.6.15) several
times. First, replace the monomorphism \( \{G_j\} \to \{H_k\} \) by an inverse system of monomorphisms \( \{G_j(\ell)\} \to \{H'_k\} \), and take the levelwise cokernel \( \{C_k\} \) to obtain

\[
(4.6.16) \quad 0 \to \{G_j(\ell)\} \to \{H'_k\} \to \{C_k\} \to 0
\]

Again, we may assume that \( L = \{\ell\} \) is a cofinite directed set. Because \( J \) and \( L \) are cofinite directed sets, we may choose elements \( \ell(j) \) for each \( j \) in \( J \) so that \( j \leq j' \) implies \( \ell(j) \leq \ell(j') \) and \( \{j(\ell(j))\} \) is cofinal in \( J \). We now replace (4.6.16) by

\[
(4.6.17) \quad 0 \to \{G_j(\ell(j))\} \to \{H'_j(\ell(j))\} \to \{C_j(\ell(j))\} \to 0.
\]

Now use the maps \( G_j(\ell(j)) \to G_j \) to push out (4.6.17) and obtain

\[
(4.6.18) \quad 0 \to \{G_j\} \to \{H''_j\} \to \{C'_j\} \to 0.
\]

Finally, the map \( Z = \{Z_j = Z\} \to \{C'_j\} \) is a pro-isomorphism, so pulling back (4.6.18) by this map yields the required extension

\[
(4.6.19) \quad 0 \to \{G_j\} \to \{H''_j\} \to \{Z_j\} \to 0
\]

isomorphic to (4.6.15). Hence, \( \tau : \Ext^1_J(Z, \{G_j\}) \to \Ext^1_{\text{pro}}(Z, \{G_j\}) \) is an epimorphism. Similar techniques imply that for each cofinite strongly directed set \( J \) and \( \{G_j\} \) in \( AG^J \),
\( \tau: \text{Ext}_j^s(Z, \{G_j\}) \to \text{Ext}_{\text{pro}}^s(Z, \{G_j\}) \) is an isomorphism for all \( s \).

The crucial point is that \( Z \) is a constant inverse system. \( \square \)

Because we have already identified \( \lim^s \) as \( \text{Ext}^s \), Theorem (4.6.14) implies the following.

\[ \begin{array}{c}
\text{(4.6.20) Theorem B. Let } J \text{ be a cofinite directed set.} \\
\text{Then the diagrams}
\end{array} \]

\[ \begin{array}{ccc}
\text{AG} & \xrightarrow{\pi} & \text{pro-AG} \\
\downarrow \text{lim}^s_J & & \downarrow \text{lim}^s_{\text{pro}} \\
\text{AG} & \xrightarrow{\text{lim}^s_J} & \text{AG}
\end{array} \]

commute up to natural equivalence. \( \square \)

Hence we shall write \( \lim^s \) for \( \lim^s_J = \lim^s_{\text{pro}} \).

\[ \begin{array}{c}
\text{(4.6.21) Definition. An inverse system of groups } \{G_j\} \text{ is called stable if it is isomorphic in } \text{pro-}G \text{ to a group.}
\end{array} \]

If \( \{G_j\} \) is stable, the natural map \( \lim \{G_j\} \to \{G_j\} \) is an isomorphism in \( \text{pro-}G \) by functoriality of \( \lim \). Theorem B then immediately implies the following vanishing theorem. (If \( \{G_j\} \) is not abelian, everything works for \( s = 0 \) or \( 1 \). See, for example, [B-K] for \( \lim^1 \) of an inverse system of non-abelian groups. In
this case, $\lim^1$ is only a pointed set.)

\[ (4.6.22) \textbf{Theorem C.} \text{ Let } \{G_j\} \text{ be a stable pro-group. Then} \]

\[ \lim^s_j \{G_j\} = \lim^{s \text{ pro}}_j \{G_j\} = \lim^{s \text{ pro}}_j (\lim \{G_j\}) = \begin{cases} \lim \{G_j\}, & s = 0 \\ 0, & s \neq 0 \end{cases} \]

§4.7. **Topological description of** $\lim^s_j$.

In the last section we showed that

\[ \lim^1_j \{G_j\} = \text{Ext}^1(Z, \{G_j\}) \]

where $J = \{j\}$ is a cofinite directed set. Because $\text{Ext}^1(Z, \{G_j\})$ is the set of short exact sequences

\[ 0 \longrightarrow \{G_j\} \longrightarrow \{H_j\} \longrightarrow \{Z\} \longrightarrow 0, \]

and a short exact sequence in $\text{pro-AG}$ is a fibration sequence in $\text{pro-SSAG}$ (see (4.7.1) - (417.13), below), one is led to ask whether

\[ \text{Ext}^1(Z, \{G_j\}) = \text{Ho}(\text{pro-SSAG})(Z, B \{G_j\}) \]

for a suitable classifying space $B\{G_j\}$. We shall carry out the above program in this section. The first step is to relate the abelian structure of $\text{pro-AG}$ and the closed model structure of $\text{pro-SSAG}$. 
(4.7.1) **Definition.** Associate to an abelian group $G$ the discrete simplicial abelian group $SG$ with $(SG)_n = G$ for all $n \geq 0$, and all face and degeneracy maps the identity. Associate to a simplicial abelian group $H$ the abelian group $TH = H_0$, the group of $0$-simplices of $H$.

(4.7.2) **Proposition.** Then $S$ and $T$ extend to functors

$$S : AG \rightarrow SSAG,$$

$$T : SSAG \rightarrow AG,$$

with $S$ coadjoint to $T$, $S$ a full embedding, and $TS = 1_{AG}$.

The proof is easy and omitted.

Now prolong $S$ and $T$ to functors

(4.7.3) $S : pro-AG \rightarrow pro-SSAG$

$T : pro-SSAG \rightarrow pro-AG$

(4.7.4) **Proposition.** $S$ is coadjoint to $T$, $S$ is a full embedding and $TS = 1_{pro-AG}$.

**Proof.** Immediate from Proposition (4.7.2). □

We shall frequently identify $pro-AG$ with its image in $pro-SSAG$ under $S$. Artin and Mazur [A-M, §A.4] showed that $pro-A$ is an abelian category if $A$ is an abelian category. See §4.6 for the case $A = AG$. Consider the following situation.
(4.7.5) SSAG is an abelian category. The required structures, namely 0, kernels, cokernels, and direct sums are defined degreewise. Addition in SSAG(G,H) is defined degreewise. The functors S and T (4.7.2) preserve abelian structures.

(4.7.6) The normalization NG of a simplicial abelian group \( G = \{G^n, d^n_i, s^n_i\} \) is the chain complex \( NG = \{N^n_n G, d^n_n\} \) with

\[
N^n_n G = \begin{cases} 
G_0, & n=0; \\
\ker(d^n_i: G^n \to G^n-1), & n \neq 0,
\end{cases}
\]

\[d^n_n = d^n_0|_{N^n_n G}.
\]

Then N extends to a functor on SSAG. Moore showed that

\[\pi_\ast G = H_\ast(N^n_n G, d^n_n),\]

the homology of the chain complex NG.

(4.7.7) Call a simplicial abelian group G discrete if G is in the image of \( S:AG \to SSAG \), that is, if the only non-degenerate simplices of G have dimension 0. If G is a discrete simplicial abelian group, \( G^n = G_0 \) for all n, and \( d^n_i = 1_{G_0} \) for all n and i; hence

\[
N^n_n G = \begin{cases} 
G_0, & n = 0 \\
0, & n \neq 0
\end{cases}
\]

if G is discrete.
Proposition (4.7.6) yields the following relationship between the abelian and closed model structures of SSAG.

(4.7.7) Proposition (Quillen [Q-1, Proposition II.3.1]). A map $f: G \to H$ is a fibration in SSAG (hence in SS) if and only if $N_n f$ is surjective for $n > 0$.

(4.7.8) Remarks. We shall need two special cases:

(a) Any map of discrete simplicial abelian groups is a fibration;

(b) Any (levelwise) surjection of simplicial abelian groups is a fibration.

The fibre of a fibration $f$ in SSAG is just the (levelwise) kernel of $f$.

(4.7.9) S. Eilenberg and S. MacLane defined a classifying space construction on SSG (see [May, p 21] for a description). To a simplicial abelian group $G$ their construction associates a fibration sequence in SSAG

$$G \to W G \to \bar{W}G;$$

$W G$ is a contractible (in SSAG) simplicial abelian group, and $\bar{W}G \in SSAG$. In fact $W G$ is always a simplicial group, even if $G$ is not abelian. Their construction immediately yields the following.

(4.7.10) Proposition. $\bar{W}$ takes a short exact sequence $0 \to K \to G \to H \to 0$ in SSAG into a short exact sequence
\(0 \to \overline{WK} \to \overline{WG} \to \overline{WH} \to 0\) in SSAG. \(\square\)

(4.7.11) **Proposition.** A map \(g : G \to H\) in SSAG is a surjection if and only if \(\overline{Wf}\) is a fibration.

**Proof.** The "if" part follows from Proposition (4.7.10) and Remarks (4.7.8). For the "only if" part, \((\overline{WG})_0\) and \((\overline{WH})_0\) each consist of trivial group \(0\), so the map \(\overline{NWf} : \overline{NWG} \to \overline{NWH}\) is a degree-wise surjection by Proposition (4.7.7). By [Q-1, Lemma II.3.5], the map \(\overline{Wf}\) is a surjection. Hence, \(f\) is a surjection by the construction of \(\overline{W}\). \(\square\)

We shall now extend the above discussion to pro-SSAG. Call a pro-(simplicial abelian group) **discrete** if it is pro-isomorphic to one in the image of \(S : \text{pro-AG} \to \text{pro-SSAG}\). Prolong the \(\overline{W}\)-construction levelwise to pro-SSAG.

(4.7.12) **Proposition.** All discrete simplicial abelian groups are fibrant.

**Proof.** If \(\{G_j\}\) is discrete, the maps \(G_j \to \lim_{k < j} \{G_k\}\) are fibrations by Remarks (4.7.8). \(\square\)

(4.7.13) **Proposition.** \(\overline{W}\) takes a short exact sequence

\(0 \to K \to H \to G \to 0\) in pro-SSAG into a short exact sequence
0 \rightarrow \overline{W}K \rightarrow \overline{W}G \rightarrow \overline{W}H \rightarrow 0 \text{ in pro-SSAG.}

**Proof.** Use Propositions (4.6.3) and (4.7.10).

The analogue of Proposition (4.7.11) is difficult to state; the ideas will be used in the latter part of this section.

We shall now use pro-SSAG and Ho(pro-SSAG) to classify extensions in pro-AG.

(4.7.14) **Theorem.** For G and H in pro-AG, S induces a natural isomorphism

$$\sigma: \text{Ext}^s_{\text{pro-AG}}(G,H) \rightarrow \text{Ext}^s_{\text{pro-SSAG}}(G,H).$$

**Proof.** Because S is full,

$$\text{HOM}^s_{\text{pro-AG}}(G,H) = \text{HOM}^s_{\text{pro-SSAG}}(G,H).$$

Also, because S is full and TS = 1_{\text{pro-AG}}, the induced natural transformation

$$\sigma: \text{Ext}^s_{\text{pro-AG}}(G,H) \rightarrow \text{Ext}^s_{\text{pro-SSAG}}(G,H)$$

is a monomorphism. To show that $\sigma$ is an epimorphism, consider an extension

$$E: 0 \rightarrow G \rightarrow X \rightarrow \cdots \rightarrow X' \rightarrow H \rightarrow 0$$

in pro-SSAG, where G and H are discrete. Applying T to E yields an extension

$$TE: 0 \rightarrow G \rightarrow TX \rightarrow \cdots \rightarrow TX' \rightarrow H \rightarrow 0$$
in pro-AG. Because there is a (natural) map $STE \to E$ which is the identity on $G$ and $H$, $E \cong STE \in \text{Im} \circ$, as required.

We shall henceforth simply write Ext for Ext$_{\text{pro-AG}}$ and Ext$_{\text{pro-SSAG}}$.

(4.7.15) **Theorem D.** For $\{G_j\}$ (levelwise), free (abelian) in pro-AG, and $\{H_k\}$ in pro-AG,

$$\text{Ext}^S(\{G_j\}, \{H_k\}) = \text{Ho}(\text{pro-SSAG})(\{G_j\}, \{W^S H_k\}).$$

**Proof.** First, consider the case $s = 0$, where

$$\text{Ext}^0(\{G_j\}, \{H_k\}) = \text{Hom}(\{G_j\}, \{H_k\}).$$

Because $\{G_j\}$ is free, hence cofibrant (see §2.3, §3.4, and [Q-1,§II.3]), and $\{H_k\}$ is discrete, hence fibrant (Proposition (4.7.12)),

$$\text{Ho}(\text{pro-SSAG})(\{G_j\}, \{H_k\}) = \{\{G_j\}, \{H_k\}\},$$

homotopy classes of maps with respect to the cocylinder

$$\{H_k\}^{0,1} = \{H_k^{0,1}\}. \text{ Because } \{H_k\}^{0,1} \text{ is discrete,}$$

$$\{H_k^{0,1}\} = \{H_k\}. \text{ Hence,}$$

$$\text{Ho}(\text{pro-SSAG})(\{G_j\}, \{H_k\}) = \text{pro-SSAG}(\{G_j\}, \{H_k\}),$$

as required.
For $s > 0$, we shall use one of B. Mitchell's characterizations of derived functors [Mit−1, p. 198, case III]. Suppose first that 

$$0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0$$

is a short exact sequence of level maps in pro-$SSAG$, that is, a short exact sequence in the appropriate level category $SSAG^J$. Consider a fixed $j$ in $J$. Then there are fibre sequences

$$A_j \to B_j \to C_j,$$

and

$$\overline{WA}_j \to \overline{WB}_j \to \overline{WC}_j.$$

We may obtain a connecting morphism in $\operatorname{Ho}(SSAG)$, $\delta: C_j \to \overline{WA}_j$, and thus a co-Puppe (fibration) sequence (each map is the fibre of the next map)

$$\begin{array}{cccccc}
A_j & \to & B_j & \to & C_j & \delta \to \overline{W}_A_j & \to \overline{W}_B_j & \to \overline{W}_C_j
\end{array}$$

as follows. The homomorphisms $B_j \to 0$ and $WA_j \to 0$ induce the fibrations $f$ and $g$ in the solid-arrow diagram

$$\begin{array}{cccccc}
& & & & \overline{W}_A_j & \\
& & & & \downarrow & \\
B_j & \to & WA_j & \times & B_j & \to & WA_j & \times & 0 \equiv \overline{W}_A_j \\
& & & & \downarrow & \\
& & & & \delta & \\
& & & & \overline{O}_A_j & \equiv & C_j
\end{array}$$
Further, \( g \) is an equivalence. Let \( \delta \) be the composite in 
\[ \text{Ho(SSAG)} : \]
\[ C_j \cong O \times B_j \xrightarrow{[\gamma]^{-1}} W_{A_j} \times B_j \xrightarrow{[\varepsilon]} W_{A_j} A_j \times O = \overline{W}_{A_j}. \]

Then \( \delta \) is the required connecting morphism, and it is easy to show that sequence (4.7.16) is a co-Puppe sequence. Therefore, the sequence

(4.7.17)

\[ \{A_j\} \rightarrow \{B_j\} \rightarrow \{C_j\} \rightarrow \{\overline{W}_{A_j}\} \rightarrow \{\overline{W}_{B_j}\} \rightarrow \{\overline{W}_{C_j}\} \]

is an inverse system of long fibration sequences, hence a long fibration sequence in \( \text{Ho(pro-SSAG)} \) by Proposition (3.4.17).

For any \( G \) in \( \text{Ho(pro-SSAG)} \), (4.7.17) induces a long exact sequence

(4.7.18)

\[ [G,\{A_{j}\}] \rightarrow [G,\{B_{j}\}] \rightarrow [G,\{C_{j}\}] \xrightarrow{\delta} [G,\{\overline{W}_{A_{j}}\}] \rightarrow \ldots \]

of abelian groups, where \([\cdot,\cdot]\) \( \in \text{Ho(pro-SSAG)}(-,-) \). Compare (3.4.16).

More generally, if \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) is a short exact sequence in \( \text{pro-SSAG} \), Proposition (4.6.3) and the above techniques
yield a long exact sequence

\[(4.7.19)\]

\[ [G, A] \longrightarrow [G, B] \longrightarrow [G, C] \overset{\delta^*}{\longrightarrow} [G, \overline{W}A] \longrightarrow \cdots \]

analogous to \((4.7.18)\). Thus

\[(4.7.20)\]

\((H, H^1) = ([G, -], [G, \overline{W}(-)])\)

is a connected pair of exact functors on \(\text{pro-SSAG}\). If \(G\) is discrete (in \(\text{pro-AG}\)) and free, and we restrict the functors \((4.7.20)\) to \(\text{pro-AG}\), then \((H, H^1)\) is a connected pair of functors with \(H = \text{Ho}(\text{pro-SSAG})(G, -) = \text{pro-AG} (G, -)\). Further, \(\text{Ho}(\text{pro-SSAG})(G, \overline{W}(-))\) vanishes on objects of the form \(\overline{W}B = \overline{W}B_i\) since such objects are levelly contractible (see \((4.7.9)\)). Finally, given any short exact sequence \(0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\) in \(\text{pro-SSAG}\), there is a diagram

\[(4.7.21)\]

\[
\begin{array}{cccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& | & | & & | & | \\
0 & \longrightarrow & A & \longrightarrow & \overline{W}B & \longrightarrow & \overline{W}B/A & \longrightarrow & 0
\end{array}
\]

in which the bottom row is exact. Thus, sequences of the form \(0 \longrightarrow A \longrightarrow \overline{W}B \longrightarrow \overline{W}B/A \longrightarrow 0\) are cofinal in the directed set of all sequences \(0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\). Since \(H^1(\overline{W}B) = 0\), \(H^1\) is the derived functor of \(H\), that is, \(H^1(A) = \text{Ext} (G, A)\), by Mitchell's criterion.
By iterating the above construction, we obtain the required isomorphisms

\[ \text{Ext}^S(\{G_j\}, \{H_k\}) = \text{Ho}(\text{pro-SSAG})(\{G_j\}, \{W^S H_k\}) \]

for \( \{G_j\} \) levelwise free in \( \text{pro-AG} \) and \( \{H_k\} \) in \( \text{pro-AG} \). \( \square \)

The above results hold on \( \text{pro-G} \) for \( s = 0 \) and \( 1 \). Details are omitted.

We may use the homotopy inverse limit (§4.2) to reformulate the above theorem (for (a) and (b) see Bousfield and Kan [B-K, §XI.7]). Part (c) is our Theorem F (§4.5).

(4.7.22) **Theorem.**

(a) Let \( \{G_i\} \) be an inverse system of groups. Then

\[ \pi_n(\text{holim} \ W G_i) = \begin{cases} 
\text{lim}^1 \{G_j\} & \text{if } n = 0, \\
\text{lim} \{G_j\} & \text{if } n = 1, \\
0 & \text{if } n > 1.
\end{cases} \]

(b) Let \( \{G_j\} \) be an inverse system of abelian groups. Then

\[ \pi_n(\text{holim}(W^S G_j)) = \begin{cases} 
\text{lim}^{s-n} \{G_j\} & \text{if } 0 \leq n \leq s, \\
0 & \text{if } n > s.
\end{cases} \]

(c) Let \( \{G_j\} \) be an inverse system of abelian groups, and let \( KG_j \) be the simplicial spectrum obtained from the
simplicial prespectrum \( \{ \overline{w}^s G_j | s \geq 0 \} \). Then using stable homotopy groups,

\[
\pi_n (\text{holim} \{ K G_j \}) = \begin{cases} 
\lim_{n \to -\infty} G_1 & n \leq 0, \\
0 & n > 0.
\end{cases}
\]

Proof. (a) and (b) follow from the observation

\[
\pi_n (\text{holim} \overline{w}^s G_j) = \text{Ho(pro-SS)} (\overline{s}^n, \overline{w}^s G_j) \\
= \text{Ho(pro-SS)} (\overline{s}^0, \overline{w}^{s-n} G_j) \\
= \text{Ho(pro-SSAC)} (\mathbb{Z}, \overline{w}^{s-n} G_j).
\]

together with Theorem E. For (c), use analogous computations with simplicial spectra \( \text{Sp} \) and simplicial abelian group spectra \( \text{SpAC} \) [Kan-1,2,3], or use (b) and consider each \( K G_j \) as a simplicial prespectrum. \( \square \)

§4.8. \( \lim^s \) vanishes on Strongly Mittag-Leffler pro-groups.

We shall give an appropriate generalization to inverse systems indexed by uncountable indexing sets of the following well-known results.

(4.8.1) Let \( \{ G_n \} \) be a tower of groups such that the bonding maps are all surjections. Then \( \lim^s \{ G_n \} = 0 \) for \( s > 0 \).

(4.8.2) Suppose further that \( \lim \{ G_n \} = 0 \), then \( \{ G_n \} \cong 0 \).
A pro-group is said to satisfy the Mittag-Leffler (M-L) condition if it is pro-isomorphic to an inverse system of groups whose bounding maps are surjections.

Keesling [Kes] has exhibited a M-L pro-(abelian group) \( \{G_i\} \), indexed by an uncountable directed set, such that \( \lim \{G_i\} = 0 \) but \( \{G_i\} \neq 0 \) in pro-\( G \). Thus (4.8.2) fails in general for M-L pro-groups. Keesling [Kes] also constructed a movable (Definition (4.8.5), below) inverse system of long exact sequences of abelian groups such that the inverse limit sequence is not exact. We shall use this example to prove (Proposition (4.8.6), below) that (4.8.1) also fails, in general, on M-L pro-groups.

In a positive direction, we suggest the following definition as the appropriate generalization of the M-L condition to uncountable inverse systems.

(4.8.3) Definition. A pro-group is said to satisfy the strong-Mittag-Leffler (S-M-L) condition if it is pro-isomorphic to a pro-group \( \{G_j\} \) such that \( J = \{j\} \) is strongly directed, \( \alpha \) cofinite and for all \( j \in J \), the natural maps \( G_j \to \lim_{k < j} G_k \) are surjections.
Clearly, $S-M-L$ implies $M-L$. Also, for towers, $M-L$ implies $S-M-L$.

(4.8.4) **Theorem C.** Let $G_j$ be a strongly-Mittag-Leffler pro-group. Then

(a) $\lim^{1} \{G_j\} = 0$;

(b) If $\{G_j\}$ is abelian, then $\lim^{s} \{G_j\} = 0$ for $s > 0$;

(c) If $\lim \{G_j\} = 0$, then $\{G_j\} \approx 0$ in pro-$G$.

**Proof.** We may assume that $J = \{j\}$ is strongly directed and that for each $j$ the induced map $p:G_j \rightarrow \lim_{k < j} \{G_k\}$ is a surjection.

Then, the induced maps $\overline{W}p: \overline{WG}_j \rightarrow \overline{W}(\lim_{k < j} \{G_k\})$ are fibrations by Proposition (4.7.11). Further, since $\overline{W}$ is an adjoint functor (see, for example, [M, Theorem 27.1]), $\overline{W}(\lim_{k < j} \{G_k\}) \approx \lim_{k < j} \{\overline{WG}_k\}$.

Hence the inverse system $\{\overline{WG}_j\}$ is fibrant (§3.3), that is, the induced maps

$$\overline{WG}_j \rightarrow \lim_{k < j} \{\overline{WG}_k\}$$

are fibrations (in $SSG$ or $SSAG$).

By Theorem E (§4.6),

$$\lim^{1} \{G_j\} = \pi_0(\overline{WG}_j) = \pi_0(\text{holim} \{\overline{WG}_j\})$$

(see §4.2 for holim). Because $\{\overline{WG}_j\}$ is fibrant,
\[ \text{holim } \{ \overline{\text{WG}}_j \} \cong \lim \{ \overline{\text{WG}}_j \} \text{ (see (4.2.11))} \]
\[ \cong \overline{W}(\lim \{ G_j \}), \]

because \( \overline{W} \) is an adjoint. Hence, \( \pi_0(\text{holim } \{ \overline{\text{WG}}_j \}) = 0 \). Part (a) follows.

The proof of part (b) uses the formula
\[ \lim^S \{ G_j \} = \pi_0(\overline{W}^S G_j) \]
in a similar way. Details are omitted.

For part (c), assume that for some \( j \) in \( J \), \( G_{j_0} \neq 0 \).

Choose \( g \neq e \) (the identity) in \( G_j \). For \( k \leq j_0 \) in \( J \), let \( g_k \)
be the image of \( g \). Otherwise use induction on the number of predecessors of \( j \) in \( J \) to define an element \( \{ g_j \} \) in \( \lim \{ G_j \} \)
with \( g_{j_0} = g \neq e \). Hence, \( \lim \{ G_j \} \neq 0 \). This contradiction establishes part (c).

\[ \square \]

(4.8.5) Definition. An object \( \{ X_j \} \) of \( \text{pro-}C \) is said to be movable if for each \( j \) there exists a \( k > j \) such that for all \( \ell > k \) there exists a filler in the diagram
It is easy to check that movable pro-groups satisfy the Mittag-Leffler condition.

\[(4.8.6) \textbf{Proposition.} \text{ In general, } \lim^1 \text{ need not vanish on Mittag-Leffler pro-groups.}\]

\textbf{Proof.} If \( \lim^1 \{G_j\} = 0 \) for all M-L pro-groups \( \{G_j\} \), then any short exact sequence

\[
0 \rightarrow \{A_j\} \rightarrow \{B_k\} \rightarrow \{C_j\} \rightarrow 0
\]

of M-L pro-groups would yield a short exact sequence under the functor \( \lim \).

In [Kes] Keesling constructs a movable pair, \( (X_j, A_j) \) in pro-\( ANR \) pairs, hence a movable system of long exact sequences

\[(4.8.7)\]

\[
\{\cdots \rightarrow H_1(X_j) \rightarrow H_1(X_j, A_j) \rightarrow \cdots \}.
\]

such that the induced sequence
\[(4.8.8)\]
\[
\lim H_1(x_j) \to \lim H_1(x_j, A_j) \to \lim \{H_0(A_j)\}
\]

is not exact at the middle term. Since the kernels and images in sequence \((4.8.7)\) are movable, and hence \(M-L\), sequence \((4.8.8)\) would then be exact, contradicting Keesling.

Therefore, \(\lim^1\) cannot vanish on all \(M-L\) pro-groups.

§4.9. The Bousfield-Kan spectral sequence

In this section we discuss the Bousfield-Kan spectral sequence for the homotopy groups of the homotopy inverse limit of a pro-(simplicial set) [B-K, Chapter XI]. Although their model structure on \(SS^J\) differs from ours, the resulting homotopy categories are isomorphic (see §3.2). Also, if \(X\) is a fibrant object in our model structure, then \(X\) is fibrant in the Bousfield-Kan structure.

Let \(X \in SS^J\). Because our holim preserves weak equivalences, we may replace \(X\) by a fibrant object \(X'\) with \(\text{holim} X = \text{holim} X'\). Hence, we may assume that \(X\) is fibrant. In this case, the Bousfield-Kan homotopy inverse limit, temporarily denoted \(\text{holim}_{B-K}\), satisfies
Ho(pro - SS)(W,X) \equiv Ho(SS^J)(cW,X)

= Ho(SS)(\overline{W}, \operatorname{holim}_{B-K} X)

(cW denotes the constant diagram with \( cW_j = W \) for all \( j \) in \( J \))

[B-K, Proposition XI.8.1]. By uniqueness of adjoints (see, e.g.,
[Min-1]), \( \operatorname{holim} X \cong \operatorname{holim}_{B-K} X. \)

Hence the following result of Bousfield and Kan holds for our
homotopy inverse limit.

