MATH 4780/6780: Mathematical Biology

Problem Set 2

The assignment is due **Friday 2/7 by 4pm**. Place your assignment in the mailbox of Nicole Song, located in Boyd 434A. Show your work on all problems. Correct answers without the necessary work will not get any credit. Submit your solutions in order (do not place all codes or figures at the end.)

1. (31 pts.) Recall the following problem you did in the previous assignment:

   Consider the modified discrete logistic equation for the Allee effect:

   $$N_{t+1} = rN_t^2(1 - N_t)$$

   Use Desmos (link available at the course website) to observe the listed cases below, print a representative cobweb figure for each case, and write down the approximate values for a, b, c, d, e.
   - For $0 < r < a$, no stable positive steady-state exists.
   - For $a < r < b$, a single single stable positive steady-state exists.
   - For $b < r < c$, a stable 2-cycle exists.
   - For $c < r < d$, a stable 4-cycle exists.
   - For $d < r < e$, a stable 8-cycle exists.
   - For $r > e$, chaotic solution exists.

   (a) (3 pts.) Even though the solution is chaotic for higher values of r, there are exceptions. For example, there exists a stable 3-cycle for $r = 6.18$. In other words, there exists values u, v, w such that

   $$\begin{aligned} v &= 6.18\,u^2(1-u) \\ w &= 6.18\,v^2(1-v) \\ u &= 6.18\,w^2(1-w) \end{aligned}$$

   Use Desmos (link available at the course website) to draw the cobweb diagram of this 3-cycle, and print this figure.

   (b) (3 pts.) Write down the third-iterate map explicitly.

   (c) (3 pts.) Use Desmos to draw the cobweb diagram of the third iterate map with the initial condition $N_0 = 0.5$, and print this figure.

   (d) (10 pts.) Using the cobweb diagram of the third iterate map, find and write down the approximate values of all positive fixed-points of the third iterate map (accurate up to three significant figures[1]). Identify the stability of each fixed point. Note that u, v, w are all stable fixed points of the third iterate map. Identify which of these fixed points correspond to u, v and w.

[1] You can zoom into any part of the figure using the scroll wheel of your mouse to increase accuracy.

(e) (5 pts.) For $a < r < b$, express the single positive stable fixed-point N^* in terms of r.

(f) (7 pts.) As r increases past b, this single stable positive steady-state becomes unstable. We can use this fact to find the exact value of b, but the equation is hard to solve by hand. So, find the equation, the solution of which is b. In other words, find a function $G(x)$ that satisfies $G(b) = 0$.

2. (15 pts.) Ricker model has the advantage that N_t never takes negative values:

$$N_{t+1} = N_t e^{r(1-N_t)}$$

Using your knowledge as to how we analyzed the previous models, draw a rough sketch of the bifurcation diagram of this model by hand for $1 \leq r \leq 2.6$. Note that almost all equations about the fixed points and their stability involve transcendental equations, which are impossible to solve by hand. So sketch this diagram using your observations of the cobwebs. Feel free to utilize the second iterate map if need be.