(4.9.1) Theorem [B-K, §XI.7.1, §IX.5.4-6]. For a pro-
(simplicial set) \( X \), there is a spectral sequence \( \{E_r(X), d_r\} \), with

\[
E_2(X) = \{E_2^{p,q}(X) = \lim_p \{\pi_q(X_j)\}\}, \text{ bidegree } d_r = (r, r-1).
\]

Under suitable conditions, \( \{E_r(X)\} \) converges completely to
\( \pi_* \operatorname{holim} X. \)

In particular, suppose for some \( s \), \( X = \{X_j\} \) satisfies

\[
\lim_p \{\pi_q(X_j)\} = 0 \quad \text{for} \quad p > s, \quad E_2^{p,q}(X) = 0 \quad \text{unless} \quad 0 \leq p \leq s.
\]

Then \( E_{s+1}^{p,q}(X) = E_\infty^{p,q}(X) \) because for \( r > s \), the differentials
\( d_r \) (of bidegree \( (r, r-1) \)) either begin or end at a 0-group, so we
obtain complete convergence. There are two important special cases.
(4.9.2) If $X$ in $\mathrm{pro-SS}_\mathcal{X}$ is equivalent in $\mathrm{pro-Ho}(\mathcal{SS})$ to a simplicial set, then

$$E_1^{p,q}(X) = E_\infty^{p,q}(X) = \begin{cases} \pi_q \mathrm{holim} X, & p = 0 \\ 0, & p \neq 0 \end{cases}. $$

(4.9.3) If $X = \{X_j\}$ in $\mathrm{pro-SS}_\mathcal{X}$ is equivalent in $\mathrm{pro-Ho}(\mathcal{SS})$ to a tower $\{X_n^i, n = 0, 1, \cdots\}$, then $E_2(X) = E_\infty(X)$ and the spectral sequence collapses to the short exact sequences

$$0 \rightarrow \lim_j \pi_{q+1} X_j \rightarrow \pi_q \mathrm{holim} X \rightarrow \lim_j \pi_q X_j \rightarrow 0.$$
§5. ALGEBRAIC TOPOLOGY.

§5.1. Introduction.

In §3 we showed that if $C$ is a nice model category then so is pro-$C$. In §4 we developed the theory of homotopy inverse limits for pro-$C$. In this section our main goal is to describe the algebraic topology of pro-$C$ and to compare $\text{Ho}(\text{pro}-C)$ with pro-$\text{Ho}(C)$.

In §5.2 we prove comparison theorems which relate maps and isomorphisms in $\text{Ho}(\text{tow}-C)$ to maps and isomorphisms in tow-$\text{Ho}(C)$.

§5.3 contains some remarks about completions.

In §5.4 we review the Artin-Mazur theory of pro-$\text{Ho}(\text{SS}_\ast)$.

§5.5 is concerned with Whitehead and stability theorems and counter-examples.

In §5.6 we introduce strong homotopy and homology theories and prove a Brown theorem for cohomology theories.

§5.2. $\text{Ho}(\text{tow}-C_\ast)$ versus tow-$\text{Ho}(C_\ast)$.

Let $C_\ast$ be a pointed nice simplicial closed model category and let $\pi: \text{Ho}(\text{tow}-C_\ast) \to \text{tow}-\text{Ho}(C_\ast)$ denote the natural functor.

(5.2.1) Comparison Theorem. There is a natural short exact sequence of pointed sets:
(5.2.2) \[0 \to \lim_{k} \colim_{j} \{ \text{Ho}(C_*)(\Sigma X_j, Y_k) \}\]
\[\to \text{Ho}(\text{tow-}C_*)(\{X_j\}, \{Y_k\})\]
\[\to \text{tow-} \text{Ho}(C_*)(\{X_j\}, \{Y_k\}) \to 0.\]

(5.2.3) **Remarks.** It is easy to see that the natural map
\[\text{Ho}(\text{tow-}C_*)(\{X_j\}, \{Y_k\}) \to \text{tow-} \text{Ho}(C_*)(\{X_j\}, \{Y_k\})\]
is always surjective; we need base points only to make a more precise statement.

(5.2.4) **Remarks.** J. Grossman [Gros 2] obtained the above sequence in his coarser homotopy theory of tow-SS_+.

**Proof of Theorem (5.2.1).** By Axiom M2 for C_+^N or an easy inductive argument we can replace any tower by an equivalent (in Ho(\text{tow-}C_*)) fibrant tower, i.e., tower of fibrations of fibrant objects. We may therefore assume that \{Y_k\} is fibrant. Define \(Y_{-1} = \ast\); then the map \(Y_0 \to Y_{-1}\) is a fibration.

Consider the following function spaces (all of which take values in SS_+; see §2.4):

\[
\text{HOM} (X_j, Y_k), \quad j \geq 0, \quad k \geq -1;
\]

(5.2.5) \[\text{HOM} (\{X_j\}, Y_k) \equiv \colim_{j} \{ \text{HOM} (X_j, Y_k) \}, \quad k \geq 0;\]
\[\text{HOM} (\{X_j\}, \{Y_k\}) \equiv \lim_{k} \{ \text{HOM} (\{X_j\}, Y_k) \}.\]
The fibrations $Y_k \rightarrow Y_{k-1}$ induce fibrations

$\text{HOM} \left( X_j, Y_k \right) \rightarrow \text{HOM} \left( X_j, Y_{k-1} \right)$

by the simplicial structure on $C_\ast$, also fibrations $\text{HOM} \left( \{X_j\}, Y_k \right) \rightarrow \text{HOM} \left( \{X_j\}, Y_{k-1} \right)$ by the simplicial structure on $\text{pro-} C_\ast$ or a direct argument (the colimit of a sequence of Kan fibrations is a Kan fibration by the "small-object argument" [Q-1, §II.3]).

Because (5.2.5) expresses $\text{HOM} \left( \{X_j\}, \{Y_k\} \right)$ as the limit of a tower of fibrations, the Bousfield-Kan spectral sequence for a tower (4.9.3)) yields an exact sequence of pointed sets

$$0 \rightarrow \lim_k^\perp \pi_1 \left( \text{HOM} \left( \{X_j\}, Y_k \right) \right) \rightarrow \pi_0 \left( \text{HOM} \left( \{X_j\}, \{Y_k\} \right) \right)$$

$$\rightarrow \lim_k \pi_0 \left( \text{HOM} \left( \{X_j\}, Y_k \right) \right) \rightarrow 0.$$

By the above simplicial structures,

$$\pi_1 \left( \text{HOM} \left( \{X_j\}, Y_k \right) \right) \equiv \left[ S^1, \text{HOM} \left( \{X_j\}, Y_k \right) \right]$$

$$\equiv \pi_0 \left( \text{HOM} \left( S^1, \text{HOM} \left( \{X_j\}, Y_k \right) \right) \right)$$

$$\equiv \pi_0 \left( \text{HOM} \left( \{X_j\} \otimes S^1, Y_k \right) \right)$$

$$\equiv \pi_0 \left( \text{HOM} \left( \{X_j \otimes S^1\}, Y_k \right) \right)$$

$$\equiv \pi_0 \left( \text{HOM} \left( \{\Sigma X_j\}, Y_k \right) \right)$$

$$\equiv \text{Ho}(\text{pro-} C_\ast) \left( \{\Sigma X_j\}, Y_k \right)$$

$$\equiv \text{colim}_j \left[ \Sigma X_j, Y_k \right].$$
([-,-] denotes $\text{Ho}(C_\mathcal{K})(-,-)$; the last isomorphism follows by the homotopy extension property). Also,

$$\pi_0(\text{HOM}((X_j),(Y_k))) \cong \text{Ho}(\text{tow-}C_\mathcal{K})(\{X_j\},\{Y_k\}),$$

and

$$\pi_0(\text{HOM}((X_j),(Y_k))) \cong \text{colim}_j\{[X_j,Y_k]\},$$

as above. Because the above isomorphisms are natural (§2.4), and

$$\text{tow-}\text{Ho}(C_\mathcal{K})(\{X_j\},\{Y_k\}) \cong \text{lim}_k\{\text{colim}_j\{[X_j,Y_k]\}\},$$

the conclusion follows from (5.2.6). \qed

(5.2.7) Remarks. In the above proof, one can replace the Bousfield-Kan spectral sequence by a direct argument dual to Milnor's "lim 1 argument" [Mil-3, Lemma 1]; see, for example [B-K, p. 254].

Theorem (5.2.1) suggests the following.

(5.2.8) Question. Let $f$ be a map in $\text{Ho}(\text{tow-}C)$ whose image in $\text{tow-}\text{Ho}(C),\pi f$, is invertible. Is $f$ invertible in $\text{Ho}(\text{tow-}C)$?

This question appears quite difficult, and its analogue in proper homotopy theory (see §6.2) has attracted recent interest (Chapman and Siebenmann [C-S]). At present we can only offer a partial answer (Theorem (5.2.9)). The analogue of (5.2.9) in proper homotopy theory (§6.5) answers another question in [C-S].
(5.2.9) **Theorem.** Let \( f: \{X_j\} \rightarrow \{Y_k\} \) be a map in \( \text{Ho(tow-} C) \) whose image \( \pi f \) in \( \text{tow-Ho(C)} \) is invertible. Then there is an isomorphism \( g: \{X_j\} \rightarrow \{Y_k\} \) in \( \text{Ho(tow-} C) \) with \( \pi f = \pi g \) in \( \text{tow-Ho(C)} \).

**Proof.** As in the proof of Theorem (5.2.1), we may assume that both \( \{X_j\} \) and \( \{Y_k\} \) are fibrant. By reindexing if necessary, we may then realize both \( \pi f \) and its inverse in \( \text{tow-Ho(C)} \) in the following homotopy-commutative diagram over \( C \).

\[
\begin{array}{ccc}
X_{k+1} & \xrightarrow{f_{k+1}} & Y_{k+1} \\
\downarrow & & \downarrow \\
X_k & \xrightarrow{f_k} & Y_k \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\end{array}
\]

(5.2.10)

\[
\begin{array}{ccc}
X_0 & \xleftarrow{h_1} & Y_1 \\
\vdots & & \vdots \\
X_k & \xleftarrow{h_{k+1}} & Y_{k+1} \\
\vdots & & \vdots \\
\end{array}
\]

Let \( Z \) be the tower

\[
\begin{array}{cccccccc}
X_0 & \xleftarrow{h_1} & Y_1 & \xleftarrow{f_1} & \cdots & \xleftarrow{f_k} & X_k & \xleftarrow{h_{k+1}} & Y_{k+1} & \xleftarrow{f_{k+1}} & X_{k+1} & \xleftarrow{h_{k+1}} & \cdots \\
\end{array}
\]

Form the homotopy commutative diagram.
Diagram (5.2.11) factors \( \pi f \) in \( \text{tot-} \mathbf{Ho}(\mathbf{C}) \). By inductively deforming the maps \( \text{id}: X_k \to X_k \) and \( \text{id}: Y_k \to Y_k \) above, we may obtain a strictly commutative diagram

(5.2.12).

such that diagrams (5.2.11) and (5.2.12) are equivalent over \( \mathbf{Ho}(\mathbf{C}) \).
The composite map \( g : X \rightarrow Y \rightarrow Z \) in diagram (5.2.12) is invertible in \( \text{Ho}(\text{tow}-\text{C}) \) because it is the composite of two levelwise weak equivalences, and satisfies \( \pi_g = \pi_f \), as required.

(5.2.13) **Corollary.** If either \( X \) or \( Y \) is stable in \( \text{tow-Ho}(\text{C}) \) (i.e., isomorphic in \( \text{tow-Ho}(\text{C}) \) to an object of \( \text{Ho}(\text{C}) \)), then \( f \) above is an isomorphism in \( \text{Ho}(\text{tow}-\text{C}) \).

**Proof.** A first application of the theorem shows that \( X \) and \( Y \) are stable in \( \text{Ho}(\text{tow}-\text{C}) \), say isomorphic to objects \( X' \) and \( Y' \) of \( \text{C} \), respectively. Consider the composite map

\[
(5.2.14) \quad X' \xrightarrow{\tilde{e}} X \xrightarrow{f} Y \xrightarrow{\tilde{e}} Y'.
\]

For any \( Z \) in \( \text{C} \),

\[
\text{Ho}(\text{C})(Z,X') \cong \text{tow-Ho}(\text{C})(Z,X') \cong \text{tow-Ho}(\text{C})(Z,Y') \cong \text{Ho}(\text{C})(Z,Y')
\]

Thus the composite (5.2.14) is an isomorphism in \( \text{Ho}(\text{C}) \), hence in \( \text{Ho}(\text{tow}-\text{C}) \). \( \square \)

(5.2.15) **Remarks.** If \( \text{Ho}(C_\ast) \) is abelian, the Comparison Theorem implies that \( f \) is an isomorphism in \( \text{Ho}(\text{tow}-C_\ast) \).

We now make the following observation.

(5.2.16) **Rigidification.** Each object in \( \text{tow-Ho}(\text{C}) \) is equivalent to the image of an object in \( \text{Ho}(\text{tow}-\text{C}) \). To see this,
given \( \{X_j\} \) in \( \text{tow-} \text{Ho}(C) \), replace \( \{X_j\} \) by a tower of fibrant objects \( \{X'_j\} \), and choose representatives for the bonding maps of the latter tower.

(5.2.17) **Corollary.** The isomorphism classification is the same in \( \text{tow-} \text{Ho}(C) \) and \( \text{Ho}(\text{tow-} C) \).

**Proof.** Use (5.2.9) and (5.2.16). \( \square \)

The above results give a usable relationship between the weak and strong homotopy theories of towers. See §6, especially §6.2, where towers are used in the proper homotopy theory of \( \sigma \)-compact spaces, and §8 for a similar application to the shape theory of compact metric spaces.

Off towers, little is known. The **Comparison Theorem** (5.2.1) is replaced by the following extension of the Bousfield-Kan spectral sequence (see §4.9).

(5.2.16) **Theorem.** For \( \{X_j\} \) and \( \{Y_k\} \) in \( \text{pro-} C_* \), there is a spectral sequence with

\[
E^2_{p,q} = \lim^p_k \{ \colim_j [Z^q X_j, Y_k] \},
\]

which is closely related to \( \text{Ho} (\text{pro-} C_*) (\{X_j\}, \{Y_k\}) \).

**Proof.** We may assume that \( \{Y_k\} \) is fibrant. Because \( \{j\} \) is filtering,
colim_j ([[Σ^q X_j, Y_k]]) = \text{Ho}(\text{pro-}C^\ast)([[Σ^q X_j, Y_k]])

(each map on the left may be represented by a map \(Σ^q X_j \rightarrow Y_k\) for some \(j\)). By the simplicial closed model structure on \(\text{pro-}C^\ast\),

\[ \text{Ho}(\text{pro-}C^\ast)([[Σ^q X_j, Y_k]]) \cong \pi_q(\text{HOM} ([X_j, Y_k])), \]

and

\[ \text{Ho}(\text{pro-}C^\ast)([X_j, Y_k]) \cong \pi_0(\text{HOM} ([X_j, Y_k])), \]

where \(\text{HOM}\) is the "function space" of §3.5. Finally,

\[ \text{HOM} ([X_j, Y_k]) = \lim_k \text{HOM} ([X_j, Y_k]) \]

(essentially by "enriched adjunction", §2.4)

\[ \text{holim}_k \text{HOM} ([X_j, Y_k]) \]

(because \(\text{holim} \Rightarrow \lim\) on fibrant objects).

Applying the Bousfield-Kan spectral sequence to \(\text{HOM} ([X_j, Y_k])\) gives the desired result. \(\Box\)

Unfortunately, we cannot conclude that

\[ \pi: \text{Ho}(\text{pro-}C^\ast)([X_j, Y_k]) \rightarrow \text{pro-Ho}(C^\ast)([X_j, Y_k]) \]

is onto.
The rigidification question - which objects of $\text{pro-Ho}(C)$ "come from" $\text{Ho}(\text{pro-C})$ - is unanswered and appears quite hard. Alex Heller asked a similar question about simplicial objects over $\text{Ho}(SS)$ and over $SS$. Also, the isomorphism classification question (compare (5.2.9) - (5.2.15)) is unanswered off towers.

Thus, our present knowledge suffices for the proper homotopy theory of $\sigma$-compact spaces (see §6) and the shape theory of compact metric spaces (see §8), but not for more general spaces.

§5.3. Remarks on Completions.

In 1965, Artin and Mazur [A-M, Chapter 3] introduced the pro-finite completion $\hat{X} \epsilon \text{pro-Ho}(CW_0)$ of an object $X \epsilon \text{Ho}(CW_0)$ in order to prove comparison theorems in etale homotopy theory. The inverse system $\hat{X}$ is indexed by the category which has as objects based homotopy classes of maps $X \rightarrow \hat{X}_\alpha$ where each $\hat{X}_\alpha$ has finite homotopy groups, and has as morphisms homotopy commutative triangles:

$$
\begin{array}{c}
\hat{X}_\alpha \\
\downarrow \\
\hat{X}_\beta \\
\end{array}
$$

Then the association $(X \rightarrow \hat{X}_\alpha) \mapsto \hat{X}_\alpha$ determines the inverse system $\hat{X}$. Sullivan's work on the Adams conjecture and the homotopy type of spaces such as $G/PL$ led him to study the functor $\lim_\alpha [\cdot, \hat{X}_\alpha]$ [Sul-1], [Sul-2]. By suitably topologizing the functors $[\cdot, \hat{X}_\alpha]$,
Sullivan showed that the functor $\lim_a \{-, \hat{X}_a\}$ satisfied the Mayer-Vietoris (exactness) axiom as well as the wedge axiom, and hence $\lim_a \{-, \hat{X}_a\}$ is representable by Brown's Theorem [Bro]. Sullivan then concentrated on the complex $\overline{X}$ which represented $\lim_a \{-, \hat{X}_a\}$,

\[(5.3.1) \quad \{-, \overline{X}\} \approx \lim_a \{-, \hat{X}_a\},\]

as being a simpler and more familiar object than $\{\hat{X}_a\}$. In 1972, Bousfield and Kan [B-K], motivated by their work on the Adams spectral sequence, defined for every commutative ring $R$ and pointed simplicial set $X$ a functorial $R$-completion, $R_\infty X$. They obtain $R_\infty X$ as the simplicial inverse limit of a tower of fibrations $\{R_s X\}$. In this situation it is no longer true that the functors $\{-, R_\infty X\}$ and $\lim_s \{-, R_s X\}$ are naturally equivalent. Instead, one has a short exact sequence of pointed sets [B-K] (see §4.9)

\[(5.3.2) \quad 0 \rightarrow \lim^1_s \{[W, R_s X]\} \rightarrow [W, R_\infty X] \rightarrow \lim_s \{[W, R_s X]\} \rightarrow 0.\]

We shall briefly compare (5.3.1) and (5.3.2), modulo rigidification problems (see the end of §5.2). In (5.3.2), we always have $[W, R_\infty X] \approx \text{Ho(pro-SS}_s)(W, \{R_s X\})$ because $R_\infty X = \text{holim}_s \{R_s X\}$ (see §4.2). In this sense, $\{-, R_\infty X\} \neq \lim_s \{-, R_s X\}$ because
\( \text{Ho(pro-} \text{SS}_R(\{-, [R, X]\} \neq \lim^1_s \{-, R_s X\}] \), and this difference is measured by the \( \lim^1 \) term of (5.3.2). For \( R = Z \), and \( X \) a simply-connected (or even nilpotent [B-K, Chapter 3]) finite complex, \( \{Z_s X\} \) is cofinal in \( \{X_n\} \). Also, for \( W \) finite, the groups \( [\Sigma W, Z_s X] \) are finite, so \( \lim^1_s ([\Sigma W, Z_s X]) \) vanishes. This suggests the following.

(5.3.3) **Proposition.** Let \( \{X_n\} \) be a tower of pointed, connected (SS or CW) complexes. If \( \lim_n \{-, X_n\} \) is representable and \( \lim^1_n \{-, X_n\} \) vanishes, then

\[ \{-, \text{holim} \{X_n\}\} \cong \lim_n \{-, X_n\} \]

on all pointed complexes.

**Proof.** Let \( Q \) represent \( \lim_n \{-, X_n\} \). Consider the diagram

\[
\begin{array}{ccc}
\{-, Q\} & \xrightarrow{\sim} & \lim_n \{-, X_n\} \\
\downarrow & & \downarrow \\
\{-, \text{holim} \{X_n\}\}
\end{array}
\]

(5.3.4)

Evaluation of the top row on \( \text{holim} \{X_n\} \) yields the filler which makes (5.3.4) commute. Vanishing of \( \lim^1_n \{-, X_n\} \) implies that the group homomorphisms \( \pi_i(\text{holim} \{X_n\}) \to \lim_n \pi_i(X_n), \ i \geq 1 \), are
isomorphisms (see §5.2; also (5.3.2)). Then diagram (5.3.4) yields isomorphisms $\pi_i(\text{holim } \{X_n\}) \cong \pi_i(Q)$, $i \geq 1$. Because vanishing of $\lim^1$ implies $\text{holim } \{X_n\}$ is connected, $\text{holim } \{X_n\}$ and $Q$ have the same weak (singular) homotopy type by the Whitehead Theorem. The conclusion follows.

§5.4. Some basic functors.

Artin and Mazur [A-M, §§1-4] introduced the homology and homotopy pro-groups of an object in $\text{pro-Ho}(SS_0)$, as well as Postnikov decompositions, the Hurewicz Theorem, and a type of Whitehead Theorem. In this section we shall review the above results, except for the Whitehead Theorem. The Whitehead Theorem will be discussed in §5.5.

Recall that any covariant functor $T: C \rightarrow D$ may be prolonged to a functor $\text{pro-T}: \text{pro-C} \rightarrow \text{pro-D}$. We may therefore define the pro-homotopy and pro-homology functors on $\text{pro-Ho}(SS_\ast)$ by the formulas

$$(5.4.1) \quad \text{pro-} \pi_i(\{X_j\}) \equiv \{\pi_i(X_j)\},$$

$$\text{pro-} \tilde{H}_i(\{X_j\};A) \equiv \{\tilde{H}_i(X_j;A)\}$$

where $A$ is an abelian group. A similar formula holds for any generalized homology theory. These functors induce pro-homotopy and pro-homology functors on $\text{pro-SS}_\ast$ which satisfy the usual properties with respect to the closed model structure. (Note that fibre sequences and related constructions are not functorial on $\text{Ho}(SS_\ast)$, hence it is difficult to describe homotopy and homology theories on
pro-Ho(SS_\ast)$. Artin and Mazur even define homology with twisted coefficients; we shall not need these formulas in our work.

Because cohomology is contravariant, the analogue of formula (5.4.1) for cohomology would take values in a category \textit{inj-AG} of direct systems of abelian groups. Because \text{colim}: \textit{inj-AG} \rightarrow \text{AG} is exact, Artin and Mazur define the cohomology groups by

\begin{equation}
\tilde{H}^i(\{X_j\};A) \cong \text{colim}_j\{\tilde{H}^i(X_j;A)\}.
\end{equation}

The category \(K_0\) of pointed, connected Kan complexes (\(=\)fibrant simplicial sets) admits functorial Postnikov-type resolutions (see, e.g. [May], or [A-M;51]). We shall describe these resolutions and the induced Postnikov-type resolutions on \textit{pro-Ho(SS_\ast)}.

Let \(\Delta^n_p\) denote the \(p\)-skeleton of the standard simplicial \(n\)-simplex \(\Delta^n\). By analogy with the formula \(X_n = SS(\Delta^n,X)\) for the set of \(n\)-simplices of a simplicial set \(X\), for \(X\) in \(K_0\) let \(\text{cosk}_p^X\) be the simplicial set whose \(n\)-simplices are given by

\[(\text{cosk}_p^X)_n = SS(\Delta^n_p,X),\]

together with face and degeneracy maps induced by the coface and codegeneracy maps \(d^i:\Delta^{n-1}_p \rightarrow \Delta^n_p\) and \(s^i:\Delta^{n+1}_p \rightarrow \Delta^n_p\) for \(0 \leq p \leq n\). Roughly, \(\text{cosk}_p^X\) is obtained from \(X\) by adjoining additional \(n\)-cells for \(n > p\) corresponding to maps from \(\Delta^n_p\) to \(X\).
The inclusions $\Delta^n_p \to \Delta^n_{p+1}$ and $\Delta^n_p \to \Delta^n$ induce compatible maps $\cosk_p X \to \cosk_{p-1} X$ and $X \to \cosk_p X$. Caution: These maps are not fibrations. We may define the coskeleta of an arbitrary simplicial set $X$ by the formulas $\cosk_p^\infty X$ (recall that $\cosk^\infty X$ is a Kan complex naturally weakly equivalent to $X$). For $X$ in $K_0$, let $X(p)$ and $X(p)$ be the homotopy-theoretic fibres of the maps below:

$$X^{\langle p \rangle} \to X \to \cosk_p X,$$

(5.4.3)

$$X(p) \to \cosk_{p+1} X \to \cosk_p X.$$

Because $SS_0$ admits canonical factorizations of maps as trivial cofibrations followed by fibrations ([(Q-1, §II.3), see §4.3), the sequences (5.4.3) are functorial in $X$. Further, $\cosk_p X$ is $(p-1)$-connected, and $X(p)$ is an Eilenberg-Maclane space of type $K(\pi_p(X),p)$. We therefore regard sequences (5.4.3) as the canonical Postnikov resolution of $X$.

In fact, the above constructions are functorial on $\text{Ho}(SS_0)$; the $p^{\text{th}}$ coskeleton of $X$ is characterized by the properties:

(i) $\pi_i(\cosk_p X) = 0$ for $i \geq p$;

(ii) The canonical map $X \to \cosk_p X$ is universal with respect to maps into objects $Y$ with $\pi_i(Y) = 0$ for $i \geq p$. 
Similarly, the fibre $X^{(p)}$ is $(p - 1)$-connected, the composition $X^{(p)} \rightarrow X \rightarrow \cosk_p X$ is trivial, and the map $X^{(p)} \rightarrow X$ is universal for these properties.

Following Artin and Mazur, we define the Postnikov system of an inverse system $X = \{X_j\}$ in either $\text{pro-}SS_0$ or $\text{pro-}Ho(SS_0)$ to be the inverse system

$$X^\# = \{\cosk_p X_j\}$$

indexed by $\{(p,j)\}$. Clearly $\#$ extends to functors from $\text{pro-}SS_0$ to $\text{pro-}SS_0$ and $\text{pro-}Ho(SS_0)$ to $\text{pro-}Ho(SS_0)$. If $\{X_n\}$ is a tower, then

$$X^\# = \{\cosk X_n\},$$

so we may restrict (5.4.4) to a functor from towers to towers. A map $f:X \rightarrow Y$ in $\text{pro-}Ho(SS_0)$ (respectively, $Ho(\text{pro-}SS_0)$) is called a $\#$-isomorphism if it induces an isomorphism on Postnikov systems $f^\#:X^\# \rightarrow Y^\#$.

By using the above machinery and a spectral sequence argument, Artin and Mazur proved the following.

(5.4.6) Hurewicz Theorem for $\text{pro-}Ho(SS_0)$. Let $\text{pro-}\pi_i(X) = 0$ for $i < n$, where $n$ is an integer $> 1$. Then the
canonical map

\[ \text{pro-} \pi_n(X) \longrightarrow \text{pro-} \tilde{\pi}_n(X) \]

is an isomorphism of pro-groups.

§5.5. Whitehead and Stability Theorems.

In this section let \( C \) be any of SS, SSG, SSAG, \( S_p \) (simplicial spectra). Then the Whitehead Theorem (5.5.1) holds in the category \( C_0 \) of pointed, connected objects in \( C \).

(5.5.1) **Whitehead Theorem in \( \text{Ho}(C_0) \).** A map \( f : X \longrightarrow Y \) in \( \text{Ho}(C_0) \) which induces isomorphisms \( f_* : \pi_i(X) \longrightarrow \pi_i(Y) \) for \( i < 1 \) is an isomorphism in \( \text{Ho}(C_0) \).

A natural question is whether (5.5.1) can be extended to \( \text{pro-} \text{Ho}(C_0) \) and \( \text{Ho}(\text{pro-} C_0) \) if the homotopy groups \( \pi_i(X) \) \((X \in C_0)\) are replaced by the homotopy progroups \( \{\pi_i(X_j)\} \) \((\{X_j\} \in \text{pro-} C_0)\). The stability problem (i.e., when is an object of \( \text{pro-} \text{Ho}(C_0) \) or \( \text{Ho}(\text{pro-} C_0) \) isomorphic to an object of \( \text{Ho}(C_0) \)) will also be studied using homotopy pro-groups. The following example shows that additional hypotheses are needed for a Whitehead Theorem in pro-homotopy.
(5.5.2) Example. Let $S^\infty = \{ V_i \geq n S^1 \}_{n > 0}$. Then
\[ \text{pro-} \pi_i (S^\infty) = 0 \quad \text{for all } i \geq 1, \] but $S^\infty$ is not equivalent to a point in $\text{pro-} \text{Ho}(SS_0)$ (in fact, because $\pi_i(S^3) \neq 0$ infinitely often, see e.g. [Spa, Corollary 9.7.6]), there is an essential map $S^\infty \rightarrow S^3$.

(5.5.3) Whitehead Theorem in $\text{pro-} \text{Ho}(C_0)$. Suppose that $f : X \rightarrow Y$ in $\text{pro-} \text{Ho}(C_0)$ induces isomorphisms $f_* : \text{pro-} \pi_i (X) \rightarrow \text{pro-} \pi_i (Y)$ for $i \geq 1$. Then $f$ induces an isomorphism $f^* : X^\circ \rightarrow Y^\circ$ of Postnikov systems in $\text{pro-} \text{Ho}(C_0)$.

Under either of the following additional conditions, $f$ is an isomorphism in $\text{pro-} \text{Ho}(C_0)$:

(a) $\sup_j, k \{ \dim(X_j), \dim(Y_k) \} < \infty$;

(b) For each $j$, $\dim(X_j) < \infty$, for each $k$, $\dim(Y_k) < \infty$, and $f$ is movable.

Proof. For $C = SS$, the first part is due to Artin and Mazur [A-M, Corollary (4.4)]. Their proof uses a spectral sequence argument and easily extends to the other $C$.

Similarly, for $C = SS$, the second part is due to Edwards and Geoghegan [E-G-1,2]; see also [A-M; Theorem (12.5)], [Mats 1],
We shall sketch the proof. By reindexing as in 2.1, we may assume that \( f \) is a level map \( \{ f_j : X_j \to Y_j \} \)
indexed by a directed set. For case (a), let \( n = \sup_j \{ \dim(X_j), \dim(Y_j) \} \). Consider a fixed \( j \). Choose \( j_1 \) such that the diagram

\[
\begin{array}{ccc}
\pi_1(X_{j_1}) & \longrightarrow & \pi_1(Y_{j_1}) \\
\downarrow & & \downarrow \\
\pi_1(X_j) & \longrightarrow & \pi_1(Y_j)
\end{array}
\]

admits a filler. Convert \( f_j \) and \( f_{j_1} \) into cofibrations \( f'_j \) and \( f'_{j_1} \). Then the map \( \pi_1(Y'_{j_1}, X'_{j_1}) \longrightarrow \pi_1(Y'_j, X'_j) \) is 0.

Then the composite map

\[
\begin{array}{ccc}
Y'_{j_1} & \longrightarrow & Y'_{j_1} \\
\downarrow & & \downarrow \\
Y'_j & \longrightarrow & Y'_j
\end{array}
\]

(\( Y'_{j_1} \) denotes the 1-skeleton of \( Y'_j \)) may be "deformed" relative to \( X'_{j_1} \) into \( X'_{j_1} \), similarly for all \( k \) with \( k \geq j_1 \). Now choose \( j_2 \geq j_1, j_3 \geq j_2, \ldots, j_n \geq j_{n-1} \) so that similar results hold for \( \pi_2, \pi_3, \ldots, \pi_n \). We conclude that the map \( Y'_j \to Y'_j \) can be deformed relative to \( X'_{j_n} \) into \( X'_{j_n} \) (because \( \dim(Y_{j_n}) \leq n \)). This argument is due to Mardesic [Mar-1] and yields a homotopy inverse to
f in case (a) by using the following lemma of Morita.

**Lemma.** (Morita [Mor-1]). Let \( X = \{X_\lambda, p_{\lambda \mu} : \lambda \leq \mu \} \) and \( Y = \{Y_\lambda, q_{\lambda \mu} : \lambda \leq \mu \} \) be inverse systems in a category \( C \) over the same directed set \( \Lambda \), and let \( \{X_\lambda \xrightarrow{f_\lambda} Y_\lambda \} \) be a level morphism in pro-\( C \). Then \( f \) is an isomorphism in pro-\( C \) iff for any \( \lambda \in \Lambda \) there is some \( \mu \in \Lambda \) such that \( \lambda \leq \mu \) and there exists \( \psi_{\lambda \mu} : Y_\mu \to X_\lambda \) for which \( \psi_{\lambda \mu} f_\mu = p_{\lambda \mu} \) and \( f_\lambda \psi_{\lambda \mu} = q_{\lambda \mu} \); i.e., a filler exists in the following solid arrow commutative diagram

\[
\begin{array}{ccc}
X_\mu & \xrightarrow{f_\mu} & Y_\mu \\
\downarrow^{p_{\lambda \mu}} & & \downarrow^{q_{\lambda \mu}} \\
X_\lambda & \xrightarrow{f_\lambda} & Y_\lambda
\end{array}
\]

Case (a) follows.

In case (b), consider a fixed \( j \). There exists a \( k \geq j \) such that for all \( k \geq k \), homotopy fillers exist in the diagram
Now use the above argument with \( n = \dim(Y_k, X_k) \) to obtain a deformation of the map \( Y_k \to Y_j \) into \( X_j \) relative to \( X_k \). As above, the conclusion follows. □

An object \( X \in \text{pro}-\mathcal{C} \) is said to be stable if it is isomorphic in \( \text{pro}-\mathcal{C} \) to an object of \( \mathcal{C} \). The stability problem is the problem of giving criteria on \( X \) which imply that \( X \) is stable. If \( X \in \text{pro}-\text{Ho}(\mathcal{C}_0) \) is stable, then so are its homotopy pro-groups \( \text{pro}-\pi_1(X) \).
(5.5.5) **The Stability Theorem in pro-**Ho(C₀). Let
\[ X \in \text{pro-} C₀ \] and let \( h: \text{holim} \ (X) \to X \) be the canonical map in \( \text{pro-} \text{Ho}(C₀) \). If \( \text{pro-} \pi_i(X) \) is stable for all \( i \geq 1 \), then \( h^\ast \) is an isomorphism. \( h \) is an isomorphism if \( X \) satisfies either of the following conditions:

(a) \( \sup_j \{ \dim \ (X_j) \} < \infty \);

(b) \( X \) is dominated in \( \text{pro-} \text{Ho}(C₀) \) by an object of \( \text{Ho}(C₀) \).

**Proof.** See [E-G-1]. We supply a sketch. Because \( \text{pro-} \pi_i(X) \) is stable, \( \text{Ext}^s_{_H} \pi_i(X_j) = 0 \) for \( s > 0 \) (§4.5, Theorem C). By the Bousfield-Kan spectral sequence (see §4.9), \( h \) induces isomorphisms

\[ \pi_i(\text{holim} \ (X)) \cong \text{pro-} \pi_i(\text{holim} \ (X)) \to \text{pro-} \pi_i(X). \]

Therefore \( h^\ast \) is an isomorphism by Theorem (5.5.3).

To show that \( h \) is an isomorphism in case (a), Edwards and Geoghegan applied Wall's finite-dimensionality criterion [Wall] to the homology and cohomology groups of \( \text{holim} \ (X) \) (which are isomorphic to the homology pro-groups and cohomology groups of the finite-dimensional system \( X \) because \( h^\ast \) is an isomorphism by [A-M]). This shows that \( \text{holim} \ (X) \) has the homotopy type of a finite-
dimensional simplicial set. The conclusion then follows by Theorem (5.5.3).

In case (b), let \( u: X \to Y \) and \( d: Y \to X \) be the domination maps with \( Y \in \mathcal{C}_0 \), and \( d u = 1_X \). One then applies Brown's Theorem [Bro] to split the homotopy idempotent \( u d: Y \to Y \)

\[(ud)^2 = u(du)d = ud \]

through \( Z \). One easily checks that

\[ X = \{ Y \leftarrow ud \ Y \leftarrow ud \ Y \leftarrow \cdots \} \cong Z. \]

Then the ordinary Whitehead Theorem implies that the composite map

\[ \text{holim} \ X \to Z \]

is an isomorphism. The conclusion follows. \( \square \)

(5.5.5.a) **Remarks.** The argument given in part (b) shows that

if \( X \) is only assumed to be in \( \text{pro} \cdot \text{Ho}(\mathcal{C}_0) \), it still follows that \( X \)

is stable. Dydak has recently shown that the same conclusion holds in

part (a).
So far, we have only been able to prove the following strong tower versions of Theorems (5.5.3) and (5.5.5).

\[(5.5.6) \] The Whitehead Theorem in $\mathbf{Ho}(\text{tow} \cdot \mathcal{C}_0)$. Suppose $f : X \to Y$ in $\text{tow} \cdot \mathcal{C}_0$ induces isomorphisms $f_* : \text{pro-}_i(X) \xrightarrow{\sim} \text{pro-}_i(Y)$ for all $i \geq 1$. Then $f$ induces an isomorphism $f^* : X^\tau \to Y^\tau$ in $\mathbf{Ho}(\text{tow} \cdot \mathcal{C}_0)$. If $f$ is itself an isomorphism in $\mathbf{Ho}(\text{tow} \cdot \mathcal{C}_0)$ if $f$ satisfies either of the following additional conditions:

\begin{itemize}
  \item[a)] $\sup \{\dim (X_j), \dim (Y_k)\} < \infty$;
  \item[b)] $f$ is movable.
\end{itemize}

**Proof.** For $\mathcal{C} = \mathcal{SS}$ the first part of this theorem is due to Grossman [Gros - 2]. For other $\mathcal{C}$, the proof is similar and omitted. The proof of the second part follows from §3.6 and appropriate filtered Whitehead theorems, which are proved in an identical manner to the proper Whitehead theorem occurring in [Br].
(5.5.7) The Stability Theorem in $\text{Ho}(\text{tow-}C_0)$. Let $X \in \text{tow-}C_0$ and let $h : \text{holim}(X) \to X$ be the canonical map in $\text{Ho}(\text{tow-}C_0)$. If $\text{pro-}^n_i(X)$ is stable for all $i \geq 1$, then $h$ is an isomorphism in $\text{Ho}(\text{tow-}C_0)$. $h$ is an isomorphism in $\text{Ho}(\text{tow-}C_0)$ if $X$ satisfies either of the following conditions:

a) $\sup \{\text{dim}(X_j)\} < \infty$;

b) $X$ is dominated in $\text{pro-Ho}(C_0)$ by an object of $\text{Ho}(C_0)$.

Proof. The first part follows from the first part of (5.5.6). For the second part, Theorem (5.5.5) implies that $X$ is isomorphic in $\text{tow-Ho}(C_0)$ to an object $Q$ of $\text{Ho}(C_0)$. Theorem (5.2.9) then implies that $X$ is isomorphic to $Q$ in $\text{Ho}(\text{tow-}C_0)$. The properties of the functor $\text{holim}$ now implies that $h$ is an isomorphism in $\text{Ho}(\text{tow-}C_0)$.

(5.5.8) Remarks. Porter [For-2] has given a simple argument which shows that if $X \in \text{Ho}(\text{pro-}SS_0)$ is dominated in $\text{Ho}(\text{pro-}SS_0)$ by an object in $\text{Ho}(SS_0)$, then $h$ is an isomorphism in $\text{Ho}(\text{pro-}SS_0)$. In fact, Porter's argument depends only on the functorial properties of $\text{holim}$, and hence shows that if $X \in \text{Ho}(\text{pro-}C)$.
is dominated in $\text{Ho}(\text{pro-}C)$ by an object of $\text{Ho}(C)$, then the canonical map $h: \text{holim}(X) \to X$ is an isomorphism in $\text{Ho}(\text{pro-}C)$.

If $X \in \text{pro-}\text{Ho}(C)$ is dominated by $P \in \text{Ho}(C)$ (i.e., we are given maps $P \xrightarrow{d} X$ in $\text{pro-}\text{Ho}(C)$ with $du = 1_X$ in $\text{pro-}\text{Ho}(C)$), then $X$ is isomorphic in $\text{pro-}\text{Ho}(C)$ to $Y = \{P \xleftarrow{ud} P \xleftarrow{ud} \ldots\}$. $Y$ is easily seen to be dominated in $\text{tow-}\text{Ho}(C)$ by $P$. Recent work of Chapman and Ferry [C-F] implies that if $Y \in \text{tow-}C$ is dominated in $\text{tow-}\text{Ho}(C)$ by a $P \in \text{Ho}(C)$, then $Y$ is also dominated by $P$ in $\text{Ho}(\text{tow-}C)$, and hence stable by Porter's argument. Thus, $X$ will also be stable in $\text{pro-}\text{Ho}(C)$. Thus, an object of $\text{pro-}\text{Ho}(C)$ is stable if and only if it is dominated by a stable object.

$f: P \to P$ is said to be a homotopy idempotent if $f^2 = f$ in $\text{Ho}(C)$. $f$ is said to split through $Q$ if there exists maps $P \xleftarrow{u} Q$ such that $du = 1_Q$ and $ud = f$ in $\text{Ho}(C)$. The above shows that every homotopy idempotent splits. This may be used to show that every homotopy idempotent on an $\ell^2$-manifold is homotopic to a strict idempotent and every homotopy idempotent $f$ on a compact $Q$-manifold $M$ is homotopic to a strict idempotent if and only if a
certain Wall obstruction \( W(f) \in \tilde{k}_0(\pi_1(M)) \) vanishes.\(^+\)

The development of coherent prohomotopy theory should enable one to extend Theorems (5.5.6) and (5.5.7) to \( \text{Ho(pro-}C_0) \) (see [Por-3]).

\(^+\)These observations were obtained in a conversation of the first author with T. Chapman.
Below we present a number of examples which show the precision of the above results.

(5.5.9) Example. Example (5.5.2) showed the need of some extra condition, such as conditions a) or b) of (5.5.3), in order that a \( \mathfrak{d} \)-isomorphism be an isomorphism. In this example we show that it is insufficient to require \( X \) and \( Y \) to be movable. Such an example was first constructed by Draper and Keesling in [D-K].

Let \( S_n = V_{i \geq n} S_i \) and \( i_n : S_n \to S_{n-1} \) be the natural inclusion.

Let \( \prod_{k \leq n} S_n = Y_n \). Let \( p_n : X_n \to X_{n-1} \), \( b_n : Y_n \to Y_{n-1} \), and \( f_n : X_n \to Y_n \) be defined by the following diagram

\[
\begin{array}{c}
S_1 \times \cdots \times S_{n-1} \times S_n \xrightarrow{1 \times \cdots \times 1 \times 0} S_1 \times \cdots \times S_{n-1} \times S_n \\
| \quad | \quad | \\
\downarrow 1 \quad \downarrow 1 \quad \downarrow i_n \\
S_1 \times \cdots \times S_{n-1} \xrightarrow{1 \times \cdots \times 0} S_1 \times \cdots \times S_{n-1}
\end{array}
\]

Then \( X = \{ X_n, p_n \} \) and \( Y = \{ Y_n, b_n \} \) are movable and \( f = \{ f_n \} \) is a \( \mathfrak{d} \)-isomorphism (easy), but \( f \) is not an isomorphism in \( \text{pro-Ho}(CW_0) \); in fact, there is an obvious essential map \( S^\infty \to X \) such that the composition \( S^\infty \to X \xrightarrow{f} Y \) is inessential.

(5.5.10) Examples. We will show in the examples below that the phenomena exhibited by examples (5.5.2) and (6.6.9) can be realized by
inverse systems of finite complexes. These are much deeper examples, though all the depth comes from J. F. Adams. Recall Adams' essential map of Moore spaces \( Y \) (these are pointed finite complexes of the form \( S^k \cup q D^{k+1} \)) [Adams - 2]

\[ A: \Sigma^{2r} Y \to Y; \]

this map is detected by the isomorphism

\[ A^*: \tilde{K}(Y) \to \tilde{K}(\Sigma^{2r} Y) \not= 0, \]

where \( \tilde{K} \) denotes reduced \( K \)-theory. Hence, the composites

\[ A_n^m: \Sigma^{2rm} Y \to \Sigma^{2rm} Y \]

are all essential, and thus the inverse system \( Z \equiv \{ \Sigma^{2rm} Y, A_n^m \} \)

is not equivalent to a point in \( \text{pro-} \text{Ho}(CW_0) \) even though \( \text{pro-} \pi_i(Z) = 0 \) for all \( i \geq 1 \). Applying the construction of Example (5.5.11) to \( Z \) in place of \( S^\infty \) yields a map between movable towers of finite complexes which is a \( \mathcal{U} \)-isomorphism but not an isomorphism in \( \text{pro-} \text{Ho}(CW_0) \).

The following example provides counter-examples to many conjectures (see [E-H-4]).

Let \( \{ B_n \} \) be the inverse system with \( B_0 \) a point, and
\[ B_n = \prod_{i=1}^{n} S^{2r} \text{ for } n \geq 1, \] and with bonding maps \( B_n \to B_{n-1} \) given by projection onto the last \( n-1 \) \( S^{2r} \) factors.

Let \( E_n = Y \times B_n \), and let \( p_n : E_n = Y \times B_n \to B_n \) be the projection; thus \( F_n \), the fibre of \( p_n \), is \( Y \). We shall define "twisted" bonding maps \( f_n : E_n \to E_{n-1} \) so that the restrictions \( f_n|_{F_n} : F_n \to F_{n-1} \) are null-homotopic, yet no composite \( E_n \to E_{n-k} \) factors through a \( B_{n-1} \).

Form the commutative solid-arrow diagrams

\[
\begin{array}{ccc}
E_n & \xrightarrow{f_n} & E_{n-1} \\
\downarrow Y \times (S^{2r})_n & \downarrow & \downarrow Y \times (S^{2r})_{n-1} \\
Y \times (S^{2r})_n & \xrightarrow{p_n} & Y \times (S^{2r})_{n-1} \\
\downarrow & \downarrow & \downarrow \\
B_n & \xrightarrow{p_n} & B_{n-1}
\end{array}
\]

\( (E'_n \) is the pullback, and \( p'_n \) is the projection). Define the filler \( \phi_n \) by requiring that \( p'_n \phi_n = p_n \), and letting the composite mapping

\[
Y \times (S^{2r})_n \xrightarrow{f_n} Y \times (S^{2r})_{n-1} \to Y
\]
\[ Y \times (S^{2r})^n \xrightarrow{\text{id} \times \pi} Y \times S^{2r} \xrightarrow{\_} Y \wedge S^{2r} \xrightarrow{A} Y; \]

here \( \pi \) is the projection onto the first \( S^{2r} \) factor.

Finally, let the bonding map \( f_n : E_n \to E_{n-1} \) be given by the composite \( E_n \xrightarrow{\phi_n} E'_n \xrightarrow{E_{n-1}} \). This yields the tower \( \{E_n\} \) and tower of fibrations \( \{F_n \xrightarrow{p_n} B_n\} \).

**Claim 1.** The tower \( F_n \) is contractible; since the bonding maps \( f_n |_{F_n} : F_n \to F_{n-1} \) are given by the composites

\[ F_n = Y \to Y \times * \to Y \times S^{2r} \to Y \]

\( \{F_n\} \) is isomorphic in \( \text{Pro-Top} \) to \( * \).

We may use the basepoints in the \( F_n (= Y) \) to define a section \( \{s_n : B_n \to E_n\} \). Since \( \{p_n\} \{s_n\} = \text{id} \{B_n\} \), to show that \( \{p_n\} \) is not invertible, it suffices to verify

**Claim 2.** \( \{s_n\} \{p_n\} \neq \text{id} \{E_n\} \) in \( \text{Pro-Ho(Top)} \). Assume otherwise, then for arbitrarily large \( n \) and suitable \( m \) (depending upon \( n \), but with \( n - m \geq 0 \)) the diagram
would commute up to homotopy.

Consider the subdiagram

\[
\begin{align*}
E_n &= Y \times (S^2)^n \xrightarrow{p_n} (S^2)^n \xrightarrow{s_n} Y \times (S^2)^n \\
E_{n-m} &= Y \times (S^2)^{n-m} \xrightarrow{p_{n-m}} (S^2)^{n-m} \xrightarrow{s_{n-m}} Y \times (S^2)^{n-m} \\
\end{align*}
\]

Note that all of the above maps are products with \( \text{id}_{(S^2)^{n-m}} \).

Hence, by this fact, and projecting the lower right corner onto \( Y \), we obtain a homotopy commutative diagram
where $g$ is induced from the bonding maps. By construction, $g$ is the composite

$$Y \times (S^2)^m \longrightarrow Y \wedge (S^2)^m = \bigvee_{Y}^{2rm} A \rightarrow Y.$$ 

Since $\tilde{K}(Y \wedge (S^2)^m)$ is a direct summand in $\tilde{K}(Y \times (S^2)^m)$, $g^{*}: \tilde{K}(Y) \longrightarrow \tilde{K}(Y \times (S^2)^m)$ is non-zero, hence Claim 2 holds. \(\square\)

Proposition. \(\prod_{(\alpha)}^{\pi} (\{ p_{\omega} \}; \{ \pi_\omega \}(F) \rightarrow \{ \pi_\omega \}(B))\) is a pro-isomorphism.

Proof. This follows from chasing in the commutative solid arrow diagram

\[
\begin{array}{ccccccc}
\pi_\omega(F_n) & \longrightarrow & \pi_\omega(E_n) & \xrightarrow{\varphi_n} & \pi_\omega(B_n) & \longrightarrow & \pi_\omega(F_n) \\
\downarrow & & \downarrow & \varphi_n^* & \downarrow & & \downarrow \\
0 & & 0 & & & & 0 \\
\pi_\omega(F_{n-1}) & \longrightarrow & \pi_\omega(E_{n-1}) & \xrightarrow{p_{n-1}^*} & \pi_\omega(B_{n-1}) & \longrightarrow & \pi_\omega(F_{n-1}).
\end{array}
\]
(5.5.10b) Proposition. An infinite dimensional Whitehead Theorem fails in shape theory.

Proof. Let $E = \lim E_n$, $\overline{B} = \lim B_n$; then $\overline{p}: E \to \overline{B}$ is a map of compact metric spaces which is an isomorphism on Čech pro-homotopy (pro-$\pi_*$) but not a shape equivalence. \qed

(5.5.10c) Remarks. $\overline{p}$ is even a C–E map, see J. L. Taylor \cite{Tay-1}

(5.5.10d) Proposition. An infinite dimensional Whitehead Theorem fails in proper homotopy theory.

Proof. Let Tel $E$ be the infinite mapping cylinder (telescope)

Tel $(* \leftarrow E_0 \leftarrow E_1 \leftarrow \cdots)$; define Tel $B$ and Tel $p$:

Tel $E \to$ Tel $B$ similarly. Then Tel $p$ is an ordinary homotopy equivalence and a pro-$\pi_*$ isomorphism at $\infty$.

$(E(Tel p): E(Tel E) \to E(Tel B)$ is given by $\{p_n\}: \{E_n\} \to \{B_n\}$ up to homotopy where $E$ is the ends functor) but Tel $p$ is not a proper homotopy equivalence (at $\infty$). \qed

(5.5.11) Example. We will construct below an inverse system $X$ of simplicial sets such that pro-$\pi_\infty(X) = 0$, but $X$ is not contractible. We do not know whether $X$ can be chosen to be an inverse
system of finite complexes or even the end of a locally finite complex.

Let $K(Z_2, n)$ denote the simplicial Eilenberg-MacLane space (see [May]). The direct system

$$
K(Z_2, 1) \xrightarrow{Sq^1} K(Z_2, 2) \xrightarrow{Sq^2} K(Z_2, 4) \xrightarrow{Sq^4} \cdots
$$

has the property that all composite maps

$$
\phi = Sq^{2n+k-1} \cdots Sq^n : K(Z_2, 2^n) \xrightarrow{\text{ess}} K(Z_2, 2^{n+k})
$$

are essential (evaluate $\phi$ on the class $x^{2^n}$ in $H^{2^n}(K(Z_2, 1); Z_2)$ where $x$ is the generator of $H^1(K(Z_2, 1); Z_2)$; see N. E. Steenrod and D. B. A. Epstein [S-E]) but each bonding map kills $\pi_0$.

Form the inverse system $X$ shown below.

\[
X = \left\{ \begin{array}{c}
X_3 = K(Z_2, 1) \times K(Z_2, 2) \times K(Z_2, 4) \times \cdots \\
X_2 = K(Z_2, 1) \times K(Z_2, 2) \times K(Z_2, 4) \times \cdots \\
X_1 = K(Z_2, 1) \times K(Z_2, 2) \times (K(Z_2, 4) \times \cdots
\end{array} \right.
\]

Then $X$ has the required properties.
(5.5.12) Example. If \( \pi = \{ \pi_i \} \) is a pro-group, we shall call \( K(\pi,1) \equiv \{ K(\pi_i,1) \} \) the standard Eilenberg-MacLane pro-space with fundamental pro-group \( \pi \). An Eilenberg-MacLane pro-space \( X = \{ X_n \} \) (i.e., \( \{ \tilde{X}_n \} \) is contractible) such that \( X \) is not equivalent to \( \{ K(\pi_1(X_n),1) \} \) in pro-\( \text{Ho}(SS_0) \) will be called exotic.

There exist exotic Eilenberg-MacLane pro-spaces! Below we shall construct an exotic \( K(\mathbb{Z}_2,1) \) by (roughly) using Example (5.5.11) as the fibre of a fibration over the standard \( K(\mathbb{Z}_2,1) \), and twisting the bonding maps as in Example (5.5.10). If \( K(\pi,1) \) is not finitely dominated, then there does not exist an inverse system of finite complexes \( X \) such that \( \text{pro-}^{-1}(X) = \pi \) and \( \tilde{X} \) is contractible.

Whether such an exotic \( X \) can come from the end of a locally finite complex is an important question in infinite dimensional topology (see §7).

Let \( F_n = \Pi_{m=1}^\infty K(\mathbb{Z}_2,2^m) \) for all \( n \), let \( X_n = F_n \times K(\mathbb{Z}_2,1) \) for all \( n \), and define twisted bonding maps \( X_n \to X_{n-1} \) so that the diagrams

\[
\begin{array}{ccc}
X_n & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
K(\mathbb{Z}_2,1) & \xrightarrow{id} & K(\mathbb{Z}_2,1), \\
\end{array}
\quad
\begin{array}{ccc}
X_n & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\text{pt} & \longrightarrow & K(\mathbb{Z}_2,2)
\end{array}
\]
and

\[
\begin{array}{ccc}
X_n & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
K(\mathbb{Z}_2, 2^m) \times K(\mathbb{Z}_2, 1) & \longrightarrow & K(\mathbb{Z}_2, 2^{m+1}) \\
\downarrow & & \downarrow \\
y \times x^{2^m} & \longrightarrow & \text{map} \\
\end{array}
\]

(5.5.13)

\(m \geq 1\)

commute. In diagram (5.5.13), \(x \in H^1(K(\mathbb{Z}_2, 1); \mathbb{Z}_2)\) and \(y \in H^2_K(\mathbb{Z}_2, 2^m); \mathbb{Z}_2\) are the generators, and the map \(y \times x^{2^m}\) represents the cohomology class \(y \times x^{2^m}\) in

\[H^{2m+1}(K(\mathbb{Z}_2, 2^m) \times K(\mathbb{Z}_2, 1); \mathbb{Z}_2).\]

As in Example (5.5.10), we obtain an inverse system of fibrations

\[
\begin{array}{ccc}
\cdots & \longrightarrow & X_n \\
\downarrow & & \downarrow \\
\text{bond} & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \text{map} \\
\downarrow & & \downarrow \\
P_n & \longrightarrow & P_{n-1} \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & K(\mathbb{Z}_2, 1) \\
\end{array}
\]

It is shown in [E-H-3] that \(X = \{X_n\}\) is an exotic Eilenberg-MacLane pro-space.
(5.5.14) Example. Let $X_n = \prod_{i=1}^{n} S^i$. Then $\text{pro-} \pi_1[X_n]$ is stable for all $i \geq 1$. Hence, the map $\text{holim} \{X_n\} \longrightarrow \{X_n\}$ is a $\ast$-isomorphism. Also, $X = \{X_n\}$ is movable. But $X$ is not stable. Dydak [Dyd-1] claims to have shown that if $X$ is movable and $\text{pro-} \pi_\omega(X)$ is stable, then $X$ is stable.

§5.6. Strong homotopy and homology theories.

We shall define strong homotopy and homology theories on $\text{pro-} SS_\ast$. At present our main application is the development of generalized Steenrod homology theories on compact metric spaces (see §8).

$$\pi_i \{X_j\} = \text{Ho}(\text{pro-} SS_\ast)(S^i, \{X_j\}) \cong \text{Ho}(SS_\ast)(S^i, \text{holim} \{X_j\})$$

$$\cong \pi_i (\text{holim} \{X_j\}).$$

These satisfy the usual properties, in particular a fibration sequence in $\text{pro-} SS_\ast$ yields a long-exact-sequence of homotopy groups.

(5.6.2) Strong stable homotopy groups. For any $\text{pro-} (\text{simplicial spectrum}) \{X_j\}$, define

$$\pi_i^S \{X_j\} = \text{Ho}(\text{pro-} Sp)(S^i, \{X_j\}) \cong \text{Ho}(Sp)(S^i, \text{holim} \{X_j\})$$

$$\cong \pi_i^S (\text{holim} \{X_j\}).$$
(5.6.3) Proposition. \( \{ \pi_i^S \} \) forms a generalized homology theory on \( \text{pro-} \text{Sp} \).

Proof. Clearly the functors \( \pi_i^S \) on \( \text{pro-} \text{Sp} \) are homotopy invariant. We begin by verifying the exactness axiom. Because each cofibration \( A \rightarrow X \) is isomorphic to a levelwise cofibration \( \{A_j\} \rightarrow \{X_j\} \) (Proposition (3.3.36)) it suffices to show that an inverse system of cofibration sequences \( \{A_j \rightarrow X_j \rightarrow X_j/A_j\} \) induces three-term exact sequences \( \pi_i^S \{A_j\} \rightarrow \pi_i^S \{X_j\} \rightarrow \pi_i^S \{X_j/A_j\} \). We may functorially replace \( \{X_j/A_j\} \) by a fibrant pro- (simplicial spectrum) \( \{Y_j\} \) \( \{X_j/A_j \rightarrow X_j\} \), replace the map \( \{X_j\} \rightarrow \{Y_j\} \) by a fibration \( \{X'_j\} \rightarrow \{Y_j\} \) \( \{X_j \rightarrow X'_j\} \) in \( \text{pro-} \text{Sp} \) with (levelwise) fibre \( \{F_j\} \), and form compatible commutative diagrams

\[
\begin{array}{ccc}
A_j & \rightarrow & X_j \rightarrow X_j/A_j \\
\downarrow \cong & & \downarrow \cong \\
F_j & \rightarrow & X'_j \rightarrow Y_j
\end{array}
\]

(5.6.4)

(The left weak equivalences arise because \( \pi_i^S \) is exact on both cofibration and fibration sequences in \( \text{Ho}(\text{Sp}) \); loosely, cofibration and fibration sequences are "the same" in \( \text{Ho}(\text{Sp}) \)\.) Therefore the sequences \( \pi_i^S \{A_j\} \rightarrow \pi_i^S \{X_j\} \rightarrow \pi_i^S \{X_j/A_j\} \) and
$$\pi_i^S F_j \longrightarrow \pi_i^S X'_j \longrightarrow \pi_i^S Y_j$$ are isomorphic. But the latter sequence is isomorphic to the sequence

$$\pi_i^S (\text{holim } \{F_j\}) \longrightarrow \pi_i^S (\text{holim } \{X'_j\}) \longrightarrow \pi_i^S (\text{holim } \{Y_j\})$$

which is exact because an inverse system of fibrations is a fibration sequence in \(\text{Ho(pro-Sp)}\), \(\text{holim} \equiv \text{lim} \) on the fibrant objects \(\{F_j\}, \{X'_j\}\) and \(\{Y_j\}\) (§4.2) and \(\text{lim} \) preserves fibration sequences (use Theorem (3.3.4)). The exactness axiom follows. We may iterate this process to obtain a long-exact sequence.

For the suspension axiom, consider a cofibration sequence of the form \(\{X_j\} \longrightarrow \{CX_j\} \longrightarrow \{EX_j\}\). By regarding this sequence as a fibration sequence, we obtain an exact sequence

$$0 = \pi_{i+1}^S \{CX_j\} \longrightarrow \pi_{i+1}^S \{EX_j\} \longrightarrow \pi_i^S \{X_j\} \longrightarrow \pi_i^S \{CX_j\} = 0;$$

hence the suspension axiom holds. The conclusion follows. \(\square\)

(5.6.5) **Strong homotopy groups and homotopy pro-groups.** Both unstably and stably these are related by the Bousfield-Kan spectral sequence (§4.9).

(5.6.6) **Strong (ordinary) homology groups.** Let \(R\) be a commutative ring with identity. We shall develop a strong homology theory \(S^h_R(\cdot;R)\) on \(\text{pro-SS}_*\). Bousfield and Kan [B-K] associate with \(R\) a free simplicial \(R\)-module functor; for \(X\) in \(\text{SS}_*\), \((RX)_n\) is the
free \( R \)-module with \( X_n \) as basis, mod \( R \) where \( * \) is the basepoint of \( X \). There results a simplicial \( R \)-module, \( RX = \{ (RX_n, d_i s_i) \} \), which depends functorially on \( X \). Because \( R \) maps a cofibration sequence \( A \to X \to X/A \) in \( \text{SS}_* \) into a fibration sequence \( RA \to RX \to R(X/A) \), the functor \( \pi_* (R-) \) is a (reduced) homology and Thom's theory on \( \text{SS}_* \). As with Bold/√∞ infinite symmetric product \([D-T]\),

\[ \pi_* (R-) \equiv \tilde{H}_*(\cdot; R). \]

We prolong the Bousfield-Kan functor \( R \) to \( \text{pro}\text{-}\text{SS}_* \) by defining \( R(X_j) = \{ RX_j \} \), and define

\[ S_{\tilde{H}}(\{ X_j \}; R) = \pi_*(R(X_j)). \]

Observe that \( R \) takes an inverse system of cofibration sequences into an inverse system of fibration sequences, which is a fibration sequence in \( \text{Ho}(\text{pro}\text{-}\text{SS}_*) \) by Proposition (3.4.17). This yields the exactness axiom for \( S_{\tilde{H}}(\cdot; R) \).

For the suspension axiom, first observe that natural map

\[ \{ X_j \} \to R(X_j) \] (induced by the identity of \( R \)) yields maps

\[ \{ EX_j \} \to \{ ERX_j \} \to \{ WRX_j \} \] where \( E \) is the simplicial suspension (see e.g. [May]). But \( \{ WRX_j \} \) is a pro-(simplicial \( R \)-module) so we obtain a map
$$R\{\text{EX}_j\} \cong \{\text{REX}_j\} \longrightarrow \{\text{WRX}_j\}.$$ 

It is easy to check that this map is a level weak equivalence. Hence

$$S\hat{H}_\ast(\{X_j\}; R) \cong \pi_\ast(\text{RX}_j)$$

$$\cong \pi_{\ast+1}(\text{WRX}_j)$$

(use the fibration sequence

$$\{\text{RX}_j\} \longrightarrow \{\text{WRX}_j\} \longrightarrow \{\text{WRX}_j\}$$

(3.4.17), and (5.6.1))

Therefore, $S\hat{H}_\ast(-; R)$ is a homology theory on $\text{pro-SS}_\ast$. On $\text{SS}_\ast$,

$$S\hat{H}_\ast(-; R) \cong \hat{H}_\ast(-; R).$$

(5.6.7) **Strong (generalized) homology groups.** Let $\hat{h}_\ast$ be a generalized reduced homology theory on $\text{CW}_\ast$, which is represented by a CW spectrum $E$. That is, $E = \{E_n | n \geq 0\}$, together with cellular inclusions $\Sigma E_n \rightarrow E_{n+1}$, and

$$\hat{h}_\ast(X) = \pi_\ast S(X \wedge E) \cong \pi_\ast S(X \wedge E_n)$$

on $\text{CW}_\ast$. We prolong the smash product $- \wedge E$ to a functor from $\text{pro-SS}_\ast$ to $\text{pro-Sp}$ by defining

$$\{X_j\} \wedge E = \{\text{Sin ((RX)_j} \wedge E)\},$$
and set

\[ S_{\tilde{h}_*}(X_j) = \pi_* S_{\tilde{h}_*}(\{X_j\} \wedge E). \]

Because \(-\wedge E\) takes cofibration sequences over \(\text{Ho}(\text{pro-SS}_*)\) into cofibration sequences over \(\text{Ho}(\text{pro-Sp})\), and similarly, \(-\wedge E\) preserves suspensions, \(S_{\tilde{h}_*}(X_j)\) is a generalized reduced homology theory on \(\text{pro-SS}_*\). As above, \(S_{\tilde{h}_*} \equiv \tilde{h}_*\) on \(\text{SS}_*\).

(5.6.8) **Proposition.** For \(\{X_j\}\) in \(\text{SS}_*\), there is a Bousfield-Kan spectral sequence

\[ E^2_{p,q} = \lim_{\to} \tilde{h}_*(X_j), \]

which converges under suitable conditions to \(S_{\tilde{h}_*}(X_j)\). If \(\{X_j\}\) is isomorphic to a tower in \(\text{Ho}(\text{pro-SS}_*)\), the spectral sequence collapses to the short exact sequences

\[ 0 \rightarrow \lim_{\to} \tilde{h}_{n+1}(X_j) \rightarrow S_{\tilde{h}_*}(X) \rightarrow \lim \tilde{h}_n(X_j) \rightarrow 0. \]

**Proof.** Use (5.6.6), (5.6.7) and the Bousfield-Kan spectral sequence (§4.9). \(\square\)

We conclude this section with several remarks about cohomology theories.
(5.6.9) Cohomology groups. If $\check{h}^*$ is a generalized, reduced cohomology theory on $SS^*_h$, then $\check{h}^*$ induces a cohomology theory on pro-$SS^*_h$:

$$\check{H}^*(X_j) \cong \operatorname{colim}_j \check{h}^*(X_j)$$

because colim is exact. Further, if $E_n$ represents $\check{H}^n$ on $SS^*_h$, then

$$\check{H}^*(X_j) = \text{pro}-\operatorname{Ho}(SS^*_h)((X_j), E_n)$$

$$\quad = \operatorname{Ho}(\text{pro}-SS^*_h)((X_j), E_n).$$

The latter isomorphism exists because $E_n$ is stable.

(5.6.10) Representable theories. For $\{Y_k\} \in \text{pro}-SS^*_h$ there is an associated generalized cohomology theory defined by

$$\check{h}^n(-) = \operatorname{Ho}(\text{pro}-SS^*_h)(-, \Omega^{-n}(Y_k)), n \leq 0.$$  

Conversely, Alex Heller [Hel-1, §11] showed that any group-valued cohomology theory on an $h$-category (an abstraction of $Cd_h$, analogous to pointed closed model categories as abstractions of $SS^*_h$) is the colimit of a directed system of representable theories. Because Heller's proof only uses factorization through a stable category (such as pro-$Sp$), and properties of the homotopy relation and cofibrations, his result also holds for pro-$SS^*_h$ and pro-$Sp$. 
§6. PROPER HOMOTOPY THEORY

§6.1 Introduction.

In this section we shall apply pro-homotopy theory to the study of the proper homotopy theory of locally compact, $\sigma$-compact Hausdorff spaces via a functor (the end) from such spaces into pro-Top. The study of the end functor is closely related to proper homotopy theory at $\omega$. Our approach involves breaking a question in proper homotopy theory up into two questions: a question in ordinary homotopy theory and a question in proper homotopy theory at $\omega$.

§6.2 contains the basic definitions of proper homotopy theory and the ends functor.

In §6.3 we shall embed the proper homotopy category (of $\sigma$-compact spaces) in $\text{Ho}(\text{Top}, \text{tow-Top})$ and embed the proper homotopy category at $\omega$ in $\text{Ho}(\text{Top})$. Consequently, a proper map which is both a homotopy equivalence and a homotopy equivalence at $\omega$ (in the $\text{Ho}(\text{pro-Top})$ sense) is a proper homotopy equivalence.

§6.4 contains a discussion of weak-proper-homotopy theory (the notion is due to Chapman [Chap-1]). In particular, we show that every weak-proper-homotopy equivalence is weakly-properly-homotopic to a proper homotopy equivalence. This answers a question of Chapman and Siebenmann [C-S].

Proper Whitehead theorems are discussed in §6.5. We reprove Siebenmann's finite dimensional Whitehead theorem [Sieb-1] and show that an infinite dimensional analogue (claimed by E.M.Brown [Br] and F.T.Farrell, L.R.Taylor and
J.B.Wagoner [F-T-W]) fails in general.

§6.2. The end of a space.

We shall show how Chapman's formulation of proper homotopy theory leads to an embedding of the proper category in a closed model category via an \textit{ends} functor.

(6.2.1) \textbf{Proper homotopy theory following Chapman [Chap-1].} Define a continuous map \( f: X \to Y \) to be proper if for each compactum \( B \subseteq Y \) there exists a compactum \( A \subseteq X \) such that \( f(X \setminus A) \subseteq Y \setminus B \). (This is just a reformulation of the usual notion of a proper map.) Then proper maps \( f, g: X \to Y \) are said to be \textbf{weakly-properly-homotopic} if for each compactum \( B \subseteq Y \), there exists a compactum \( A \subseteq X \) and a homotopy (dependent upon \( B \)) \( H = \{ H_t \}: X \times I \to Y \) with \( H_0 = f \), \( H_1 = g \), and \( H((X \setminus A) \times I) \subseteq Y \setminus B \). If, in fact, there exists a proper map \( H: X \times I \to Y \) with \( H_0 = f \) and \( H_1 = g \), then we say that \( f \) and \( g \) are \textbf{properly-homotopic}. The notions of \textbf{weak-proper-homotopy equivalence} and \textbf{proper-homotopy equivalence} are now defined in the obvious way.

(6.2.2) \textbf{The proper categories.} Let \( P \) be the category of locally compact, Hausdorff spaces, and proper maps; let \( \text{Ho}(P) \) and \( \text{whO}(P) \) be the quotient categories obtained by identifying properly-homotopic maps, and weakly-properly-homotopic maps, respectively. Let \( P_\sigma \), \( \text{Ho}(P_\sigma) \) and \( \text{whO}(P_\sigma) \) be the respective subcategories of \( \sigma \)-compact spaces.

(6.2.3) \textbf{Definition.} For \( X \) in \( P \), the \textit{end} of \( X \) is the \textbf{pro-space}

\[
E(X) = \{ \overline{(X \setminus A)} \mid A \text{ a compactum in } X \},
\]

bonded by inclusion (--- denotes closure in \( X \)).
Observe that for $X$ in $P$, and any compactum $A$ in $X$, there is a compactum $B$ in $X$ such that $(X \setminus B) \subset X \setminus A$. We may thus replace complements of compacta by their closures in (6.2.1).

(6.2.4) **Proposition.** The end construction extends to functors as follows:

\[
\begin{array}{ccc}
P & \overset{E}{\longrightarrow} & \text{pro-Top} \\
\downarrow & & \downarrow \\
\text{Ho} \left( \overset{E}{P} \right) & \overset{\left(1,E\right)}{\longrightarrow} & \text{Ho} \left( \text{pro-Top} \right) \\
\downarrow & & \downarrow \\
\omega \text{Ho} \left( \overset{E}{P} \right) & \overset{\left(1,E\right)}{\longrightarrow} & \text{pro-} \text{Ho} \left( \text{Top} \right) \\
\end{array}
\]

where $(1,E)(X)$ is the pair $(X,E(X))$.

**Proof.** Immediate from the definitions. □

(6.2.7) **Proposition.** The functor $(1,E): P \longrightarrow (\text{Top, pro-Top})$ is a full embedding.

**Proof.** $(1,E)$ is an embedding because $P$ is a subcategory of $\text{Top}$. $(1,E)$ is full because a continuous map is proper if and only if it induces a map of the ends in pro-Top by (6.2.1). □

(6.2.8) **The proper categories at $\infty$.** On the other hand, $E:P \longrightarrow \text{pro-Top}$ is not an embedding, but rather reflects $P$ at $\infty$. 
A proper map at \( \infty \) (or germ of a proper map) \( f : X \to Y \) consists of a compactum \( A \subseteq X \) and a proper map \( f : (X \setminus A) \to Y \). Two such maps are equal if they agree on \( (X \setminus B) \) for some compactum \( B \) in \( X \). Let \( P_{\infty} \) be the category of locally compact Hausdorff spaces and proper maps at \( \infty \), and let \( P_{\sigma, \infty} \) be the full subcategory of \( \sigma \)-compact spaces. There are also proper-homotopy and weak-proper-homotopy categories at \( \infty \), \( \text{Ho}(P_{\infty}), \text{Ho}(P_{\sigma, \infty}), \text{wh}(P_{\infty}), \text{wh}(P_{\sigma, \infty}) \), defined as in (6.2.2).

(6.2.9) **Proposition.** The functors \( E \) factor as follows:

\[
\begin{array}{ccc}
P_{\infty} & \xrightarrow{E} & \text{pro-Top} \\
\downarrow & & \downarrow \\
\text{Ho}(P_{\infty}) & \xrightarrow{E} & \text{Ho}(\text{pro-Top}) \\
\downarrow & & \downarrow \\
\omega \text{Ho}(P_{1(\infty)}) & \xrightarrow{E} & \text{pro- Top}.
\end{array}
\]

(6.2.11) **Proposition.** The functor \( E : P_{\sigma} \to \text{pro-Top} \) is a full embedding.

(6.2.12) **The end of a \( \sigma \)-compact space.** We shall discuss the restriction of \( E \) to \( P_{\sigma} \). Let \( X \in P_{\sigma} \). Suppose that

(6.2.13) \( X = \bigcup_{n=0}^{\infty} K_n \),

where \( K_0 = \emptyset \), each \( K_n \) is compact, and \( K_n \subseteq \text{int} K_{n+1} \). Then set

(6.2.14) \( \varepsilon(X) \equiv \{ x_0 \supset x_1 \supset x_2 \supset \ldots \} \), where \( X_n \equiv (X \setminus K_n) \).

Then, \( \varepsilon(X) \) is a cofinal subtower of \( E(X) \); hence \( \varepsilon(X) \) is canonically isomorphic to \( E(X) \). Further, any two representations of \( \sigma \)-compactness of \( X \) (6.2.13) yield canonically isomorphic towers (6.2.14). We shall thus loosely regard \( \varepsilon \) as
a functor from $P_\sigma$ to $\text{ tow-Top }$ and call $\varepsilon(X)$ the end of $X$ when there is no chance of confusion. Similarly, $\varepsilon$ may be loosely regarded as a functor from $P_\sigma$ to $(\text{Top}, \text{ tow-Top})$, because $\varepsilon(X)_0 = X$, or as a functor from $P_\sigma$ to $\text{Filt.}$

§6.3 $P_\sigma$: towers, and telescopes.

We shall relate the proper homotopy theory of $P_\sigma$ to the pro-homotopy theory of $\text{ tow-Top }$ in Theorems (6.3.4)-(6.3.6). We shall associate to a space $X$ in $P_\sigma$ with end $\varepsilon(X)$ (as in (6.2.14)) the telescope $\text{Tel}(\varepsilon(X))$ and projection

\[ p_X : \text{Tel}(\varepsilon(X)) \to X, \quad p_X(x,t) = x. \]

Then $p_X$ is a filtered map ($X$ is filtered by $\varepsilon(X)$).

We shall need a suitable notion of naturality of $p_X$.

Let $\varepsilon(X)$ and $\varepsilon'(X)$ be two cofinal towers in the end of $X$.

Because $\varepsilon(X)$ and $\varepsilon'(X)$ are mutually cofinal, there is a natural equivalence $\text{Tel}(\varepsilon(X)) \cong \text{Tel}(\varepsilon'(X))$ in $\text{Ho}(\text{Tel})$ as in Proposition (3.6.13). Somewhat more is true. Maps $f_0, f_1 : W \to \text{Tel}(\varepsilon(X))$ with $p_Xf_0 = p_Xf_1 = f : W \to X$ are called vertically homotopic if there is a homotopy $H = \{H_t\} : W \times [0,1] \to \text{Tel}(\varepsilon(X))$ with $H_0 = f_0, H_1 = f_1$, and $p_Ht = f$ for all $t$. We call $H$ a vertical homotopy.

If $f_0, f_1$, and $H$ are also filtered maps, $f_0$ and $f_1$ are called filtered-vertically-homotopic. It is easy to prove the following.

(6.3.2) **Lemma.** (a) $\text{Tel}(\varepsilon(X))$ and $\text{Tel}(\varepsilon'(X))$ are canonically equivalent up to filtered vertical homotopy. (b) $p_X$ is natural up to filtered vertical homotopy of $\text{Tel}(\varepsilon(X))$.

**Proof.** For part (a) use the proof of Proposition (3.6.13). Part (b) follows immediately. □
(6.3.3) **Definition.** A proper section for $\varepsilon(X)$ is a filtered map $s: X \to \text{Tel}(\varepsilon(X))$ with $ps = \text{id}_X$.

(6.3.4) **Construction of proper sections.** The Tietze extension theorem yields maps $h_n: \overline{(K_n \setminus K_{n-1})} \to [n-2, n-1]$ with $h_n(\text{bd } K_{n-1}) = n-2$ and $h_n(\text{bd } K_n) = n-1$ for $n \geq 2$. We may glue these maps together to obtain a proper map $h: X \to \mathbb{R}^+$ (notation: $\mathbb{R}^+$ denotes the set of non-negative real numbers) such that $h(K_n \setminus K_{n-1}) \subset [n-2, n-1]$ for $n \geq 1$. Because $(X_{n-1} \setminus X_n) \subset \overline{(K_n \setminus K_{n-1})}$, there results a map $s: X \to \text{Tel}(\varepsilon(X))$, given by the formula

(6.3.5) $s(x) = (x, h(x))$.

Clearly $s$ is a proper section for $\varepsilon(X)$. In fact, each proper section $s'$ for $\varepsilon(X)$ comes from a suitable proper map $h: X \to \mathbb{R}^+$ and formula (6.3.5).

(6.3.6) **Proposition.** $X$ is a strong deformation retract of $\text{Tel}(\varepsilon(X))$ in $\text{Filt}$.

**Proof.** The required retraction and inclusion are given by $p_X: \text{Tel}(\varepsilon(X)) \to X$ and any proper section $s$ for $\varepsilon(X)$. The required homotopy from $\text{id}_{\text{Tel}(\varepsilon(X))}$ to $s p_X$ is given by

$H(x, t, t') = (x, (1-t')t + t' \cdot h(x))$,

where $s(x) = (x, h(x))$. For each $n$ choose $m > n$ so that $h(x_m) \subset [n, \infty)$. Then

$H(\text{Tel}(\varepsilon(X))_m \times [0,1]) \subset \text{Tel}(\varepsilon(X))_{n'}$,

so that $H$ is a filtered homotopy, as required. Note that $H$ is even vertical. □

(6.3.7) **Theorem.** The functor $\varepsilon: \text{Ho}(P_0) \to \text{Ho}(\text{Top}, \text{tow-Top})$
is a full embedding.

**Proof.** By §3.6, it suffices to prove that the composite functor

\[ \text{Tel} \circ \mathcal{E} : \text{Ho}(P_0) \to \text{Ho}(\text{Top}, \text{tow-Top}) \to \text{Ho}(	ext{Tel}) \to \text{Ho}(	ext{Filt}) \]

is a full embedding. But this is an immediate consequence of Proposition (6.3.6) and the definition of \( \text{Ho}(	ext{Tel}) \) as a full subcategory of \( \text{Ho}(	ext{Filt}) \). \( \square \)

(6.3.8) **Theorem.** The functor \( \mathcal{E} : \text{Ho}(P_0) \to \text{Ho}(\text{tow-Top}) \) is a full embedding.

**Proof.** Use the model \( \text{Ho}(	ext{Con Tel}) \) for \( \text{Ho}(\text{tow-Top}) \) and the observation that equivalence classes of maps

\[ \text{Tel} \left( \ast \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \right) \right) \to \text{Tel} \left( \ast \leftarrow Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots \right) \]

correspond bijectively with maps in \( \text{Ho}(	ext{Con Tel}) \) because

\[ \text{Tel} \left( \ast \leftarrow Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_k \right) \]

is contractible for any \( k \), as above. \( \square \)

(6.3.9) **Theorem.** Let \( f : X \to Y \) be a map in \( P \) which is a homotopy equivalence and a homotopy equivalence at \( \infty \) (in the strong sense). Then \( f \) is a proper homotopy equivalence.

**Proof.** By hypothesis, \( (f, \mathcal{E}(f)) : (X, \mathcal{E}(X)) \to (Y, \mathcal{E}(Y)) \) is an equivalence in \( \text{Ho}(	ext{Top}, \text{tow-Top}) \). Now use Theorem (6.3.7.). \( \square \)
§6.4. Weak proper homotopy theory.

In their work on compactifying $Q$-manifolds, Chapman and Siebenmann [C-S] asked the following questions.

(1) Is every weak-proper-homotopy equivalence a proper homotopy equivalence?

(2) Is every weak-proper-homotopy equivalence weakly-properly-homotopic to a proper homotopy equivalence?

Chapman and Siebenmann verified a special case of (2), the case of a proper map between $Q$-manifolds with tame ends.
In this section we shall give an affirmative answer to question (2) and discuss some related questions.

(E-H-3)

(6.4.1) Theorem. Let \( f: X \rightarrow Y \) be a proper map of \( \sigma \)-compact, locally compact, Hausdorff spaces. If \( f \) is a weak-proper-homotopy equivalence, then \( f \) is weakly-properly-homotopic to a proper homotopy equivalence.

Proof. Choose a proper map \( g: Y \rightarrow X \) which is a weak-proper-homotopy inverse to \( f \). Then choose ends

\[
\varepsilon(X) = \{ X = X_0 \supset X_1 \supset X_2 \supset \ldots \}, \text{ and } \\
\varepsilon(Y) = \{ Y = Y_0 \supset Y_1 \supset Y_2 \supset \ldots \}
\]

such that:

(i) \( X_n \subseteq \text{int}(X_{n-1}) \) and \( Y_n \subseteq \text{int}(Y_{n-1}) \) for \( n \geq 1 \);  
(ii) \( f(X_n) \subseteq Y_n \) for \( n \geq 0 \);  
(iii) \( g(Y_n) \subseteq X_{n-1} \) for \( n \geq 1 \);  
(iv) there exist homotopies \( H_n : X \times I \rightarrow X \) with \( H_n|_0 = \text{id} \) and \( H_n|_1 = gf \) for \( n \geq 0 \), and further, \( H_n(X_n \times I) \subseteq X_{n-1} \) for \( n \geq 1 \); and  
(v) there exist homotopies \( K_n : Y \times I \rightarrow Y \) with \( K_n|_0 = \text{id} \) and \( K_n|_1 = fg \) for \( n \geq 0 \), and further, \( K_n(Y_n \times I) \subseteq Y_{n-1} \) for \( n \geq 1 \).

(Such ends are easily obtained by successively passing to cofinal subsystems of any ends of \( X \) and \( Y \).)

We shall use the above data to construct a suitable proper homotopy equivalence \( f' : X \rightarrow Y \). Write \( f_n \) for \( f|_{X_n} \) and \( g_n \) for \( g|_{Y_n} \).

Let \( \bar{a} \) be the inverse system.

\[
\begin{array}{cccccc}
Y_0 & \overset{f_0}{\leftarrow} & X_0 & \overset{e_4}{\leftarrow} & Y_1 & \overset{f_1}{\leftarrow} & X_1 & \overset{e_2}{\leftarrow} & \cdots \\
\end{array}
\]
Form the homotopy-commutative diagram

\[ \begin{array}{cccccccc}
& & & & & & & \\
& \varepsilon(X) & \downarrow & & \varepsilon(Y) & \downarrow & & \\
& 1 & \downarrow & & 1 & \downarrow & & \\
& & \downarrow & & & \downarrow & & \\
(6.4.2) & \delta & \downarrow & & \delta & \downarrow & & \\
& Y_0 \xleftarrow{f_0} X_0 \xleftarrow{g_1} Y_1 \xleftarrow{f_1} X_1 \xleftarrow{g_2} Y_2 \xleftarrow{f_2} X_2 \xleftarrow{\ldots} & & \\
\end{array} \]

in which the vertical arrows denote the appropriate identity maps, and the required homotopies are given by conditions (iv) and (v) above.

Diagram (6.4.2) together with the homotopies (iv) and (v), yields filtered maps of mapping telescopers

\[ (6.4.3) \quad \text{Tel} (\varepsilon(X)) \xrightarrow{F} \text{Tel} (\delta) \xrightarrow{G} \text{Tel} (\varepsilon(Y)). \]

In order to give explicit formulas for these maps, we regard \( \text{Tel} (W_0 \xleftarrow{h_0} W_1 \xleftarrow{h_1} W_2 \xleftarrow{h_2} \ldots) \) as the union of the "levels"

\[ W_{n-1} \times 0 \cup W_n \times [0,1]. \]

Then \( F \) maps \( X_{n-1} \times 0 \cup X_n \times [0, 3/4] \) to \( X_{n-1} \times 0 \cup Y_n \times [0,1] \) by the formula

\[
F(x, t)= \begin{cases} 
(h_n(x, 2t), 0) \in X_{n-1} \times 0 \text{ for } 0 \leq t \leq 1/2, \\
(g_n f_n(t), 0) \in X_n \times 0 \text{ for } t = 1/2, \\
(f_n(x), 4t - 2) \in Y_n \times [0,1] \text{ for } 1/2 \leq t \leq 3/4,
\end{cases}
\]
and $F$ maps $X_n \times [3/4, 1]$ to $Y_n \times 0 \cup \bigcup_{f_n} X_n \times [0,1]$ by the formula

$$F(x,t) = \begin{cases} 
(f_n(x),0) & \in Y_n \times 0 \text{ for } t = 3/4, \\
(x,4t-3) & \in X_n \times [0,1] \text{ for } 3/4 \leq t \leq 1.
\end{cases}$$

See Figure I. The map $G$ in diagram (6.4.3) has an analogous description.

$\text{Diagram:}$

$\text{Figure I}$
The map $F': \text{Tel}(\tilde{\mathcal{E}}) \longrightarrow \text{Tel}(\mathcal{E}(X))$ maps
\[ X_{n-1} \times 0 \cup_{g_n} X_n \times [0,1] \text{ into } X_{n-1} \times 0 \cup X_n \times [0,1] \text{ according to the formula:} \]
\[
\begin{cases}
F'(x,0) = (x,0), \\
F'(y,t) = (g_n(y),0),
\end{cases}
\]
and maps $X_n \times 0 \cup_{f_n} X_n \times [0,1]$ into
\[ X_{n-1} \times 0 \cup X_n \times [0,1] \text{ according to the formula:} \]
\[
\begin{cases}
F'(y,0) = (g_n(y),0), \\
(g_n f_n(x),0) \text{ for } t = 0, \\
F'(x,t) = \begin{cases}
(H_n(x,1-2t),0) \text{ for } 0 \leq t \leq 1/2, \\
(x,2t-1) \text{ for } 1/2 \leq t \leq 1.
\end{cases}
\end{cases}
\]

See Figure II. The function $(,t) \longrightarrow (H_n(,1-2t),0)$ used in the definition of $F'$ is indicated by "-H" in Figure II to indicate that $1-2t$ decreases as $t$ increases. The map $G'$ in diagram (6.4.2) has an analogous description.
\( F': \text{Tel} (\overline{Z}) \longrightarrow \text{Tel} (\varepsilon(X)) \)
Our construction also yields filtered homotopies

\[(6.4.4) F'F - \text{id}_{\text{Tel} \{\varepsilon(X)\}''}, \quad FF' - \text{id}_{\text{Tel} \mathbb{A}''}, \quad \text{and} \]

\[G'G - \text{id}_{\text{Tel} \{\varepsilon(Y)\}''}, \quad GG' - \text{id}_{\text{Tel} \mathbb{A}''}.\]

These homotopies arise from first deforming maps \(H: X_n \times I \longrightarrow X_{n-1}\)
of the form \(H_n + "-H_n":\)

\[
H(x, t) = \begin{cases} 
H_n(x, 2t) & \text{for } 0 \leq t \leq 1/2, \\
H_n(x, 2-2t) & \text{for } 1/2 \leq t \leq 1 
\end{cases}
\]

to the constant homotopy while keeping the top and bottom of the cylinder fixed, and then moving at most two levels up within the telescopes. See Figure III for a representation of the homotopy \(F'F - \text{id}_{\text{Tel} \{\varepsilon(X)\}''}\) which illustrates this process. The formulas are complicated and not enlightening, and therefore omitted.
H_{n+(-H_n)} pulled back to the identity

\text{id}\ _{\text{Tel}}(\varepsilon(X))

Deforming F'F to \text{id}\ _{\text{Tel}}(\varepsilon(X))

Figure III
Now write \( p: \text{Tel} (\varepsilon(X)) \to X \) and \( p': \text{Tel} (\varepsilon(Y)) \to Y \) for the projections. Choose proper maps \( h: X \to \mathbb{R}^+ \) and \( h': Y \to \mathbb{R}^+ \) with the property that \( h(X \setminus X_{n-1}) \subseteq [n-2, n-1] \) and \( h'(Y \setminus Y_{n-1}) \subseteq [n-2, n-1] \) as in (6.3.4). (This is possible by property (i) of \( \varepsilon(X) \) and \( \varepsilon(Y) \).)

As in formula (6.3.5), there result proper sections \( s \) for \( \varepsilon(X) \) and \( s' \) for \( \varepsilon(Y) \), respectively. Define maps \( f': X \to Y \) and \( g': Y \to X \) to be the composites

\[
\begin{align*}
X \xrightarrow{s} \text{Tel} (\varepsilon(X)) & \xrightarrow{F} \text{Tel} (\overline{X}) \xrightarrow{G'} \text{Tel} (\varepsilon(Y)) \xrightarrow{p'} Y, \quad \text{and} \\
Y \xrightarrow{s'} \text{Tel} (\varepsilon(Y)) & \xrightarrow{G} \text{Tel} (\overline{Y}) \xrightarrow{F'} \text{Tel} (\varepsilon(X)) \xrightarrow{p} X,
\end{align*}
\]

respectively. Because \( f' \) and \( g' \) are composites of filtered maps, they are filtered maps, and thus proper maps.

To complete the proof, we shall verify the following claims.

**Claim 1.** The maps \( f' \) and \( g' \) are proper homotopy inverses, hence the map \( f': X \to Y \) is a proper homotopy equivalence.

**Claim 2.** The maps \( f', f : X \to Y \) are weakly-properly-homotopic.
Verification of Claim 1. Consider the commutative solid-arrow diagram

\[
\begin{array}{cccccc}
\text{Tel } (e(X)) & \xrightarrow{G'F} & \text{Tel } (e(Y)) & \xrightarrow{id} & \text{Tel } (e(Y)) & \xrightarrow{F'G} & \text{Tel } (e(X)) \\
\vdash & & \downarrow{p'} & & \downarrow{s'} & & \downarrow{p} \\
X & \xrightarrow{f'} & Y & \xrightarrow{g'} & X.
\end{array}
\]

There exists a vertical homotopy \( H \) between the maps

\[
H_0 = s' \circ p' \circ G'F \circ s, \quad H_1 = G'F \circ s: X \rightarrow \text{Tel } (e(Y));
\]

given by the formula

\[
H(x, \tau) = (f'(x), (1-\tau) \cdot \pi (H_0 (x)) + \tau \cdot \pi (H_1 (x)))
\]

where \( \pi \) denotes the projection \( \text{Tel } (F(Y)) \rightarrow R^+ \). Because

\[
| \pi H_0 (x) - h' (x) | \quad \text{and} \quad | \pi H_1 (x) - h (x) | \leq 2
\]

by construction, \( \pi H_1 \) is a proper map and \( H \) is a filtered vertical homotopy. Hence the maps

\[
g' f', \quad p \circ F' G \circ G' F \circ s: X \rightarrow X
\]

are properly homotopic. But the composite \( F' \circ GG' \circ F \) is filtered-homotopic to \( \text{id}_{\text{Tel } (e(X))} \) by composites of the filtered homotopies of formula (6.4.4). Hence the map \( g'f' \) is properly homotopic to \( \text{id}_X \).
Similarly, \( f'g' \) is properly homotopic to \( \text{id}_Y \), and Claim 1 follows.

**Verification of Claim 2.** The construction of the telescopes \( \text{Tel} (\varepsilon(X)), \text{Tel} (\varepsilon(Y)) \) and \( \text{Tel} (\tilde{\varepsilon}) \) is readily extended to give telescopes

\[
\text{Tel} (\tilde{X}) = \text{Tel} (X \leftarrow \text{id} X \leftarrow \text{id} \ldots) \sim X \times \mathbb{R}^+;
\]

\[
\text{Tel} (\tilde{Y}) = \text{Tel} (Y \leftarrow \text{id} Y \leftarrow \text{id} \ldots) \sim Y \times \mathbb{R}^+; \text{ and}
\]

\[
\text{Tel} (\tilde{\varepsilon}) = \text{Tel} (Y \leftarrow \frac{f}{f} X \leftarrow \frac{g}{g} Y \leftarrow \frac{f}{f} X \leftarrow \mathbb{R} \ldots).
\]

Also, because the homotopies \( H_n \) and \( K_n \) (used to define the maps \( F: \text{Tel} (\varepsilon(X)) \rightarrow \text{Tel} (\tilde{\varepsilon}) \) and \( G': \text{Tel} (\tilde{\varepsilon}) \rightarrow \text{Tel} (\varepsilon(Y)) \) (see formula (6.4.3) and the following discussion)) are defined on all of \( X \times I \) and \( Y \times I \) respectively (see conditions (iv) and (v), above) we obtain maps \( \tilde{F}: \text{Tel} (\tilde{X}) \rightarrow \text{Tel} (\tilde{\varepsilon}) \) and \( \tilde{G}': \text{Tel} (\tilde{\varepsilon}) \rightarrow \text{Tel} (\tilde{Y}) \)

and a commutative diagram

\[
\begin{array}{ccc}
\text{Tel} (\varepsilon(X)) & \xrightarrow{F} & \text{Tel} (\tilde{\varepsilon}) \\
\downarrow & & \downarrow \\
\text{Tel} (\tilde{X}) & \xrightarrow{\tilde{F}} & \text{Tel} (\tilde{\varepsilon}) \\
\downarrow & & \downarrow \\
\text{Tel} (\varepsilon(Y)) & \xrightarrow{G'} & \text{Tel} (\tilde{Y}).
\end{array}
\]

(6.4.7)

Now let \( B \) be a compactum in \( Y \). We shall first define an auxiliary map \( f'' : X \rightarrow Y \) depending upon \( B \), a homotopy \( H: X \times I \rightarrow Y \) with \( H_0 = f' \) and \( H_1 = f'' \), and a compactum \( A \) in \( X \) such that
\[ H((X \setminus A) \times I) \subset (Y \setminus B). \] Both \( H \) and \( A \) depend upon \( B \). We shall then define a similar homotopy from \( f'' \) to \( f \).

To define \( f'' \), choose an integer \( n \) so that \( Y_n \subset (Y \setminus B) \). Let

\[ h_n : X \longrightarrow \mathbb{R}^+ \] be the map

\[ h_n(x) = \min \{ h(x), n \} \]

where \( h \) is the map used to construct a proper section for \( \varepsilon(X) \).

Let \( f'' : X \longrightarrow Y \) be the composite

\[ p' \tilde{G'} \tilde{F}(x, h_n(x)). \]

(Recall that \( p' \) is the projection \( \text{Tel}(\varepsilon(Y)) \longrightarrow Y \). See Figure IV.

![Figure IV](image-url)
The required homotopy is given by

$$H(x, \tau) = p' \circ G'F(x, (1-\tau)^n h_n(x) + \tau \cdot h_n(x)).$$

Then, $H(X_{n+1} \times I) \subset (Y_n \setminus A)$. If $A$ is the compactum $cl(X \setminus X_{n+1})$, then $H((X \setminus A) \times I) \subset (Y \setminus B)$, as required.

It remains to find a compactum $A'$ in $X$ and a homotopy $H': X \times I \rightarrow Y$ such that $H'_0 = f''$, $H'_1 = f$, and $H'((X \setminus A') \times I) \subset (Y \setminus B)$. To do this, observe that

$$p' \circ G'F(x, n) = f(x),$$

so that $H'$ may be given by the formula

$$H'(x, \tau) = p' \circ G'F(x, (1-\tau) \cdot h_n(x) + \tau \cdot n).$$

See Figure V. Finally, $H'^{-1}(Y \setminus B) \supset H'^{-1}(Y_n) \supset X_{n+1} \times I$ so that we may take $A' = cl(X \setminus X_{n+1})$. □
Tel (ε(X))

\( f \) at level \( n \)

graph of \( h_n \)

\( \tilde{F} : X \times R^+ \to Y \times R^+ \)

Figure V
The above result and Theorem (6.3.9) suggest a question. Let \( f: X \to Y \) be a proper map of \( \sigma \)-compact, locally compact Hausdorff spaces such that \( f \) is a weak equivalence at \( \infty \) (either in the sense that the induced map \( \varepsilon(f):\varepsilon(X) \to \varepsilon(Y) \) is invertible in \( \text{pro-Ho(Top)} \), or more strongly, that \( f \) is invertible in \( \text{Ho(P}_\infty\text{)} \)) and an ordinary homotopy equivalence. Is \( f \) a weak-proper-homotopy equivalence? The following theorem provides a partial answer.

(6.4.8) **Theorem.** Let \( f: X \to Y \) be a map in \( \text{P}_\sigma \) such that \( \varepsilon(f) \) is invertible in \( \text{pro-Ho(Top)} \) and \( f \) is invertible in \( \text{Ho(Top)} \). Then there is a proper homotopy equivalence \( g: X \to Y \) such that \( \varepsilon(g) \cong \varepsilon(f) \) in \( \text{pro-Ho(Top)} \) and \( g \cong f \) in \( \text{Ho(Top)} \).

**Proof.** By Theorem (5.2.9) there is a map \( g': \varepsilon(X) \to \varepsilon(Y) \) in \( \text{pro-Top} \) such that \( g' \cong \varepsilon(f) \) in \( \text{pro-Ho(Top)} \) and \( g' \) is invertible in \( \text{Ho(pro-Top)} \). Use theorem (6.3.8) to realize \( g' \) as a map \( g'': (X \setminus K) \to Y \) for some compactum \( K \subseteq X \); i.e., \( \varepsilon(g'') = g' \). For suitable \( K \), \( g'' \cong f | (X \setminus K) \) in \( \text{Ho(Top)} \). Let \( H: (X \setminus K) \times [0,1] \to Y \) be a homotopy from \( f \) to \( g'' \). Then choose a compactum \( K_1 \subseteq X \) with \( K_0 \subseteq \text{int } K_1 \) and Urysohn function \( h: K_1 \to [0,1] \) with \( h(K_0) = 0 \) and \( h(\text{bd } K_1) = 1 \). The required map \( g \) is given by

\[
g(x) = \begin{cases} 
  f(x), & \text{for } x \in K_0, \\
  H(x,h(x)), & \text{for } x \in K_1 \setminus K_0, \\
  g''(x), & \text{for } x \in X \setminus K_1.
\end{cases}
\]

The required properties are easily verified. \( \square \)

(6.4.9) **Remarks.** The above results suggest the following analogy.
<table>
<thead>
<tr>
<th>Weak theory</th>
<th>Proper homotopy theory at $\infty$</th>
<th>Pro-homotopy theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong theory</td>
<td>$\text{who}(P_\infty)$</td>
<td>$\text{pro-Ho}(\text{Top})$</td>
</tr>
<tr>
<td></td>
<td>$\text{Ho}(P_\infty)$</td>
<td>$\text{Ho}(\text{pro-Top})$</td>
</tr>
</tbody>
</table>

How closely is $\text{who}(P_\infty)$ (defined analogously with $\text{who}(P)$) related to $\text{pro-Ho}(\text{Top})$ via the end functor?

§6.5 **Whitehead Theorems.**

Siebenmann [Sieb] gave a criterion for a proper map of finite-dimensional, one-ended (see (6.5.1) below), countable, locally finite simplicial complexes to be a proper homotopy equivalence involving "homotopy pro-groups at $\infty$." We shall show that Siebenmann's criterion fails without the finite-dimensional restriction and discuss Siebenmann's and another positive result. The latter result is related to work of the first author and Geoghegan [E-G-1,4].

(6.5.1) **Basepoints.** Let $X$ be a non-compact space in $P$ (if $X$ is compact, $E(X) \cong \emptyset$, so basepoints for $E(X)$ are irrelevant). To each proper map $w: [0, \infty) \to X$, we associate the inverse system

$$
E_*(X) = \{ (X \setminus A) \cup w[0, \infty), w(0)) \mid A \text{ a compactum in } X \}
$$

in $\text{pro-Top}_*$. (Suppressing $w$ from the notation). Note:

$E(X) \cong E_*(X)$ in $\text{Ho}(\text{pro-Top})$. A somewhat different construction is used in §7 (see (7.4.7)).

Call $X$ one-ended if there is at most one proper homotopy class $[0, \infty) \to X$ in $\text{Ho}(P_\infty)$.

(6.5.2) **Homotopy pro-groups at $\infty$.** These are the pro-groups

$$
\text{pro-$\pi_i(E_*(-))$, and}
$$

$$
\text{pro-$\pi_\omega(E_*(-))$}
$$
where
\[ \pi_\omega(-) \equiv \prod_{i=1}^{\infty} \pi_i(-) \equiv [\mathcal{V}_{i=1}^{\infty} S^i, -]; \]
see §5.6.

(6.5.3) **Remarks.** Siebenmann uses essentially these pro-groups; E.M. Brown \([\text{Br }]\) introduced equivalent (see \([\text{Br }]\) or \([\text{Gros-3}]\)) but more complicated homotopy groups at \(\omega\).

(6.5.4) **An example.** In §5.5 we constructed a tower of fibrations of finite, pointed, 1-connected CW complexes
\[ \{F_n \to E_n \to B_n\} \]
with the following properties:

1. \(\{F_n\} * \text{ in } \text{Ho(pro-Top)}\); hence \(\{p_n\}\) induces isomorphisms
   \[ \text{pro-} \pi_\omega \{E_n\} \cong \text{pro-} \pi_\omega \{B_n\}; \]
2. \(\{p_n\}\) is not invertible in \(\text{pro-} \text{Ho(Top)}\).

The induced proper map
\[ \text{Tel}\{p_n\} : \text{Tel}\{\star \leftarrow E_0 \leftarrow E_1 \leftarrow \ldots\} \to \text{Tel}\{\star \leftarrow B_0 \leftarrow B_1 \leftarrow \ldots\} \]
thus induces isomorphisms on all homotopy pro-groups at \(\omega\)
\[ (E(\text{Tel}\{\star \leftarrow E_0 \leftarrow E_1 \leftarrow \ldots\}) \cong \{E_n\}; \text{ similarly for } \{B_n\}) \]
but is not a proper homotopy equivalence (by Theorem (6.3.7)).

However, by suitably restricting either the dimension of the domain and range, or, introducing movability assumptions, we obtain the following positive result.

(6.5.5) **Theorem.** Let \(f : X \to Y\) be a proper map of one-ended, connected, countable, locally finite simplicial complexes which is an ordinary homotopy equivalence and induces isomorphisms \(\text{pro-} \pi_\omega E(X) \to \text{pro-} \pi_\omega E(Y)\). Then \(f\) is a proper homotopy equivalence if either of the following additional conditions holds:

- (a) \([\text{Sieb}]\) \(\text{dim } X < \infty\) and \(\text{dim } Y < \infty\);
- (b) \(f\) is movable.
Proof. By Theorem (6.3.9) it suffices to verify that the induced map $E(f)$ is invertible in $Ho(pro\text{-}Top)$. Because $X$ and $Y$ are countable, $E(X)$ and $E(Y)$ admit cofinal subtowers $\mathcal{C}(X)$ and $\mathcal{C}(Y)$. Now use Theorem (5.5.6), with basepoints defined as above. ☐
§7. GROUP ACTIONS ON INFINITE DIMENSIONAL MANIFOLDS.*

§7.1. Introduction.

We shall discuss the classification of actions of compact Lie groups on $s = \prod_{n=1}^{\infty} (-1/n, 1/n)$ and on $Q = \prod_{n=1}^{\infty} [-1/n, 1/n]$.

In §7.2 we shall review the theory of $s$-manifolds and $Q$-manifolds.

Standard group actions will be constructed in §7.3, following Jim West [West-1]. We shall also show that all principal actions on $s$ are standard up to equivariant homeomorphism.

§7.4 contains a classification theorem (largely due to West) for semifree actions on $Q$. Whether such actions are unique is an open, interesting, and deep question.

§7.2. Basic theory of $s$-manifolds and $Q$-manifolds.

An $s$-manifold (respectively, $Q$-manifold) is a separable metric space locally homeomorphic to an open set in $s$ (respectively, $Q$).

(7.2.1) $s$-manifolds. Work of Kadec, Bessaga, and Pelczynski culminated in Anderson's proof that all separable Fréchet spaces are homeomorphic (see [A-B]).

The classification theory of $s$-manifolds is due to Henderson [Hend ]. First, every $s$-manifold $M$ can be triangulated, that is, there is a locally finite simplicial

*This section represents joint work with Jim West and much of the material is taken from his unpublished notes [West-1].
complex $K$ such that $M$ is homeomorphic to $K \times S$. Second, every homotopy equivalence $f: M_1 \to M_2$ between $s$-manifolds is homotopic to a homeomorphism.

Torunczyk [Tor-1, 2] has shown that $X \times S$ is an $s$-manifold if and only if $X$ is an ANR (separable, metric).

(7.2.2) **$Q$-manifolds.** Many of the above results hold for $Q$-manifolds if homotopy equivalence is replaced by $\omega$-simple homotopy equivalence (see §7.2.3). Keller [Kel] showed that $Q$ is homogeneous.

The classification theory of $Q$-manifolds is due to Chapman [Chap-1-7] and West [West-2, 3]. West [West-2] showed that $K \times Q$ is a $Q$-manifold if $K$ is a locally finite simplicial complex. Chapman [Chap-4] proved the converse: every $Q$ manifold is triangulable, i.e., homeomorphic to $K \times Q$ for some locally finite simplicial complex $K$. If a map $f: K \to L$ of locally finite simplicial complexes is an $\omega$-simple homotopy equivalence, then $f \times \text{id}_Q: K \times Q \to L \times Q$ is properly homotopic to a homeomorphism [West-2]. The converse also holds [Chap-3] and implies the topological invariance of Whitehead torsion.

West [West-3] showed that every locally compact ANR is the $CE$-image of a $Q$-manifold; thus every compact ANR has finite homotopy type. Chapman [Chap-7] observed that West's result can be used to extend $\omega$-simple homotopy theory locally to compact ANR's; in particular, every such ANR has the $\omega$-simple homotopy type of a locally finite simplicial complex. Hence, a proper map $f: M_1 \to M_2$ between $Q$-manifolds is properly
homotopic to a homeomorphism if and only if $f$ is $\infty$-simple.

Bob Edwards [Edw] has recently shown that $X \times \mathbb{Q}$ is a metric $\Omega$-manifold if (and only if) $X$ is a locally compact ANR.

We shall conclude this section by giving West's [West-1] treatment of "Infinite simple homotopy theory after Siebenmann".

(7.2.3) The basic idea of (finite) simple homotopy theory [Coh], is to single-out and study maps of finite cell complexes which are homotopic to finite compositions of maps which are of the form of an inclusion $i: L \rightarrow K$ where

$$K = L \cup e^{n-1} \cup e^n$$

with $e^{n-1}$ an $n$-cell which is a face of $e^n$ or are of the form of a homotopy inverse to such an inclusion. In generalizing this notion to locally finite and not necessarily finite-dimensional complexes, one replaces the map $i$ above by an inclusion $j: L \rightarrow K$ in which $K \setminus L$ is the union $\bigcup_i K_i$ of disjoint complexes $K_i$ each of which collapses to $K_i \cap L$, i.e.,

$$(K_i \cap L) \rightarrow K_i$$

is the result of a finite sequence of inclusions such as $i$, and one restricts oneself to proper mappings and proper homotopies.

(7.2.4) $\mathcal{S}(K)$. In his treatment [Sieb] of infinite simple homotopy theory, Siebenmann introduces the group $\mathcal{S}(K)$ of simple structures on $K$. Each element of $\mathcal{S}(K)$ is an equivalence class $[f:K \rightarrow L]$ of proper homotopy equivalences with domain $K$, where $g:K \rightarrow M$ is in the class of $f$ whenever there is a simple homotopy equivalence $s:L \rightarrow M$ such that $g = sf$.

(The group operation on $\mathcal{S}(K)$ need not concern us here, but it is (essentially) geometrically defined in [Sieb] by representing

*This survey is a quote from [West-1].
\([f]\) and \([g]\) with inclusions, say, into the tops (domains) of mapping cylinders, and then \([f][g]\) is represented by the inclusion of \(K\) into the result of identifying the two spaces along the copies of \(K\).) In particular, if there is only one simple structure on \(K\), then all proper homotopy equivalences are simple.

(7.2.5) \(\text{Wh, } K_0, \text{ and limits.}\) To examine \(A_0(K)\), Siebenmann uses the Whitehead functor \(\text{Wh}\) and the projective class group functor \(K_0\) in several limiting constructions at the end of \(K\). (In essence, all that is needed for this paper is that these are functors.) The limiting constructions are as follows. Let the end of \(K\) be
\[
\varepsilon(K) = \{K = W_0 \supset W_1 \supset W_2 \supset \ldots\},
\]
(see (6.2.14)), where the \(W_n\) are subcomplexes of \(K\) whose complements have compact closure. Choose a proper base ray \(a: [0, \infty) \to K\) such that \(a([n, \infty)) \subseteq W_n\) (as in §6.5). This fixes base points and change of basepoint isomorphisms for \(\pi_1\).

Now consider the inverse system
\[
\pi_1 \varepsilon(K) \cong \{ \pi_1(W_n, a(n)), i_\ast \},
\]
where \(i_\ast: \pi_1(W_{n+1}, a(n+1)) \to \pi_1(W_n, a(n))\) is the inclusion homomorphism followed by change-of-base-point from \(a(n+1)\) to \(a(n)\) along \(a\) (\([n, n+1]\)). Now define
\[
K_0 \pi_1 \varepsilon(K) \cong \lim \{ K_0 \pi_1(W_n, a(n)), K_0(i_\ast) \},
\]
\[
\text{Wh } \pi_1 \varepsilon(K) \cong \lim \{ \text{Wh } \pi_1(W_n, a(n)), \text{Wh } (i_\ast) \},
\]
and the attenuation
\[
\text{Wh } \pi_1 \varepsilon^{-1}(K) \cong \lim^1 \{ \text{Wh } \pi_1(W_n, a(n)), \text{Wh } (i_\ast) \}.
\]
Observe that the attenuation is zero if $\tau_1 \varepsilon(K)$ is stable, i.e., pro-isomorphic to a group.

**Exact sequences.** Siebenmann gives two exact sequences to aid in computing \( J(K) \). They are as follows:

\[
(7.2.6) \quad 0 \rightarrow J_b(K) \rightarrow J(K) \rightarrow K \cdot \tau_1 \varepsilon(K) \rightarrow K \cdot \tau_1 \varepsilon(K);
\]

\[
(7.2.7) \quad \text{Wh} \tau_1 \varepsilon(K) \rightarrow \text{Wh} \tau_1 \varepsilon(K) \rightarrow J_b(K) \rightarrow \text{Wh} \tau_1 \varepsilon'(K) \rightarrow 0.
\]

The group of equivalence \( J_b(K) \) is a class of proper homotopy equivalences defined analogously to \( J(K) \) but where one allows the inclusion \( K_1(L) \rightarrow K_1 \) to be any inclusion of finite complexes which is a homotopy equivalence.

§7.3. **The Standard Actions.**

Following West [West-1], we shall construct standard principal actions of any compact Lie group \( G \) on \( Q_0 \) (\( Q \) with a point deleted) and on \( s \). We shall show that all principal actions of \( G \) on \( s \) are standard. Let \( G \) act on itself by left translation. This principal action extends to a semi-principal action with unique fixed point on the cone of \( G \), \( C(G) \equiv G \times [0,1] / G \times \{0\} \). The product action on \( \prod_{i=1}^{\infty} C(G) \) is also semi-principal with unique fixed point, the infinite cone point. But, it follows from [West-2] that \( \prod_{i=1}^{\infty} C(G) \) is homeomorphic to \( Q \). Removing the unique fixed point yields the standard principal action \( \sigma_G \) of \( G \) on \( Q_0 \). Since \( Q_0 \) is contractible, and \( Q_0 \times s \) is an \( s \)-manifold, \( Q_0 \times s \) is homeomorphic to \( s \).

Thus, we also obtain a standard action \( \sigma_G \) of \( G \) on \( s \). Any principal action of \( G \) on \( Q_0 \) or \( s \) which is not conjugate to the
standard action will be called \textit{exotic}.

Let $\rho$ and $\rho'$ be two principal actions of $G$ on $s$. We will say that the actions are \textit{nice} if the quotient spaces $s/\rho$ and $s/\rho'$ are $s$-manifolds (e.g., if $G$ is finite, then this is always the case). In any case, $s/\rho$ and $s/\rho'$ are both classifying spaces for $G$. Let $f: s/\rho \to s/\rho'$ be a map such that the induced bundle $f^*(s, \rho')$ over $s/\rho$ is isomorphic to the bundle $(s, \rho)$ (see [Hus]). Since $f$ is a homotopy equivalence, if $s/\rho$ and $s/\rho'$ are both $s$-manifolds, then $f$ is homotopic to a homeomorphism $g$. One thus obtains the diagram of principal $G$-bundle isomorphisms

\[
\begin{array}{ccc}
s & \xrightarrow{\varphi} & s \\
\downarrow & & \downarrow \\
\frac{s}{\rho} & \xrightarrow{\varphi} & \frac{s}{\rho'}
\end{array}
\]

in which $h = \varphi \cdot \psi$ is a $G$-equivariant homeomorphism from $(s, \rho)$ to $(s, \rho')$; hence $\rho$ and $\rho'$ are equivalent actions of $G$ on $s$. Summarizing, we have the following theorems.

(7.3.2) \textbf{Theorem.} All nice principal actions of a compact Lie group $G$ on $s$ are standard. \hfill $\square$

(7.3.3) \textbf{Theorem.} All free actions of a finite group $G$ on $s$ are standard. \hfill $\square$

The $\mathbb{Q}_0$ case is much more subtle, since one must show that $\mathbb{Q}_0/\rho$ and $\mathbb{Q}_0/\rho'$ have the same $\omega$-simple homotopy type, and not just the same homotopy type, before one can conclude that they are homeomorphic. The main result of this section is the following theorem, which will be proved in §7.4.
(7.3.4) **Theorem.** Let $\rho$ and $\rho'$ be free actions of a finite group $G$ on $\mathcal{Q}_0$. Then the following statements are equivalent:

1. $\rho$ is equivalent to $\rho'$;
2. $\mathcal{Q}_0/\rho$ is homeomorphic to $\mathcal{Q}_0/\rho'$;
3. $\mathcal{Q}_0/\rho$ is $\omega$-simple homotopy equivalent to $\mathcal{Q}_0/\rho'$;
4. $\mathcal{Q}_0/\rho$ is proper homotopy equivalent to $\mathcal{Q}_0/\rho'$;
5. The end of $\mathcal{Q}_0/\rho$, $\varepsilon(\mathcal{Q}_0/\rho)$, is homotopy equivalent to $\varepsilon(\mathcal{Q}_0/\rho')$ in $\text{pro-}\text{Ho(Top)}$.

(7.3.5) **Remarks.** The equivalence of (1)-(4) for $G$ a finite group is due to West [West-1]. The end $\varepsilon(\mathcal{Q}_0/\rho)$ is a quotient of $\varepsilon(\mathcal{Q}_0) \cong \text{pt.}$; hence, $\varepsilon(\mathcal{Q}_0/\rho)$ is a pro-space analog of the classifying space $BG = \mathcal{Q}_0/\rho$. West showed that the natural inclusion $\varepsilon(\mathcal{Q}_0/\rho) \to \mathcal{Q}_0/\rho$ is always a homotopy equivalence if and only if $\rho$ is standard. In §5 we showed the existence of uncountably many exotic $K(\mathbb{Z}_2,1)$'s; but we still do not know of any exotic compact lie group actions on $\mathcal{Q}_0$. On the other hand, work of Tucker [Tuc-2] shows that there are uncountably many different actions of $\mathbb{Z}$ on $\mathcal{Q}_0$.

§7.4. **Proof of Theorem (7.3.4).**

The following preliminary lemmas, as well as the equivalence of statements (1)-(4) in (7.3.4) are taken from [West-1]. Our machinery (pro-spaces) is used to simplify some of the statements and arguments.

Let $G$ be a fixed finite group acting semifreely on $\mathcal{Q}$ with unique fixed point $q$. Let
\[ \alpha: G \times Q \longrightarrow Q, \quad (g, x) \mapsto gx \]
denote the action.

(7.4.1) **Lemma.** \( Q \setminus \{q\} \) is contractible, and its end \( E(Q \setminus \{q\}) \) is contractible (in \( Ho(\text{pro-Top}) \)).

**Proof.** Represent \( Q \) as the product \( \prod_{i=1}^{\infty} [0,1]_i \).
Because \( Q \) is homogeneous (see 7.2.2), we may assume that \( q = (0,0,0,...) \). \( Q \setminus \{q\} \) is then convex, hence contractible. Also,
\[ E(Q \setminus \{q\}) \cong \{ U_i = (0,1/i)^i \times \prod_{j \neq 1} (0,1)_j \mid i \geq 1 \}, \]
bonded by inclusion. Because each \( U_i \) is convex, hence contractible, \( E(Q \setminus \{q\}) \) is contractible in \( pro-Ho(\text{Top}) \), hence in \( Ho(\text{pro-Top}) \) by Corollary (5.2.17). \( \square \)

(7.4.2) **Lemma.** There is a commutative diagram of covering maps in \( pro-\text{Top} \)

\[
\begin{array}{ccc}
G & \longrightarrow & G \\
\downarrow & & \downarrow \\
E(Q \setminus \{q\}) & \longrightarrow & Q \setminus \{q\} \\
\downarrow & & \downarrow \\
E(Q \setminus \{q\})/\alpha & \longrightarrow & (Q \setminus \{q\})/\alpha.
\end{array}
\]

**Proof.** First, choose a representative tower for \( E(Q \setminus \{q\}) \) as follows. Let \( U_0 = Q \setminus \{q\} \) and \( \{ U_i \mid i \geq 1 \} \) be as in (7.4.1). For each \( i \), \( \bigcap_{j \in \mathbb{N}} U_i \) (the intersection of translates of \( U_i \) under \( \alpha \)) contracts in \( U_i \setminus \{q\} \). Let
\[ \epsilon(Q \setminus \{q\}) \equiv \{ Q \setminus \{q\} = v_0 > v_1 > v_2 > ... \} \]
be a subsequence of \( \bigcap_{j \in \mathbb{N}} U_i \) chosen so that \( v_i \) contracts in \( v_{i-1} \). Note that each \( v_i \) is invariant under \( \alpha \).

Next, rewrite diagram (7.4.3) as
where \( p \) is the covering map induced by \( \alpha \), and \( \varepsilon(p) \) is the levelwise covering map

\[
\varepsilon((\mathcal{Q}\setminus\{q\})/\alpha) \rightarrow (\mathcal{Q}\setminus\{q\})/\alpha,
\]

yielding the conclusion. \( \square \)

(7.4.6) **Corollary.** \( \varepsilon((\mathcal{Q}\setminus\{q\})/\alpha) \) is an Eilenberg-MacLane pro-space with pro- \( \pi_1 = G \); i.e., the tower of universal covers \( \{(V_i/\alpha)/\sim\} \) is contractible in \( Ho(pro\text{-}Top) \).

**Proof.** It remains only to observe that \( \{(V_i/\alpha)/\sim\} \equiv \varepsilon((\mathcal{Q}\setminus\{q\})/\alpha) \); this is an easy exercise involving covering spaces. \( \square \)

(7.4.7) **Basepoints.** To make (7.4.6) precise, choose a base ray \( a: [0,\infty) \rightarrow (\mathcal{Q}\setminus\{q\})/\alpha \), with \( a[i,\infty) \subset V_i/\alpha \). Let

\[
\pi_1\varepsilon((\mathcal{Q}\setminus\{q\})/\alpha) \equiv \{(\pi_1(V_i/\alpha), a[i,j])\}
\]

where the bonding maps are induced by the inclusions

\( V_i/\alpha \hookrightarrow V_{i-1}/\alpha \), followed by change-of-basepoint from \( a(i) \) to \( a(i-1) \) along \( a[i-1,i] \) (reversed). Then

\[
\pi_1\varepsilon((\mathcal{Q}\setminus\{q\})/\alpha) \sim \pi_1((\mathcal{Q}\setminus\{q\})/\alpha) \sim G,
\]

via the inclusion of the end.

(7.4.8) **Proof of Theorem (7.3.4).**

The following implications are easy:

(1) \( \iff \) (2) by covering space theory;

(2) \( \iff \) (3) by Chapman and West's classification of \( \mathcal{Q} \)-manifolds (see (7.2.2)).
(3) \(\Rightarrow\) (4) by definition; and
(4) \(\Rightarrow\) (5) by definition.

To verify (4) \(\Rightarrow\) (3), we shall show that for any semi-free action \(\alpha\) of a finite group \(G\) on \(Q\) with unique fixed point \(q\), 
\[\partial \left( (Q \setminus \{q\}) / \alpha \right) = 0,\]
so that any proper homotopy equivalence with domain \((Q \setminus \{q\}) / \alpha\) is \(\omega\)-simple. Triangulate
\((Q \setminus \{q\}) / \alpha\) as \(K \times Q\) for some locally finite simplicial
complex \(K\). Because the projection map \((Q \setminus \{q\}) / \alpha \to K\)
is a proper homotopy equivalence,

\[(7.4.9) \quad \pi_1 \varepsilon(K) \cong \pi_1 K \cong G,\]
via the inclusion. Hence \(K_0 \pi_1 \varepsilon(K) = K_0 \pi_1 (K)\) so that
\[\partial(K) \cong \partial_b(K)\] by (7.2.6). Also, \(Wh \pi_1 \varepsilon(K) \cong Wh \pi_1 (K)\) and
\(Wh \pi_1 \varepsilon'(K) = 0\) (by (7.4.9), see (7.2.5)), so that
\[\partial_b(K) = 0\] by (7.2.7). Hence \(\partial(K) = 0\), as required.

To verify (5) \(\Rightarrow\) (4), first consider the diagram

\[
\begin{array}{ccc}
\varprojlim \varepsilon(K) & \longrightarrow & \varprojlim K \\
\downarrow & & \downarrow \\
\varepsilon(K) & \leftarrow & K
\end{array}
\]
induced by (7.4.5) (the homotopy inverse limit, \(\varprojlim\), is
developed in §§4.1-4.2 and §4.9), where \(K\) is as above. By
Corollary (7.4.6), \(\text{pro-} \pi_i \varepsilon(K) = 0\) unless \(i = 1\), in which case
\(\text{pro-} \pi_1 \varepsilon(K) \cong G\). Applying the Bousfield-Kan spectral
sequence to \(\varprojlim \varepsilon(K)\) yields

\[
\pi_1 \varprojlim \varepsilon(K) = \begin{cases} 
G, & i = 1 \\
0, & i \geq 1
\end{cases}
\]
(note that holim ε(K) is pointed and connected). Hence
the map holim ε(K) −→ K is a homotopy equivalence by the
ordinary Whitehead theorem (K is a K(G,1) by Lemmas (7.4.1)
and (7.4.2)). Therefore K is a retract of ε(K) in Ho(pro-Top)
by the diagram
\[
\begin{array}{ccc}
\text{holim } \varepsilon(K) & \xrightarrow{\sim} & \text{holim } K \\
\downarrow & & \downarrow \sim \\
\varepsilon(K) & \xrightarrow{\sim} & K
\end{array}
\]
(7.4.10)

Now, for two semifree actions ρ and ρ' of G on Q with
unique fixed points q and q', (Q \{q})/ρ and (Q \{q'})/ρ'
are proper homotopy equivalent at ∞. Then
\[
\varepsilon((Q \{q})/\rho) \sim \varepsilon((Q \{q'})/\rho') \text{ in Ho(pro-Top) by Theorem (6.3.4). This equivalence extends to the diagram}
\]
\[
\begin{array}{ccc}
\varepsilon((Q \{q})/\rho) & \xleftarrow{\sim} & (Q \{q\})/\rho \\
\downarrow & & \downarrow \sim \\
\varepsilon((Q \{q'})/\rho') & \xleftarrow{\sim} & (Q \{q')}/\rho'
\end{array}
\]
(7.4.11) by (7.4.10). Theorem (6.3.3) now implies that (Q \{q})/ρ
and (Q \{q'})/ρ' are proper homotopy equivalent (globally),
as required. □

(7.4.12) Remarks. The implications (1) ⇒ (2) ⇔ (3)
⇔ (4) ⇔ (5) hold for arbitrary compact Lie groups G with
a similar proof. For (2) ⇒ (1) we must verify that the
maps (Q \{q})/ρ → (Q \{q'})/ρ' of statement (2) are
covered by equivariant maps Q \{q} → Q \{q'}. 

§8. SHAPE THEORY.

§8.1. Introduction.

This chapter is concerned with shape theory, shape functors, and Steenrod homology theories.

In §8.2 we review Borsuk's original approach to shape theory and the Chapman Complement Theorem. We also make some remarks about a strong form of the complement theorem.

In §8.3 we discuss the Steenrod homology theory [St-1], the Kaminker-Schochet [K-S] axioms for generalized Steenrod homology theories, and the Vietoris construction [Por-1]. We then use the Vietoris construction to define canonical Steenrod and Čech extensions, $S_{h_*}$ and $h_*$, of any generalized homology theory, $h_*$, defined on finite CW complexes. Proofs of the properties of $S_{h_*}$ occupy the next four sections.

In §8.4 we verify useful properties of the Vietoris functor.

In §8.5 we prove that $S_{h_*}$ is a homology theory on the category CM of compact metric spaces, that is, that $S_{h_*}$ is a homotopy invariant functor which satisfies the first two Kaminker-Schochet axioms for a Steenrod homology theory. We also show that products and operations associated with $h_*$ extend to $S_{h_*}$. 
We give two spectral sequences converging to $S_{h^*}$ in §8.6, and use the first spectral sequence to prove that $S_{h^*}$ satisfies the remaining Kaminker-Schochet axiom.

In §8.7 we prove a Steenrod duality theorem for $S_{h^*}$, that is, for a compactum $X$ in $S^n$, $S_{h^p}(X) = h^{n-p-1}(S^n \setminus X)$.

§8.2. Borsuk's Shape Theory and the Chapman Complement Theorem.

In the late 1960's Borsuk sparked an avalanche of interest in the study of the global homotopy properties of compacta (see [Mar-3], [Ed-1] and [E-H-2] for surveys of shape theory). Borsuk's original formulation of the shape theory of compact subsets of Hilbert space [Bor-2] lacks the flexibility of the approach to be described in §8.3, but it has the advantage of being more geometric. This added geometry was quickly capitalized upon by Chapman in [Chap-1].

Let $s = \pi_{n=1}^\infty (-1/n, 1/n)$, $Q = \pi_{n=1}^\infty [-1/n, 1/n]$, as in §7. One defines the fundamental category or shape category, $Sh$, as follows. The objects of $Sh$ are compact subsets of $s$. If $X$ and $Y$ are compact subsets of $s$, then a fundamental sequence $f:X \rightarrow Y$ is defined as a sequence of maps $f_n:Q \rightarrow Q$ with the property that for every neighborhood $V$ of $Y$ in $Q$ there exists a neighborhood $U$ of $X$ in $Q$ and an integer $n_0$ such that for $n$, $n' \geq n_0$ the
restrictions $f_n|_U$ and $f'_n|_U$ are homotopic in $V$. Note that $f_n(X)$ does not have to be contained in $Y$; it only has to be near $Y$.

Two fundamental sequences $f, f': X \to Y$ are considered homotopic, $f \simeq f'$, provided that for every neighborhood $V$ of $Y$ in $Q$ there exists a neighborhood $U$ of $X$ in $Q$ and an integer $n_0$ such that for $n \geq n_0$, $f_n|_U$ and $f'_n|_U$ are homotopic in $V$. The morphisms in $Sh$ are now taken to be homotopy equivalence classes of fundamental sequences. Two compacta $X$ and $Y$ contained in $s$ are said to have the same shape if they are isomorphic in $Sh$.

In [Chap. 1] Chapman proved the following beautiful theorems.

(8.2.1) Chapman Complement Theorem. If $X$ and $Y$ are compacta in $s$, then $X$ and $Y$ have the same shape if and only if their complements $Q \setminus X$ and $Q \setminus Y$ are homeomorphic.

(8.2.2) Theorem. There is a category isomorphism $T$ from $Sh$ to $w$-$Ho(Q)$ such that for each object $X \in Sh$, $T(X) = Q \setminus X$.

Here $P_Q$ is the full subcategory of $P$ consisting of complements in $Q$ of compacta in $s$.

These theorems suggest the following questions.

(1) What is the relationship between the shape theory of compacta in $s$ and the proper homotopy theory of their complements in $Q$?
(2) What shape theory does $Ho(P_Q)$ correspond to?

One can show that there is a natural functor $T$ from the homotopy category, $Ho(M)$, of compact subsets of $s$, to $Ho(P_Q)$ such that on objects one has $T(X) = Q \setminus X$. One can now simply define the strong shape category, $s-Sh$, to be the category whose objects are compacta in $s$ and whose morphism sets are defined by $s-Sh(X,Y) \equiv Ho(P)\ (Q \setminus X, Q \setminus Y)$. Clearly (and trivially), $T$ determines a category isomorphism from $s-Sh$ to $Ho(P_Q)$. Alternatively, the results of §6 imply that the end functor $E:Ho(P_Q) \rightarrow Ho(pro-Top)$ is a full embedding. The following diagram

\[
\begin{array}{ccc}
\{U \setminus X\} & \longrightarrow & \{V \setminus Y\} \\
\downarrow & & \downarrow \\
\{U\} & \longrightarrow & \{V\}
\end{array}
\]

has the property that the vertical maps are level homotopy equivalences (compact subsets of $s$ are $Z$-sets (see §7)) and hence isomorphisms in $Ho(pro-Top)$. This shows that the assignment $X \mapsto \{U \mid U \ldots$ open in $Q \setminus X \subset U\}$ determines a functor $S:Ho(M) \rightarrow Ho(pro-ANR)$ which is naturally equivalent to $E \circ T$. $S$ can also be defined directly – without using $Ho(P_Q)$ – by using the telescope description of $Ho(tow-Top)$ given in §3.6. If one developed "coherent" pro-homotopy theory (see [Por-31]), then a Borsuk style definition of strong shape theory could be made, and a Chapman style strong complement theorem could be proved.
§8.3. Steenrod homology theories.

Let \( h_* \) be a generalized homology theory defined on the category of finite CW complexes. In this section we shall define a canonical Steenrod extension \( S_{h_*} \) (see (8.3.3)) of \( h_* \) to the category CM of compact metric spaces. Proofs occupy §§8.4-8.6. See (8.3.1) and (8.3.2) for ordinary Steenrod homology and Steenrod K-homology.

The problem of constructing generalized Steenrod homology theories with "good properties" was posed by M. Atiyah and others at the Operator Theory and Topology Conference held at the University of Georgia in April, 1975. By "good properties" Atiyah meant that products and homology operations extended from \( h_* \) to \( S_{h_*} \). Our extension enjoys these properties.

(8.3.1) Duality and homotopy theories. Pontryagin duality states that the Čech cohomology of a compactum \( X \) in \( S^n \) is canonically isomorphic to the ordinary homology with compact supports of the complement \( S^n \setminus X \). In [St-1] Steenrod defined a homology theory \( S_{H_*} \) such that \( S_{H_*}(X) \) is canonically isomorphic to the ordinary cohomology of \( S^n \setminus X \). \( S_{H_*} \) is now called (ordinary) Steenrod homology. Milnor [Mil-1] (see also [Sky]) showed that \( S_{H_*} \) satisfies
all of the Eilenberg-Steenrod [E-S] axioms on CM, and further that $S_{H_*}$ is characterized by these axioms together with two additional axioms; namely, invariance under relative homeomorphism (generalized excision) and the strong wedge axiom (see (8.3.3)).

(8.3.2) Steenrod $K$-homology. In a series of papers [Brow], [B-D-F-1-2], L. Brown, R. Douglas, and P. Fillmore defined a functor $\text{Ext}$ on CM by taking for $\text{Ext}(X)$ unitary equivalence classes of $C^*$-algebra extensions of the compact operators by the $C^*$-algebra of complex-valued functions on $X$. Kaminker and Schochet [K-S] then set

$$S_{E_n}(X) = \begin{cases} 
\text{Ext}(X), & \text{for } n \text{ odd}, \\
\text{Ext}(\Sigma X), & \text{for } n \text{ even},
\end{cases}$$

($\Sigma$ is the unreduced suspension) and showed that $S_{E_*}$ satisfies the axioms (8.3.3) for a generalized reduced Steenrod homology theory.

L. Brown, Douglas, and Fillmore have shown that on the category of finite complexes, $S_{E_*}$ is reduced $K$-homology.

(8.3.3) The Kaminker-Schochet [K-S] axioms. A generalized (reduced) Steenrod homology theory consists of a sequence $h_* = \{h_n | n \in \mathbb{Z} \}$ of covariant, homotopy invariant functors from the
category $CM$ of compact metric spaces to the category $AG$ of abelian
groups which satisfy the following axioms.

(E) **Exactness:** if $(X,A)$ is a compact metric pair,

then the natural sequence

$$h_n(A) \rightarrow h_n(X) \rightarrow h_n(X/A)$$

is exact for all $n$.

(S) **Suspension:** there is a sequence of natural
equivalences

$$\sigma_n : h_n \rightarrow h_{n+1} \circ \Sigma$$

called suspension, where $\Sigma$ is unreduced suspension.

(W) **Strong wedge:** if $X = \vee_{j=1}^{\infty} X_j \equiv \lim_{N} \{ \vee_{n=1}^{N} X_j \}$
is the strong wedge of a sequence of pointed compact
metric spaces, then the natural projections

$X \rightarrow X_j$

induce an isomorphism

$$h_\ast(X) \cong \pi_j h_\ast(X_j).$$

(8.3.4) **Remarks.** The homology theory $h_\ast$ is reduced (see axiom (E) above) but not pointed. An unreduced theory $H_\ast$ yields such a reduced theory $\tilde{H}_\ast$ by the formula $\tilde{H}_n(X) \equiv \ker (H_n(X) \rightarrow H_n(\ast))$. 
We break the problem of constructing Steenrod extensions into two parts. The first part involves approximating a compactum by an inverse system of simplicial complexes, an idea which goes back to Alexandroff [Alex]. The second part involves prolonging a generalized homology theory defined on the category of finite CW complexes to a generalized homology theory defined on pro-$\mathcal{S}$ and satisfying suitable analogues of the Kaminker-Schochet axioms. This was done in §5.6.

We shall carry out the first part of this program using a Vietoris functor (see (8.3.7); this construction was first introduced by T. Porter [Por-1]) after giving Steenrod's original construction for motivation.

(8.3.5) Regular cycles following Steenrod [St-1]. Let $X$ be a compact metric space, $K$ an abstract countable locally finite simplicial complex (clf simplicial complex), and let $V_K$ be the set of vertices of $K$. A regular map is a map $f:V_K \to X$ such that for each $\varepsilon > 0$, the images of all but finitely many simplices have diameter less than $\varepsilon$. A regular $q$-chain on $X$ with coefficients in an abelian group $G$, $(K,f,c_q)$, consists of a clf simplicial complex $K$ and a regular map $f$ as above, together with a (possibly
infinite) q-chain on K with coefficients in G,c_q. One then obtains a chain complex \( C^R_X(X;G) \) based upon regular chains, and reduced Steenrod homology

\[
S^R_{q}(X;G) = H_{q+1}(C^R_X(X;G)).
\]

(8.3.6) Remarks. Consider a regular q-chain (K,f,c_q) on a compact metric space X. Let U be an open cover of X. Then there is an \( \epsilon > 0 \) (the Lebesque number of U) such that for any point x in X, the \( \epsilon \)-neighborhood of x is contained within a single open set \( U \in \mathfrak{U} \). Hence, for almost all simplices \( \Delta \) of K, the image under f of the vertices of K is contained within some open set U (depending upon \( \Delta \) \( \in \mathfrak{U} \)). Also, the fact that f maps \( V_K \) (and not K itself) to X conceals the "local pathology" of X.

(8.3.7) The Vietoris construction [Por-1]. Let U be an open covering of a topological space X. The Vietoris nerve of U, denoted \( VN(U) \), is the simplicial set in which an n-simplex is an ordered \( (n+1) \)-tuple \( (x_0,x_1,\ldots,x_n) \) of points contained in an open set \( U \in \mathfrak{U} \). Faces and degeneracies are given by

\[
\begin{align*}
    d_i(x_0,x_1,\ldots,x_n) & = (x_0,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n), & \text{and} \\
    s_i(x_0,x_1,\ldots,x_n) & = (x_0,x_1,\ldots,x_{i-1},x_i,x_i,x_{i+1},\ldots,x_n), & \text{for} \\
    0 \leq i \leq n.
\end{align*}
\]
Now consider open covers \( U \) and \( V \) of \( X \) where \( V \) refines \( U \) (notation: \( U \leq V \)). The identity map of \( X \) induces a canonical inclusion \( VN(V) \hookrightarrow VN(U) \). We may therefore associate to a topological space its Vietoris complex,

\[
(8.3.8) \quad VX = \{ VN(U) \mid U \text{ an open cover of } X \}. \quad VX \text{ is bonded by the canonical inclusions } VN(V) \hookrightarrow VN(U) \text{ when } U \leq V.
\]

\[
(8.3.9) \quad \text{Proposition.} \quad \text{The Vietoris construction extends to a functor } V: \text{Top} \rightarrow \text{pro-SS}.
\]

**Proof.** We define \( V \) on morphisms as follows: to a continuous map \( f:X \rightarrow Y \) and open cover \( U \) of \( Y \) we associate the open cover \( f^{-1}(U) = \{ f^{-1}(U) \mid U \in U \} \) of \( X \). Then \( f \) induces an obvious map

\[
VN(f^{-1}(U)) \rightarrow VN(U) \quad ((x_0, x_1, \ldots, x_n) \mapsto (f(x_0), f(x_1), \ldots, f(x_n))
\]

such that whenever \( U \leq V \) (and hence \( f^{-1}(U) \leq f^{-1}(V) \)), the diagrams

\[
\begin{array}{c}
VN(f^{-1}(V)) \rightarrow VN(V) \\
V(f^{-1}(U)) \rightarrow VN(U)
\end{array}
\]

commute. These diagrams induce the required morphism \( Vf: VX \rightarrow VY \) in \( \text{pro-SS} \). Clearly \( V \) preserves identity maps and

\[
V(fg) = Vf \circ Vg. \quad \square
\]
(8.3.11) **Steenrod and Čech extensions of homology theories.**

Let $h_*$ be a generalized (reduced) homology theory defined on the category of finite CW complexes. By G. W. Whitehead [Wh] there is a CW spectrum $E$ which represents $h_*$, i.e., $h_*(-) = \pi_*(-)^E$.

See also (5.6.7). Define the Steenrod and Čech extensions of $h_*$ to the category CM of compact metric spaces by the following formulas.

[We write $S h_*$ for both the extension of $h_*$ to pro-$SS$ (5.6.7) and the Steenrod extension; the usage should be clear from the context.)

(8.3.12) **Steenrod extension:**

\[
S h_*(-) = S h_* : V \\
= \pi_3^{S}(\text{holim } (\text{Sin } (RV(-)^E))) \\
= \text{Ho}(\text{pro-Sp})(S^*, \text{Sin } (RV(-)^E)).
\]

(8.3.13) **Čech extension:**

\[
h_*(-) = \lim_j \{h_* V(-)_j\} \\
= \lim_j \{\pi_3^{S}(\text{Sin } (RV(-)_j^E))\} \\
= \text{pro-Ho}(\text{Sp})(S^*, \text{Sin } (RV(-)^E)).
\]

Here, $V(-) = \{V(-)_j\}$ denotes the Vietoris functor and $\text{Sp}$ the category of simplicial spectra.
For ordinary homology these formulas become

\[(8.3.14) \quad \tilde{S}_{H_*}(-; R) = \tilde{S}_{H_*}(V(-); R) \]
\[= \pi_*(\text{holim} RV(-)) \]
\[= \text{Ho}(\text{pro-}SS)(S^*, RV(-)); \]

\[(8.3.15) \quad \tilde{H}_*(-; R) = \lim_j \{\tilde{H}_*(V(-)_j; R) \]
\[= \lim_j \{\pi_*(RV(-)_j)\} \]
\[= \text{pro-}\text{Ho}(SS)(S^*, RV(-)). \]

In (8.3.14) and (8.3.15), \( R \) denotes any commutative ring with identity as well as the free \( R \)-module functor of Bousfield and Kan [B-K-1], not the geometric realization functor.

(8.3.16) **Remarks.** Ken Brown [Brown] defined generalized sheaf cohomology theories with a similar use of simplicial spectra and smash products.

(8.3.17) **Theorem.** \( \tilde{h}_* \) is the \( \check{\text{C}} \)ech extension of \( h_* \).

**Proof.** This follows from Dowker [Dow]: the realizations of the \( \check{\text{C}} \)ech and Vietoris nerves of an open covering are canonically homotopy equivalent. Alternatively follow the proofs of Theorems (8.3.18) and (8.3.21). \( \square \)
(8.3.18) Theorem. $S_{h_*}$ is a homology theory on the category CM of compact metric spaces.

(8.3.19) Theorem. $S_{h_*} \cong h_*$ on finite CW complexes.

(8.3.20) Theorem. Products and operations associated with $h_*$ extend to $S_{h_*}$.

(8.3.21) Theorem. $S_{h_*}$ is a Steenrod homology theory on the category of compact metric spaces.

We begin the proofs by verifying properties of the Vietoris functor in §8.4. We shall prove Theorem (8.3.18) in §8.5 using strong homology theories on pro-SS, (5.6.7). Theorems (8.3.19) and (8.3.20) follow easily, see §8.4. We shall develop a Bousfield-Kan spectral sequence for $S_{h_*}$ in §8.6 (compare [Brown]) and prove Theorem (8.3.21) there.

§8.4. The Vietoris functor

We shall prove that the Vietoris functor on the category CM of compact metric spaces preserves homotopies, cofibration sequences, suspensions, and limits, at least up to canonical equivalence in $\text{Ho(pro-SS)}$. 
(8.4.1) **Proposition** (announced independently by T. Porter [Por-1]). Homotopic maps of spaces \( f,g:X \rightarrow Y \) induce homotopic maps in \( \text{pro-SS} \), \( Vf,Vg:VX \Rightarrow VY \), of their Vietoris systems, hence \( V \) induces a functor

\[ V: \text{Ho(Top)} \rightarrow \text{Ho(pro-SS)}. \]

**Proof.** Let \( H:X \times [0,1] \rightarrow Y \) be a homotopy with \( H_0 = f \) and \( H_1 = g \), where \( i_0 \) and \( i_1 \) are the inclusions of \( X \) as the ends of the cylinder \( X \times [0,1] \). It therefore suffices to show that

\[ V_i_0 = V_i_1 \text{ in } \text{Ho(pro-SS)}. \]

We shall define a homotopy \( K \) from \( RV_i_0 \) to \( RV_i_1 \) in \( \text{pro-Top} \), i.e., a map

\[ K:RVX \times [0,1] \rightarrow RV(X \times [0,1]) \]

with \( K_0 = RV_i_0 \) and \( K_1 = RV_i_1 \) (the realization functor \( R \) is applied levelwise). Because adjunction morphisms \( \text{id} \rightarrow \text{Sin } R \) are natural weak equivalences in \( \text{SS} \), we obtain a diagram

\[ (8.4.2) \]

\[ VX \times [0,1] \rightarrow \text{Sin } (RVX \times [0,1]) \]

\[ \rightarrow \text{Sin } RV(X \times [0,1]) \]

\[ \leftarrow V(X \times [0,1]). \]

Because the "wrong-way" arrow in diagram (8.4.2) is invertible in \( \text{Ho(pro-SS)} \), we shall see that \( V_i_0 = V_i_1 \), as required.
Call an open covering $U$ of $X \times [0,1]$ a stacked covering if $U$ is a union of families of open sets $U \times V_a$ where $U$ is an open set in $X$ and $V_a$ is an open covering of $[0,1]$ depending upon $U$ (see [E-S]).

Let $U$ be a covering of $X \times [0,1]$ by basic open sets, i.e.,

$$U = \{U_a \times V_a | U_a \text{ open in } X, V_a \text{ open in } [0,1]\}.$$ 

Such coverings are clearly cofinal in the inverse system of all coverings of $X \times [0,1]$. For each $x$ in $X$, consider the induced covering $U_x$ of $x \times [0,1] \subset X \times [0,1]$. Because $[0,1]$ is compact, $U_x$ admits a finite subcover, say $\{U_{x,i} \times V_{x,i} | i = 1, 2, \ldots, n_x\}$. Let $U_x = \bigcap_{i=1}^{n_x} U_{x,i}$, and form the stacked covering

$$U' = \{U_x \times V_{x,i} | i = 1, 2, \ldots, n_x, x \in X\}.$$ 

Clearly each open set of $U'$ is contained in an open set of $U$.

Hence, stacked coverings are cofinal in all coverings of $X \times [0,1]$. 
To define $K$, let $U$ be a stacked covering of $X \times [0,1]$, say

$$U = \bigcup_{u \in U'} \{ U \times V_a \mid V_a \}$$

is an open covering of $[0,1]$.

where $U'$ is an open covering of $X$. Then the homotopies

$$K_U : RVN(U') \times [0,1] \to RVN(U), \text{ with}$$

$$K_U((x_0, x_1, \ldots, x_n), t) = ((x_0, t), (x_1, t), \ldots, (x_n, t))$$

from $RV_i$ to $RV_i$ form the required homotopy

$$K = \{ K_U : RVN(U') \times [0,1] \to RVN(U) \mid U$$

a stacked covering of $X$, $U'$ as in (8.4.3))

in $\text{pro-Top}$ with $K_0 = RV_0 : RVX \to RV(X \times [0,1])$ and

$$K_1 = RV_1 : RVX \to RV(X \times [0,1]).\]$$

(8.4.) Proposition. Let $A$ be a closed subset of a topological space $X$. Then the induced map $VA \to VX$ is a cofibration in $\text{pro-SS}$.

Proof. We may represent the map $VA \to VX$ as a levelwise cofibration in an appropriate level category $SS^J$ as follows. Each open covering $U$ of $X$ induces an open covering $U|A$ of $A$,
namely

\[(8.4.5) \quad U\mid A = \{U \cap A \mid U \in U\},\]

and an inclusion of Vietoris nerves

\[\text{VN}(U\mid A) \rightarrow \text{VN}(U).\]

Because each open covering of \( A \) can be extended to an open covering of \( X \) by adjoining the open set \( X \setminus A \), the set of restrictions of open coverings of \( X \) to \( A \), \( \{U\mid A\} \) (see (8.4.5)), is cofinal in the set of all open coverings of \( A \). We obtain the required representation

\[\text{VA} = \{\text{VN}(U\mid A) \mid U \]

an open covering of \( X\}\}

\[\rightarrow \{\text{VN}U\}\]

\[\equiv \text{VX}. \quad \Box\]

\[(8.4.6) \quad \text{Proposition.} \quad \text{Given} \ A \subset X, \ \text{there is a natural map}\]

\[\text{VX/VA} \rightarrow V(X/A).\]

\textbf{Proof.} In the solid-arrow diagram

\[\begin{array}{c}
\text{VA} \rightarrow \text{VX} \rightarrow \text{VX/VA} \\
\downarrow \\
V(X/A),
\end{array}\]
the composite mapping \( VA \rightarrow VX \rightarrow V(X/A) \) is trivial. This yields
the required map. \( \square \)

(8.4.7) Proposition. Let \( A \) be a closed subset of a compact
metric space \( X \). Then there is a natural equivalence
\( VX/VA \rightarrow V(X/A) \) in \( Ho(pro-SS) \), hence the sequence
\( VA \rightarrow VX \rightarrow V(X/A) \) is a cofibration sequence in \( Ho(pro-SS) \).

The following lemma about "shape cofibrations" is a key tool in
the proof. It is analogous to the statement that a map \( A \rightarrow X \) is
a cofibration if and only if there is a neighborhood \( N \) of \( A \) in \( X \)
such that \( A \) is a strong deformation retract of \( N \) [St-3]. We
state it in somewhat greater generality than is needed now; the extra
generality is needed in Proposition (8.4.22), below.

(8.4.8) Lemma. Let \( A \) be a closed subset of a compact metric
space \( X \), and let \( U \) be an open covering of \( X \). Then there is an
open covering \( V \) of \( X \) and a neighborhood \( N \) of \( A \) in \( X \) with the
following properties:

(a) \( V \) refines \( U \);

(b) For each open set \( V \) of \( V \), either \( V \cap A = \emptyset \)
or \( V \subseteq N \);

(c) For each neighborhood \( N' \) of \( A \) in \( N \), the
inclusion of Vietoris nerves
\[ \text{VN}(U|A) \longrightarrow \text{VN}(U|N') \]

is an equivalence in \( \text{Ho(SS)} \).

**Proof of Lemma.** Because \( X \) is compact, we may assume that \( U \) is a finite open cover, \( U = \{ U_i \}_{i=1}^{n} \). From now on, in all constructions and sets indexed by \( i \), \( i \) ranges over \( \{1,2,\ldots,n\} \). Consider the restriction \( U|A = \{ U_i \cap A \} \). By a result of Kuratowski [Kur, p. 122], there are open sets \( U'_i \) in \( X \) such that \( U_i \cap A = U'_i \cap A \), and if \( U' = \{ U'_i \} \), the inclusions \( U_i \cap A \hookrightarrow U'_i \) induce an isomorphism on Čech nerves

\[ \text{CN}(U|A) \xrightarrow{\sim} \text{CN}(U'). \]

To do this, let

\[ (8.4.9) \quad U'_i = \{ x \mid d(x, U_i \cap A) < d(x, A \setminus U_i) \}, \]

Next, perform the following constructions. Let \( U''_i = U_i \cap U'_i \), let \( U'' = \{ U''_i \} \), let \( N'' = u U''_i \), and, using normality of \( X \), let \( N \) be an open set with \( A \subseteq N \subseteq N \subseteq B'' \).

Next, let

\[ (8.4.10) \quad V^1_i = U''_i \cap N, \]

\[ V^2_i = U''_i \cap (N'' \setminus A), \quad \text{and} \]

\[ V^3_i = U_i \cap (X \setminus \overline{N}), \]
and let $V_1 = \{v_1^1\}$, $V_2 = \{v_1^2\}$, and $V_3 = \{v_1^3\}$. Finally, let

$V = V_1 \cup V_2 \cup V_3$.

Clearly our construction yields an open covering of $X$ which refines $U$. Because the open sets $V_1^1$ are subsets of $N$, and the open sets $V_1^2$ and $V_1^3$ are disjoint from $A$ by construction, property (b) holds.

To check property (c), first note that the open sets $V_1^3$ are disjoint from $N$, hence for any open set $N'$ with $A \subseteq N' \subseteq N$,

$$V|_{N'} = (V_1 \cup V_2)|_{N'}.$$  

(8.4.11)

Next, observe that each open set $V_1^2 \cap N'$ of $V_2|_{N'}$ is contained in an open set of $V_1|_{N'}$, namely, $V_1^1 \cap N'$. Hence,

$$\text{VN}(V_1|_{N'}) = \text{VN}((V_1 \cup V_2)|_{N'}) = \text{VN}(V|_{N'})$$

(8.4.12)

and similarly,
(8.4.13) \( \text{VN}(V_1 | A) = \text{VN}((V_1 \cup V_2) | A) = \text{VN}(V | A) \).

Finally, \( V | A = V_1 | A = V | A \), and a look at Kuratowski's construction (8.4.9) applied within \( N' \) yields

(8.4.14)

\[
\text{CN}(V | A) = \text{CN}(V_1 | A) = \text{CN}(U | A) \cong \text{CN}(U' | N) \cong \text{CN}(V_1 | N) = \text{CN}(V | N),
\]

via the inclusion \( A \subseteq N \). Because the geometric realizations of the Čech and Vietoris nerves of an open covering are naturally equivalent in \( \text{Ho}(\text{Top}) \) by Dowker [Dow], formula (8.4.14) yields the desired equivalence \( \text{VN}(V | A) \cong \text{VN}(V | N) \), as required in (c). \( \square \)

(8.4.15) Proof of Proposition (8.4.7). Let \( U \) be an open covering of \( X \) and \( V \) any open covering of \( X \) constructed by Lemma (8.4.8) above. With \( V_2 \) and \( V_3 \) as above, let \( V' \) be the open covering \( V_2 \cup V_3 \cup \{N\} \) of \( X \). Because \( N \) is a neighborhood of \( A \) (in \( X \)) and the open sets of \( V_2 \) and \( V_3 \) are disjoint from \( A \), the projection \( \pi: X \rightarrow X / A \) induces an open covering \( V'' \cong \pi V' \) of \( X / A \). Because \( V \) refines \( V'' \), we obtain a commutative diagram

(8.4.16) \( \text{VN}(V | A) \rightarrow \text{VN}(V) \rightarrow \text{VN}(V) / \text{VN}(V | A) \)

\[
\pi_\ast \downarrow \quad \downarrow p
\]

\[
\text{VN}(V'').
\]
We shall show that $p$ is a weak equivalence.

Form the following commutative diagram.

\[
\begin{array}{ccccccccc}
\text{VN}(V|A) & \rightarrow & \text{VN}(V) & \rightarrow & \text{VN}(V)/\text{VN}(V|A) \\
\downarrow^{(\cong)} & & \downarrow & & \downarrow^{(\cong)} \\
\text{VN}(V|N) & \rightarrow & \text{VN}(V) & \rightarrow & \text{VN}(V)/\text{VN}(V|N) \\
\downarrow & & \downarrow^{q} & & \downarrow^{q} \\
\text{VN}(V'|N) & \rightarrow & \text{VN}(V') & \rightarrow & \text{VN}(V')/\text{VN}(V'|N) \\
\downarrow^{(\cong)} & & \downarrow^{q'} & & \downarrow^{q'} \\
\text{VN}(V'|A) & \rightarrow & \text{VN}(V') & \rightarrow & \text{VN}(V')/\text{VN}(V'|A) \\
\downarrow^{\pi_*} & & \downarrow^{p''} & & \downarrow^{p''} \\
\text{VN}(V'') & & & & \\
\end{array}
\]

(8.4.17).

In Diagram (8.4.17) the composite map $p''p'$ is the map $p$ in Diagram (8.4.16), and the rows are cofibration sequences in $SS$. The indicated maps are weak equivalences (or isomorphisms) for the following reasons.
(a) \( VN(V' \mid N) = VN(\{N\}) \) (because \( N \in V' \)) = \( CN(\{N\}) \) (by [Dow]) \( 
abla \). Similarly, \( VN(V' \mid A) = \nabla \). Therefore the maps
\[ VN(V') \longrightarrow VN(V')/VN(V' \mid N) \quad \text{and} \quad VN(V') \longrightarrow VN(V')/VN(V' \mid A) \]
are weak equivalences. Hence \( q'' \) is a weak equivalence (by Axiom M5 for SS).

(b) The map \( VN(V \mid A) \longrightarrow VN(V \mid N) \) is a weak equivalence by the construction of \( V \) and \( N \). Hence, the induced map of the cofibres, \( q' \), is a weak equivalence (by [Q, Prop. I.3.5] for SS, compare Proposition (3.4.12)(c)).

(c) To show that \( p'' \) is an isomorphism in SS, recall that \( V' = \{V_2\} \cup \{V_3\} \cup N \) where the open sets of \( V_2 \) and \( V_3 \) are disjoint from \( A \) and \( N \) is a neighborhood of \( A \), and \( V'' = \pi'' V' \).
Choose a point \( a \) in \( A \). Then the required inverse of \( p'' \) is given by the formula
\[ p''^{-1}(y_0, y_1, \ldots, y_n) = (x_0, x_1, \ldots, x_n) \]
where
\[ x_i = \begin{cases} 
  a & \text{if } y_i = [A] \in X/A, \\
  \pi^{-1} y_i & \text{if } y_i \in \pi(X \setminus A) \subset X/A.
\end{cases} \]

(d) To show that \( q' \) is a weak equivalence observe that
\[ (8.4.18) \quad VN(V') = VN(V) \sqcup_{VN(V \mid N)VN(\{N\})} VN(V' \mid N) \]
(and recall that $N \in \mathcal{V}'$ so that $\text{VN}(\mathcal{V}'|N) = \text{VN}([N])$). Because $\text{VN}([N])$ is contractible, the diagram of geometric realizations

\[
\begin{array}{ccc}
\text{RVN}(\mathcal{V}|N) & \longrightarrow & \text{RVN}([N]) \\
\phantom{RVN(\mathcal{V}|N)} & \downarrow & \phantom{\text{RVN}([N])} \\
\text{RVN}(\mathcal{V}|N) & \longrightarrow & \text{CRVN}(\mathcal{V}|N)
\end{array}
\]

(8.4.19)

commutes up to homotopy ($\ast$ denotes the unreduced cone). We may use the homotopy extension property to find a map $\text{RVN}([N]) \longrightarrow (\text{RVN}(\mathcal{V}|N)$ which makes diagram (8.4.19) strictly commute. This yields a map $\text{RVN}(\mathcal{V}') \longrightarrow \text{RVN}(\mathcal{V}) \cup_{\text{RVN}(\mathcal{V}|N)} (\text{RVN}(\mathcal{V}|N)$ ($R$ preserves cofibrations, cones, and quotients by $[\text{Mil} - 2]$) which extends the identity map of $\text{RVN}(\mathcal{V})$ and is a weak equivalence by an easy argument involving the homotopy extension property. Because $\text{VN}(\mathcal{V}') \simeq \text{VN}(\mathcal{V}')/\text{VN}(\mathcal{V}'|N)$ (see (a), above), the map $q$ of diagram (8.4.17) is a weak equivalence, as required.

It follows that the map $p$ in diagram (8.4.16) is a weak equivalence. It is easy to check that open coverings of $X/A$ of the form $\mathcal{V}''$ constructed above are cofinal in all open coverings of $X/A$. Therefore, the map $\text{V}(X/A) \longrightarrow \text{V}(X/A)$ factors as a level weak equivalence (use the maps $p$ of diagrams (8.4.8)) followed by a cofinal
inclusion \( \{ \text{VN}(V^n) \} \subset V(X/A) \). Thus \( VX/VA \simeq V(X/A) \) as required.

\[
\]

(8.4.20) **Proposition.** There are natural weak equivalences \( \Sigma VX \to V \Sigma X \) in Ho(pro-Top) where \( \Sigma \) denotes the appropriate unreduced suspension.

**Proof.** We use geometric realizations as in Proposition (8.4.1). Following (8.4.1), we may define a map \( CRVX \to RV CX \) which yields a solid-arrow commutative diagram

\[
\begin{array}{ccc}
RVX & \longrightarrow & CRVX \\
\downarrow & & \downarrow \\
RVX & \longrightarrow & RV CX
\end{array}
\]

in which the rows are cofibration sequences and the vertical maps are equivalences in Ho(pro-Top). Proposition (3.4.12) yields a filler (in Ho(pro-Top)) \( CRVX/\text{RVX} \to RVX/RVCX \) in diagram (8.4.21) which is also an equivalence there. But \( CRVX/\text{RVX} \simeq \Sigma RVX \simeq R \Sigma VX \) (R preserves suspensions by [Mil-2]) and \( RV CX/\text{RVX} \simeq R(V CX/VX) \) (R preserves quotients by [Mil-2]) \( \simeq RV \Sigma X \). Naturality is easy to check. The conclusion follows. \( \square \)
(8.4.22) **Proposition.** Let \( \{X_j\} \) be an inverse system of compact metric spaces, and let \( X = \lim_j\{X_j\} \). Then the projections \( X \rightarrow X_j \) induce a natural equivalence \( \{VX_j\} \rightarrow \{VX\} \) in \( \text{Ho(pro-SS)} \).

**Proof.** By first applying the Mardešić trick (Theorem (2.1.6)) if necessary, we may assume that the indexing category \( J = \{j\} \) is a strongly cofinite directed set.

Let \( \{U_{j,k} \mid k \in K_j\} \) be the inverse system of all finite open coverings of \( X_j \), where the \( k \)-index is assigned so that for \( j' > j \), \( U_{j',k} \) is the pullback of \( U_{j,k} \) to \( X_{j',k} \). Again, by applying the Mardešić trick if necessary, we may assume that each indexing category \( K_j \), where \( j \in J \), is a cofinite directed set. Because each \( X_j \) is compact, \( \{\VN(U_{j,k})\} \) is cofinal in the Vietoris system of \( X_j \), hence isomorphic to it. Assign a partial order to \( U_{j \in J\{j\} \times K_j} \) as follows: \( (j,k) \preceq (j',k') \) if \( j \preceq j' \) and \( k \preceq k' \). Then

(8.4.23)

\[
\lim_j\{VX_j\} = \lim_j\{\{\VN(U_{j,k})\}_{k \in K_j}\} = \{\VN(U_{j,k'})\}_{j \in J, k \in K_j}
\]

(an inverse system is its own limit in any pro-category). From now on, unless otherwise stated, \( (j,k) \) ranges over \( U_{j \in J\{j\} \times K_j} \).
We shall now write the natural map \( \mathbb{V}_X \to \lim_j \{ \mathbb{V}_{X_j} \} \) as a composite of several maps which will be later shown to be weak equivalences. To do this, let \( X'_j \) denote the image of \( X \) in \( X_j \).

Apply Lemma (8.4.8) inductively to obtain open coverings \( U_{j,k} \) of \( X_j \) and neighborhoods \( N_{j,k} \) of \( X'_j \) in \( X_j \) with the following properties:

(a) \( U_{j,k} \) refines \( U_{j,k'} \) for \( k' < k \), yielding inverse systems \( \{ U_{j,k} \}_{k \in K_j} \) for \( j \in J \);

(b) \( U_{j,k} \) refines \( U_{j,k} \), so that \( \{ U_{j,k} \}_{k \in K_j} \) is cofinal in \( \{ U_{j,k} \}_{k \in K_j} \) for \( j \in J \);

(c) \( U_{j,k} \) refines \( U_{j',k} \) for \( j' < j \), yielding an inverse system \( \{ U_{j,k} \} \) which is cofinal in \( \{ U_{j,k} \} \) by (b);

(d) \( N_{j,k} \subseteq N_{j,k'} \) for \( k' < k \), yielding inverse systems \( \{ VN(U_{j,k} | N_{j,k}) \}_{k \in K_j} \) for \( j \in J \);

(e) \( N_{j,k} \) is contained in the pullback of \( N_{j',k} \) to \( X_{j,k} \) for \( j' < j \), yielding an inverse system \( \{ VN(U_{j,k} | N_{j,k}) \} \);

(f) The inclusions \( VN(U_{j,k} | X'_j) \to VN(U_{j,k} | N_{j,k}) \) are equivalences in \( \text{Ho}(S) \), hence the
levelwise inclusion
\[ \{ \text{VN}(V_j,k | N_j,k) \} \leftarrow \{ \text{VN}(V_j,k | N_j,k) \} \]

is an equivalence in \( \text{Ho(pro-SS)} \).

Factor the natural map \( \mathbb{V}X \rightarrow \text{lim}_j \{ \mathbb{V}X_j \} \) as follows:

(8.4.24) \[ \mathbb{V}X \xrightarrow{\pi} \{ \text{VN}(U_j, \overline{X}_j) \} \]

\( \Rightarrow \{ \text{VN}(U_j, \overline{X}_j) \} \) (by (c), above)

\[ \xrightarrow{\delta} \{ \text{VN}(V_j,k | N_j,k) \} \] (by (f), above)

\[ \rightarrow \{ \text{VN}(V_j,k) \} \]

\[ \cong \text{lim}_j \{ \{ \text{VN}(V_j,k) \}_{k \in K_j} \} \]

\[ \cong \text{lim}_j \{ \mathbb{V}X_j \} \) (by (b), above).

We shall complete the proof by observing that the maps \( \pi \) and \( i \) above are pro-isomorphisms. For \( \pi \) this is an easy consequence of compactness and properties of the product topology. For \( i \), consider any fixed index \( (j,k) \). The images of \( X_j \), in \( X_j \) for \( j' > j \) form a family of compact sets whose intersection \( X_j \) is contained in the open set \( N_j,k \). Hence we may choose a \( j' > j \) such that the image of \( X_j \), in \( X_j \) is contained within \( N_j,k \). Because \( V_j,k \) refines \( V_j,k \) (property (c), above), we obtain a commutative diagram.
(8.4.25) \[
\text{VN}(V_j, k \mid N_j, k) \leftrightarrow \text{VN}(V_{j'}, k)
\]
\[
\text{VN}(V_j, k \mid N_j, k) \leftrightarrow \text{VN}(V_j, k).
\]

Hence, \( i \) is a pro-isomorphism, as required. The conclusion follows. \( \Box \)

§8.5. **Proofs of Theorems (8.3.19), (8.3.20), and (8.3.21).**

(8.5.1) **Proof of Theorem (8.3.19).** Because \( V \) preserves homotopies (proposition (8.4.1)) and the strong homology theory \( S_{h_n} \) on pro-SS is homotopy invariant (5.6.7), the composites \( S_{h_n} \circ V \) are homotopy invariant. Similarly, because \( V \) preserves cofibration sequences (Proposition (8.4.7)), exactness of \( S_{h_n} \circ V \) follows from exactness of \( S_{h_n} \) (5.6.7). Thus, Axiom (8.3.3)(E) holds.

For Axiom (S), the required natural equivalences are given by

\[
S_{h_n} \circ V \xrightarrow{=} S_{h_{n+1}} \circ \Sigma \circ V \quad \text{(by (5.6.7))}
\]
\[
\xrightarrow{=} S_{h_{n+1}} \circ V \circ \Sigma \quad \text{(by (5.6.7) and Proposition (8.4.20)).}
\]

The conclusion follows. \( \Box \)

(8.5.2) **Proof of Theorem (8.3.20).** Let \( X \) be a finite complex. Then \( VX \) admits a cofinal subtower \( \{\text{VN}(U_n)\} \) where each \( U_n \)
is a finite open covering of \( X \). Let \( \{\text{RCN}(U_n)\} \) be the tower of realizations of \( \check{\text{C}} \)ech nerves. Choose bonding maps

\[
\text{RCN}(U_{n+1}) \to \text{RCN}(U_n) \to \text{rigidify} \{\text{RCN}(U_n)\}.
\]

By [Dow], \( \{\text{RVN}(U_n)\} \cong \{\text{RCN}(U_n)\} \) in \( \text{tow-Ho(}\text{Top}) \). But because \( X \) is a complex, \( \{\text{RCN}(U_n)\} \cong X \) in \( \text{tow-Ho(}\text{Top}) \). But, by (5.2.13), the composite \( \{\text{RVN}(U_n)\} \to X \) is an isomorphism in \( \text{Ho(}\text{tow-Top}) \). This yields a natural isomorphism \( \eta: S_{h_*} \to h_* \) on finite complexes. \( \square \)

(8.5.2) \textbf{Proof of Theorem (8.3.21).} Interpret products and operations in terms of maps of spectra (see [Adams - 5]). The conclusion follows easily from our formula. \( \square \)

8.6. \textbf{Spectral sequences.} We develop Bousfield-Kan and Atiyah-Hirzebruch type spectral sequences which converge to \( S_{h_*} \). Ken Brown [Brown] developed similar spectral sequences for sheaf cohomology; Kaminker and Schochet [K-S, Theorem (3.10)] obtained the second spectral sequence using fundamental complexes. We shall use the Bousfield-Kan spectral sequence to verify that \( S_{h_*} \) satisfies Axiom (8.3.3)(W). This completes the proof that \( S_{h_*} \) is a generalized (reduced) Steenrod homology theory.

(8.6.1) \textbf{Theorem.} (Bousfield-Kan spectral sequence). Let

\( \{X_j\} \) be an inverse system of compact metric spaces and let
\[ X = \lim \{ X_j \}. \quad \text{Then there is a spectral sequence with} \]

\[ E_2^{p,q} = \lim_j \{ S_{h_q}(X_j) \}, \]

\[ which\ converges\ completely\ under\ suitable\ circumstaneses\ to\ \text{S}_{h^*}(X).\]

**Proof.** Recall that \( VX = \{ VX_j \} \) in pro-SS. Now apply the Bousfield-Kan spectral sequence (§4.9) for \( S_{h_q} \) applied to the inverse system \( \{ VX_j \} \) in "pro-(pro-SS)". Compare Proposition (5.6.8).

The conclusion follows. □

In particular, suppose that \( X_j \) is an inverse system of cardinality \( \leq k^n \) so that \( \lim_j \{ S_{h_q}(X_j) \} = 0 \) unless \( 0 \leq p \leq n+1 \). Then \( E_n = E_n^{*,q} \) because for \( r > n+1 \) the differentials \( \partial_r \) (of bidegree \( (r, r-1) \)) either begin or end at a 0-group. This is complete convergence. We cite an important special case.

(8.6.2) **Corollary.** (Compare (4.9.3)). Let \( \{ X_j \} \) be a tower of compact metric spaces and \( X = \lim \{ X_j \} \). Then there are short exact sequences.

\[ n \to \lim_j \{ S_{h_q+1}(X_j) \} \to S_{h_q}(X) \to \lim_j \{ S_{h_q}(X_j) \} \to 0. \quad □ \]

(8.6.3) **Corollary.** The strong wedge axiom ((8.3.3)(W)) holds for \( S_{h^*} \).
Proof. For compact metric spaces $X_1, X_2, \cdots$, consider the tower $\{v_{n=1}^{N} X_j|N = 1, 2, \cdots\}$ with $\lim_N \{v_{j=1}^{N} X_j\} = V_{j=1}^{\infty} X_j$, the strong wedge. Apply Corollary (8.5.2) to this tower. Because bonding maps $S_{q+1}^{h_j} (v_{j=1}^{N+1} X_j) \longrightarrow S_{q}^{h_j} (v_{j=1}^{N} X_j)$ in the towers $\{S_{q+1}^{h_j} (v_{j=1}^{N} X_j)\}$ are clearly surjections, the $\lim^1$ terms vanish in this case. The conclusion follows. \qed

(8.6.4) Remarks.

(a) This completes the proof of Theorem (8.3.21).

(b) If a compact metric space $X$ is represented as the limit of a tower of polyhedra $\{X_j\}$, Corollary (8.5.2) yields short exact sequences

$$0 \longrightarrow \lim^1_j \{h_{q+1}(X_j)\} \longrightarrow S_{q}^{h_j}(X) \longrightarrow \lim_j \{h_{q}(X_j)\} \longrightarrow 0$$

relating $S_{q}^{h_j}$ to $h_{q}$. Compare Milnor's characterization of ordinary (reduced) Steenrod homology [Mil-1]. Uniqueness does not follow in our case because of possible extension problems; however, any natural transformation of Steenrod extensions of $h_{q}$ is an isomorphism by the above short exact sequences.
(8.6.5) **Theorem.** (Atiyah-Hirzebruch spectral sequence) \([K - S]\).

Let \(X\) be a compact metric space of dimension \(d < \infty\). Then there is a spectral sequence with

\[
E^2_{p, q} = H_p(X; \mathbb{H}^q(S^0))
\]

and \(d^r\) of bidegree \((-r, r - 1)\) which converges to \(S_{h_\ast}(X)\) in the sense that \(E^{d+1} = E^\infty\).

Our proof is contained in (8.6.6) - (8.6.21), below. For a compact metric space \(X\) of dimension \(d < \infty\) there is a cofinal tower \(\{U_n\}\) in the Vietoris system \(VX\) such that each \(U_n\) is a finite open cover (use compactness) with \(\dim \text{CN}(U_n) \leq d\) (use the definition of covering dimension). We therefore begin by proving the following.

(8.6.6) **Theorem.** Let \(X = \{X_n\}\) be a tower of finite simplicial complexes and simplicial maps. Then there is a spectral sequence with

\[
E^2_{p, q} = H_p(X; \mathbb{H}^q(S^0))
\]

which converges to \(S_{h_\ast}(X)\) if \(\dim X\) is finite. More precisely, if \(\dim X \leq d < \infty\), \(E^{d+1} = E^\infty\).

**Proof.** The proof is broken up into several steps: (8.6.7) - (8.6.20), below.
(8.6.7) Construction of the spectral sequence. Let
\( F^p X = \{ F^p X \} \) be the levelwise \( p \)-skeleton of \( X \). Following Massey, we define an exact couple

\[
\begin{array}{ccc}
D^1_{**} & \xrightarrow{i} & D^1_{**} \\
\downarrow{k} & & \downarrow{j} \\
E^1_{**} & & \\
\end{array}
\]

where

\[
E^1_{p,q} = S_{p+q} (F^p X / F^{p-1} X),
\]

\[
D^1_{p,q} = S_{p+q} (F^p X),
\]

degree \( i = (1,-1), \) degree \( j = (0,0), \) and degree \( k = (-1,0). \)

This yields a spectral sequence \( \{ E^F_{p,q}(X) \}. \)

(8.6.9) Description of \( E^1 \) and \( E^2 \). For each \( p \) and \( n \),

\[
(F^p X / F^{p-1} X)_n = (F^p X_n / F^{p-1} X_n) = \text{VS}^p,
\]

a finite wedge of \( p \)-spheres, one for each \( p \)-cell of \( X_n \). Hence,

\[
E^1_{p,q} = S_{p+q} \{ \text{VS}^p \}_n.
\]

Also, because the bonding maps \( X_n \to X_{n-1} \) are simplicial, the composite maps

\[
S^p \overset{i}{\to} (\text{VS}^p)_{n+1} \overset{\pi}{\to} (\text{VS}^p)_n \overset{\pi}{\to} S^p
\]
(where \( i \) is a typical injection and \( \pi \) a typical projection) have degree 0 except for at most one \( \pi \) for each \( i \); in which case the degree is \( \pm 1 \). Hence, if \( S_{h_\pi}^*(S^0) \) is a graded ring, we may choose bases for the free modules \( \bigoplus_{n \geq 0} (S_{h_\pi}^* S^0) \) so that the maps

\[
(8.6.11) \quad S_{h_{p+1}}((VS^P)_n^{n+1}) \longrightarrow S_{h_{p+1}}((VS^P)_n^n)
\]

are represented by matrices of the form

\[
\begin{pmatrix}
\pm 1 \\
\cdot \\
\cdot \\
\cdot \\
\pm 1 \\
0 \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
0
\end{pmatrix}
\]

(8.6.11)

Hence, the towers

\[
(8.6.12) \quad \{h_{p+q+1}(VS^P)_n | n \geq 0\}
\]

are Mittag-Leffler, so that

\[
(8.6.13) \quad \lim_{n \to \infty} \{h_{p+q+1}(VS^P)_n\} = 0
\]

for all \( p, q \). (If \( S_{h_\pi}^*(S^0) \) is not a ring, analogues of (8.6.10) - (8.6.11) still hold, but are more difficult to describe. Thus (8.6.13) holds in general.) (8.6.13) yields the following useful calculation.
\[(8.6.14) \quad S_{h^p+q}((VS^P)_n) = h^p+q((VS^P)_n) \]
\[= \lim_{n \to \infty} h^p+q((VS^P)_n) \]
\[= \lim_{n \to \infty} \{H_p((VS^P)_n); h_q(S^0)\} \]
\[= H_p((VS^P)_n); h_q(S^0) \]
\[= S_{h^p}((VS^P)_n); h_q(S^0) \]

Now, for each fixed \(q = q_0\) consider the exact couple (8.6.8) associated with the generalized Steenrod homology theory

\[(8.6.15) \quad S_{k_n}(-) = S_{h_{n-q_0}}^{-}(-; h_{q_0}(S^0)) \]

\[
\begin{align*}
S_{k_n}(S^0) &= \begin{cases} 
  h_q(S^0) & n = q_0, \\
  0 & n \neq q_0.
\end{cases}
\end{align*}
\]

In this case, the resulting \(E^1_{p,q} = 0\) unless \(q = q_0\), so that in the resulting spectral sequence

\[
E^2_{p,q}(X) = E^\infty_{p,q}(X) = \begin{cases} 
  S_{k_{p+q}}(X) = S_{h_p}(X; h_q(S^0)), \\
  0 & \text{otherwise}.
\end{cases}
\]

By (8.6.14), and construction, the spectral sequences associated with the homology theories (8.6.15) map to the original spectral sequence associated with \(S_{h^*}\). Hence in that case,
(8.6.16) \[ E^2_{p, q}(X) = \tilde{S}_p(X; h_q(S^0)) \]

as required.

(8.6.17) **Convergence.** If \( \dim X = d < \infty \), then \( E^1_{p, q} = 0 \) unless \( 0 \leq q \leq d \), so that \( d_{d+2} = d_{d+3} = \cdots = 0 \) and \( E^{d+1}_{p, q} = E^\infty_{p, q} \).

(8.6.18) **Naturality.** Consider a weak equivalence \( X \to Y \) of towers of simplicial sets of bounded dimension. Then there is a diagram

(8.6.19) \[
\begin{array}{ccc}
X & \overset{\sim}{\longrightarrow} & Z \\
\downarrow & & \downarrow \\
Y & \overset{\sim}{\longrightarrow} & \end{array}
\]

in tow-SS, hence an isomorphism

(8.6.20) \[
E^2_{p, q}(X) \cong \tilde{S}_p(Z; h_q(S^0)) \\
\overset{\sim}{\longrightarrow} \tilde{S}_p(Y; h_q(S^0)) \\
\overset{\sim}{\longleftarrow} \tilde{S}_p(X; h_q(S^0)) \\
\cong E^2_{p, q}(Y).
\]

It is easy to see that the isomorphism (8.6.20) is independent of \( Z \) and the maps in (8.6.19). Hence naturality follows from naturality of \( \tilde{S}_{p}(\ , h_q(S^0)) \). This concludes the proof of Theorem (8.6.6). □
(8.6.21) Proof of Theorem (8.6.5). With $X$ a compact metric space of dimension $d$, choose a cofinal tower of open coverings of $X$, \( \{U_n\} \), in $V(X)$ such that each $U_n$ is finite and satisfies $\dim \text{CN}(U_n) \leq d$. Next, choose bonding maps to rigidify the Čech system $\text{CX} = \{\text{CN}(U_n)\}$. By Dowker [Dow], $\text{RVX} \cong \text{RCX}$ in $\text{tow-Ho(Top)}$. Hence, by (5.2.9), $\text{RVX} \cong \text{RCX}$ in $\text{Ho(tow-Top)}$.

Applying Theorem (8.6.6) to $\text{RCX} = \{\text{RCN}(U_n)\}$ with $X = \lim_n \{\text{RCN}(U_n)\}$ yields the required spectral sequence. Naturality follows from naturality of the Vietoris construction; replace $\text{RCX}$ by $\text{RVX}$ in the formula for $E^2_{*,*}(X)$. \( \square \)
9.8.7 Duality.

As above, $h_*$ is a generalized homology theory on finite complexes represented by a CW spectrum $E$, and $S^h_*$ is its canonical Steenrod extension. We shall prove the following duality theorem.

(8.7.1) Theorem. Let $X$ be a compactum in $S^n$. Then

$$S^n_*(X) \cong h^{n-p-1}(S^n \setminus X),$$

and the isomorphism is natural with respect to inclusion maps.

(8.7.2) Alexander and Spanier-Whitehead duality. See Adams [Adams-5, Chapter 5, 10] for details. Consider a polyhedron $K$ linearly embedded in $S^n$. Alexander duality provides an isomorphism

$$H^p(K) \cong H_{n-p-1}(S^n \setminus K).$$

This was extended by Spanier and Whitehead to state that $K$ determines the stable homotopy type of $S^n \setminus K$, and that $S^n \setminus K$ has the homotopy type of a polyhedron.

Spanier introduced the following formulation of duality. Consider two polyhedra $K$ and $L$ disjointly embedded in $S^n$. Choose a PL path $w$ from $K$ to $L$ with $w(0) \in K$, $w(1) \in L$, and $w(0,1)$ disjoint from $K$ and $L$. Regarding $S^n$ as the compactification of $R^n$ with $w(1/2)$ as the "point at $\infty$" yields disjoint embeddings of $K$ and $L$ in $R^n$. Define a map

(8.7.3) $\mu : K \times L \to S^{n-1}$

by

(8.7.4) $\mu(k,1) = \frac{k-1}{\|k-1\|}.$

It is easy to see that the maps $\mu|_{w(0) \times L}$ and $\mu|_{K \times w(1)}$ are null-homotopic, so that (8.7.3) yields a map

(8.7.5) $\mu : K \cup L \to S^{n-1}.$
Regarding \( K, L, \) and \( S^{n-1} \) as spectra and taking the adjoint of \( \mu \) in (8.7.5) yields a map

\[
(8.7.6) \quad \mu_* : K \to \text{Map}(L, S^{n-1});
\]

here \( \text{Map}(L, S^{n-1}) \) is the function spectrum \([\text{Has-4}], [\text{Has-5}]\). If the inclusion \( L \hookrightarrow S^n \setminus K \) is a homotopy equivalence, then \( \mu_* \) is a stable homotopy equivalence. The roles of \( K \) and \( L \) in (8.7.5) and hence (8.7.6) are symmetric. These remarks lead to calling \( S^{n-1} \setminus K \) the \((n-1)\)-dual of \( K \),

\[
(8.7.7) \quad D_{n-1} K \equiv S^{n-1} \setminus K.
\]

Then \( D_{n-1} \circ D_{n-1} = \text{id}. \) Finally, one can show that the natural "composition"

\[
(8.7.8) \quad D_K \wedge \text{Map}(S^{n-1}, E) \xrightarrow{\mu_*} \text{Map}(K, S^{n-1}) \wedge \text{Map}(S^{n-1}, E) \to \text{Map}(K, E)
\]

is a stable homotopy equivalence. Let \( \Sigma^{n-1} E \equiv \text{Map}(S^{n-1}, E); \) (stably) \( \Sigma^{n-1} E \) is the \((n-1)\)-fold desuspension of \( E \).

(8.7.9) Proof of Theorem (8.7.1). Choose a sequence of closed polyhedral neighborhoods of \( X, \{X_j\}_{j=0,1,2,\ldots} \) with the following properties: \( X_j \) is a PL subspace of \( \text{int} \: X_{j-1} \), and \( \bigcap_j X_j = X \).

We shall construct a sequence of isomorphisms

\[
(8.7.10) \quad S_{\pi}(X) \equiv \pi_p^s \{X_j \wedge E\} \quad (\star)
\]

\[
\cong \pi_{p-n+1}^s \{X_j \wedge \Sigma^{n-1} E\} \quad (\star)
\]

\[
\cong \pi_{p-n+1} \{\text{Map}(X_j, E)\} \quad (1)
\]

\[
\cong \{X_j \wedge \Sigma^{p-n-1} E\}, \; \text{E} \quad (\star)
\]

\[
\cong \{(S^n \setminus X_j) \wedge \Sigma^{p-n-1} E\}, \; \text{E} \quad (2)
\]

\[
\cong \text{H}(n-p-1)(S^n, X) \quad (\star),
\]

where \( X_j' \subset S^n \setminus X_j \) and the inclusion is a weak equivalence.

Isomorphisms \((\star)\) are clear. Isomorphism \((1)\) will follow from
a careful look at Spanier-Whitehead duality (8.7.11). For (2) see (8.7.15). For (3), see Lemma (8.7.16).

(8.7.11) **Construction of** $X'_j$. We shall assume that the embedding of $X$ in $S^n$ extends to an embedding of $X \cup [0,1]$, where $0 \in X$ and $(0,1]$ is disjoint from $X$. This may be done by composing the embedding $X \hookrightarrow S^n$ with the "equatorial" embedding $S^n \hookrightarrow S^{n+1}$ and having the "whisker" $[0,1]$ run from a point in $X$ to the north pole of $S^{n+1}$. One would then have to check the behavior of formula (8.7.10) with respect to suspension, but this is easy. Choose polyhedral neighborhoods of $X$ as above, but subject to the further restriction that $1 \notin X_j$ for all $j$ (it suffices to have $1 \notin X_0$). Extend these neighborhoods to closed polyhedral neighborhoods $X^*_j$ of $X \cup [0,1]$ with $X^*_j$ a PL subspace of $\text{int } X^*_{j-1}$, so that the diagrams

$$\begin{array}{ccc}
X & \xrightarrow{\alpha} & X \cup [0,1] \\
\downarrow & & \downarrow \\
X_j & \xrightarrow{\alpha} & X^*_j
\end{array}$$

commute. Finally, inductively choose polyhedra $X'_j \subset S^n \setminus X^*_j$ such that $X'_j$ is a PL subspace of $\text{int } X'_{j+1}$ and the inclusions $X'_j \hookrightarrow S^n \setminus X^*_j$ are homotopy equivalences. (This can be done by thickening $X_j$ to a homotopy equivalent open subspace of $S^n$ and triangulating the complement.) These constructions are illustrated in the following figure.
The point $a$ on $[0,1]$ is chosen so that $a \notin X_j$, consequently, $a \notin X_j$ (and $a \notin X_j'$) for all $j$. Applying Spanier-Whitehead duality as in (8.7.3) - (8.7.3) with $a$ as the "point at $\infty$" yields homotopy commutative diagrams

\[
  \begin{array}{ccc}
    X_{j+1} & \rightarrow & \text{Map}(X_{j+1}', S^{n-1}) \\
    \downarrow & & \downarrow \\
    X_j & \rightarrow & \text{Map}(X_j', S^{n-1}),
  \end{array}
\]

(8.7.13)

hence also

\[
  \begin{array}{ccc}
    X_{j+1} \wedge \Omega^{n-1} E & \xrightarrow{\alpha} & \text{Map}(X_{j+1}', E) \\
    \downarrow & & \downarrow \\
    X_j \wedge \Omega^{n-1} E & \xrightarrow{\alpha} & \text{Map}(X_j', E).
  \end{array}
\]

If diagrams (8.7.14) were actually commutative, (8.7.10)(1) and naturality would follow immediately. Instead, we must appeal to Theorem (5.2.3) to conclude that the equivalence

\[
\left\{ X_j \wedge \Omega^{n-1} E \right\} \xrightarrow{\text{equivalence}} \left\{ \text{Map}(X_j', E) \right\}
\]

in $\text{Ho}(\text{Sp})$ (use the singular functor to work in simplicial spectra) yields an equivalence in $\text{Ho}(\text{tow-Sp})$. The construction in (5.2.3) is sufficiently natural to be carried out for pairs $(X_j \wedge \Omega^{n-1} E, X_j')$, yielding the claimed naturality. Hence the map (8.7.10)(1) is an isomorphism.

(8.7.15) The commutative diagrams

\[
  \begin{array}{ccc}
    X_{j+1}' & \xrightarrow{\alpha} & S^n \setminus X_{j+1} \\
    \downarrow & & \downarrow \\
    X_j' & \xrightarrow{\alpha} & S^n \setminus X_j
  \end{array}
\]
imply that the inclusion \( \{ X'_j \} \rightarrow \{ S^n \setminus X_j \} \) is an equivalence in \( \mathbf{Ho}(\text{tow-Top}) \), hence also in \( \mathbf{Ho}(\text{tow-Sp}) \) (use the stable singular functor): (3.7.10) (2) follows. \( \Box \)

(3.7.16) **Lemma.** Let \( \{ X'_j \} = \{ X'_c \subseteq X'_1 \subseteq X'_2 \subseteq \ldots \} \)

be a direct system bonded by cofibrations. Let \( X' = \text{colim}_j X'_j \).

Then

\[
\{ \{ X'_j \} \}, E \cong [X', E].
\]

Proof. The maps \( X' \rightarrow X'_j \) induce a map \( [X', E] \rightarrow \{ \{ X'_j \} \} \).

E). Applying the homotopy extension property of the weak

Compare Propositions (4.2.2) and (4.2.6)

\((X'_j, X'_{j-1})\) yields the required inverse. \( \star \) The conclusion follows. \( \Box \)

This completes the proof of Theorem (3.7.1); the required

naturality with respect to inclusion maps follows from our

constructions. \( \Box \)

(3.7.17) **Remarks.** In the language of model categories,

\( X' \) is the homotopy colimit of \( \{ X'_j \} \). See \( \S 4 \), Bousfield and Kan

[\( \text{B-K}, \text{Chapter XII} \)], and R. Vogt [\( \text{Vogt-1} \)].
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ADDENDUM TO §2.

§2.1. Add the following definitions.

An object \( \{x_j\} \) of pro-\( C \) is called **stable** if it is isomorphic in pro-\( C \) to an object of \( C \). \( \{x_j\} \) is called **moveable** if for each \( j \) there exists a \( k>j \) such that for each \( l>k \) there exists a filler in the diagram

\[
\begin{array}{ccc}
X_k & \longrightarrow & X_l \\
\downarrow & & \downarrow \\
X_j & \longrightarrow & X_j'
\end{array}
\]

tow-\( C \) is the full subcategory whose objects are inverse systems indexed by the natural numbers. Objects in tow-\( C \) are called **towers**.

The construction of pro-\( C \) may be dualized to yield a category inj-\( C \) of direct systems:

\[
\text{inj-} C \left( \{x_j\}, \{y_k\} \right) = \lim_j \colim_k \{C(x_j, y_k)\}.
\]

Alternatively, a map in inj-\( C \) from \( \{x_j\} \) to \( \{y_k\} \) consists of a function \( \theta: \{j\} \rightarrow \{k\} \) and maps \( f_j: x_j \rightarrow y_{\theta(j)} \) such that for each bonding map \( x_j \rightarrow x_j' \) exists a \( k \) and a commutative diagram

\[
\begin{array}{ccc}
x_j & \xrightarrow{f_j} & y_{\theta(j)} \\
\uparrow & & \uparrow \\
x_j & \xrightarrow{f_j'} & y_{\theta(j')} \\
\downarrow & & \downarrow \\
x_j & \xrightarrow{\text{bond}} & y_k
\end{array}
\]
The Artin-Mazur theory of pro-$C$ \([A-M, \text{ appendix}]\) and Mardešić construction (Theorem (2.6.1)) may be dualized to inj-$C$. The co-limit functor $\text{colim}: C^J \to C$ ($J$ a direct system) factors through inj-$C$.

Suppose $C$ admits a function space construction $\text{HOM}: C^{\text{op}} \times C \to C$. Then $\text{HOM}$ extends to a functor $\text{HOM}: (\text{pro-}C)^{\text{op}} \times C \to \text{inj-}C$.

§2.2. Include definitions of finite simplicial sets and the dimension of a simplicial set.

After §2.2. Add a new section on spectra. This section will include

- CW prespectra;
- CW spectra, (following Adams \([\text{Adams-5}]\);
- The smash product on CW spectra (following \([\text{Has-4, Has-5}]\)) and the dual: function spectra;
- Representable homology and cohomology theories (following, e.g. \([\text{Adams -5}]\)), as needed for §5;
- Simplicial spectra (following \([\text{Kan-1}]\)), with a proof of the closed model structure following \([\text{Has-2}]\); the first proof of this is due to Ken Brown \([\text{Br, K}]\) as needed for §§5,8;
- Homology and cohomology operations arising from maps of spectra, following \([\text{Adams-5,6}]\), as needed for §8;
- Spanier-Whitehead duality, (following, e.g. \([\text{Adams-5}]\), as needed for §8.7.

§2.3. Cite \([\text{Br, K}]\) for the closed model structure on simplicial spectra.
ADDENDUM TO §3.

§3.2 In Remarks (3.2.5), refer to the definition of \( \mathcal{K} \)-isomorphism stated in §5.4. Also, the Bousfield-Kan [B–K] closed model structure is "natural" on direct systems and level maps.

§3.2, 3.3. Mention the induced closed model structure on \( C^N \) and \( \text{tow}-C \subset \text{pro}-C \).

Following §3.3. Add a new section on the closed model structures on \( C^J \) (\( J \) a cofinite direct system) and \( \text{ini}-C \).

Fibrations and weak equivalences in \( C^J \) are defined levelwise. Cofibrations are defined by the left lifting property of \([n-1]\), see §2.3. The explicit definition is dual to diagram (3.2.3):

\( \{X_j\} \rightarrow \{Y_j\} \) in \( C^J \) is a cofibration if and only if the induced maps \( f_j \) below are cofibrations in \( C \). \( P_j \) is the pushout below.
Proofs in this section are dual to those in §§3.2-3.3 and will be sketched or omitted. This section will affect §3.6: Pairs.
ADDITION TO §4.

§4.2 and 4.3. Give the dual construction of homotopy colimits, involving a dual to $\text{Ex}^\infty$ on inj-SS.

§4.9. Expand the Bousfield-Kan [B-K] spectral sequence in several directions. First, consider the $C$-simplicial spectra in which $\pi_0$ is a group. In fact, if $C$ is a stable category, we may replace $\pi_X$ by any generalized homology theory. Second, the sequence naturally relates $\{\pi_*X_j\}$ to $\pi_*\{X_j\}$ in the case when the $X_j$ are themselves inverse systems. In some sense §§4.9 and 5.2 belong together, i.e., pro-$\text{Ho}(C)$ vs. $\text{Ho}$(pro-$C$), for $C$ a closed simplicial model category, is closely related to the Bousfield-Kan spectral sequence. Finally, discuss, here or in §5.2, inj-$\text{Ho}(C)$ vs. $\text{Ho}$(inj-$C$).
ADDENDUM TO §5.

§5.2. The wall obstruction for homotopy idempotents (Remarks (5.5.8)) may be relevant to Heller's question about rigidifying simplicial objects over $\text{Ho}(SS)$.

§5.5. For (5.5.4), see also recent work of Morita.

Following (5.5.13), we shall prove that $X = \{x_n\}$ is an exotic Eilenberg-MacLane pro-space.

§5.6. Following (5.6.8), we shall show how the homology operations associated with $h_*$ extend to $S_{h_*}$ by using maps of spectra.