GALOIS THEORY FOR ARBITRARY FIELD EXTENSIONS

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Contents

Abstract.

1. Introduction

1.1. Kaplansky's Galois Connection and Correspondence.

For an arbitrary field extension K/F, define $\mathcal{L} = \mathcal{L}(K/F)$ to be the lattice of subextensions L of K/F and $\mathcal{H} = \mathcal{H}(K/F)$ to be the lattice of all subgroups H of $G = \mathrm{Aut}(K/F)$. Then we have

$$\Phi : \mathcal{L} \to \mathcal{H},\ L \mapsto \mathrm{Aut}(K/L)$$

and

$$\Psi : \mathcal{H} \to \mathcal{F},\ H \mapsto K^H.$$

For $L \in \mathcal{L}$, we write

$$c(L) := \Psi(\Phi(L)) = K^{\mathrm{Aut}(K/L)}.$$

One immediately verifies:

$$L \subset L' \implies c(L) \subset c(L'),\ L \subset c(L),\ c(c(L)) = c(L);$$

these properties assert that $L \mapsto c(L)$ is a **closure operator** on the lattice $\mathcal{L}$ in the sense of order theory. Quite similarly, for $H \in \mathcal{H}$, we write

$$c(H) := \Phi(\Psi(H)) = \mathrm{Aut}(K/K^H)$$

and observe

$$H \subset H' \implies c(H) \subset c(H'),\ H \subset c(H),\ c(c(H)) = c(H),$$

so c is a closure operator on $\mathcal{H}$. A subextension L (resp. a subgroup H) is said to be **closed** if $L = c(L)$ (resp. $H = c(H)$). Let us write $\mathcal{L}_c$ (resp. $\mathcal{H}_c$) for the subset of closed subextensions (resp. closed subgroups). Then:

Theorem 1. *(Kaplansky's Galois Correspondence) The antitone maps*

$$\Phi : \mathcal{L}_c \to \mathcal{H}_c,\ \Psi : \mathcal{H}_c \to \mathcal{L}_c$$

are mutually inverse.

If Theorem 1 looks profound, it is only because we are reading into it some prior knowledge of Galois theory. In fact the result is, by itself, a triviality. In the next section we will give a significant generalization of this result – to an arbitrary **Galois connection** – and include the full details of the proof.

Example 1.1: Suppose $K = \mathbb{Q}(\sqrt[n]{2})$, $F = \mathbb{Q}$ and $n \geq 3$. Then $\mathrm{Aut}(K/F) = 1$. Then for any subextension L of $K/\mathbb{Q}$, we have $\mathrm{Aut}(K/L) = 1$, so $\overline{L} = K^{\mathrm{Aut}(K/L)} = K^{\{1\}} = K$. That is, the closure operator takes every intermediate field L to the top field K. Such a closure operator will be called **trivial**. And indeed it is: there is nothing interesting happening here.

Of course the closure operator on subextensions will be trivial whenever $\mathrm{Aut}(K/F)$ is the trivial group. There are lots of field extensions having this property, e.g. $\mathbb{R}/\mathbb{Q}$.

1.2. **Three flavors of Galois extensions.**

An extension K/F is **perfectly Galois** if every subextension L of K/F and every subgroup H of $\mathrm{Aut}(K/F)$ is closed.

An extension K/F is **Dedekind** if every subextension L of K/F is closed. An extension K/F is **quasi-Dedekind** if for every subextension K of K/F, $\overline{L}/L$ is purely inseparable.

An extension K/F is **Galois** if $K^{\mathrm{Aut}(K/F)} = F$. An extension K/F is **quasi-Galois** if $K^{\mathrm{Aut}(K/F)}/F$ is purely inseparable.

An extension K/F is **normal** if every irreducible polynomial $f(t) \in F[t]$ with a root in K splits completely in K. Normality only depends on the "algebraic part" of the extension in the following sense: K/F is normal iff the algebraic closure of F in K is normal over F.

Lemma 2. *If K/F is Galois, then $\mathrm{Cl}_K(F)/F$ is separable.*

Proposition 3. *Let K/F be an extension, and let $\mathrm{Cl}_K(F)$ be the algebraic closure of F in K. Then K/F is normal iff $\mathrm{Cl}_K(F)$ is normal.*

Theorem 4. *Let K/F be a field extension.*
a) If K/F is quasi-Galois, it is normal.
b) If K/F is normal and algebraic, it is quasi-Galois.
c) (T/F?) The class of Galois (resp. quasi-Galois, resp. normal) extensions is very distinguished.

1.3. **Galois theory for algebraic extensions.**

Theorem 5. *(Algebraic Galois Theory) Let K/F be an algebraic field extension.*
a) The following are equivalent:
(i) K/F is perfectly Galois.
(ii) K/F is Dedekind and of finite degree.
(iii) K/F is Galois and of finite degree.
(iv) $\# \mathrm{Aut}(K/F) = [K : F] < \aleph_0$.
(v) K/F is Galois and $\# \mathrm{Aut}(K/F) \leq [K : F]$.
b) The following are equivalent:
(i) K/F is Dedekind.
(ii) K/F is Galois.
(iii) K/F is normal and separable.
(iv) K/F is the splitting field of a set of separable polynomials.
When these equivalent conditions holds, then a subextension L is normal over K iff $\mathrm{Aut}(K/L)$ is normal in $\mathrm{Aut}(K/F)$, and if so $\mathrm{Aut}(K/F)$ is canonically isomorphic to $\mathrm{Aut}(K/F)/\mathrm{Aut}(K/L)$.
c) The following are equivalent:
(i) K/F is quasi-Galois.
(ii) K/F is normal.

Remark: Condition (v) of part a) perhaps looks strange. Its content is in fact the following: when K/F is infinite algebraic Galois, the infinite cardinal $\# \mathrm{Aut}(K/F)$ is greater than the infinite cardinal $[K : F]$.

Mention something about **Krull topology** here?

1.4. **Transcendental Extensions.**

Theorem 6. *For a field extension K/F, TFAE:*
(i) K/F is perfectly Galois.
(ii) K/F is finite and Galois.

One of the key features of algebraic Galois theory is that a Galois extension is automatically Dedekind. This is no longer the case for transcendental extensions: e.g., for every infinite field K, the pure transcendental extension $K(t)/K$ is Galois but not Dedekind. In fact the next result records all known Dedekind extensions.

Theorem 7. *Let K/F be a field extension.*
a) Suppose that either *K/F is algebraic and Galois or that K is algebraically closed of characteristic zero. Then K/F is Dedekind.*
b) Suppose that either *K/F is algebraic and quasi-Galois or that K is algebraically closed. Then K/F is quasi-Dedekind.*

2. Galois Connections

2.1. The basic formalism.

Let $(X, \leq)$ and $(Y, \leq)$ be partially ordered sets. A map $f : X \to Y$ is **isotone** (or **order-preserving**) if for all $x_1, x_2 \in X$, $x_1 \leq x_2 \implies f(x_1) \leq f(x_2)$. A map $f : X \to Y$ is **antitone** (or **order-reversing**) if for all $x_1, x_2 \in X$, $x_1 \leq x_2 \implies f(x_1) \geq f(x_2)$.

Let $(X, \leq)$ and $(Y, \leq)$ be partially ordered sets. An **antitone Galois connection between X and Y** is a pair of maps $\Phi : X \to Y$ and $\Psi : Y \to X$ such that:

(GC1) Φ and Ψ are both antitone maps, and
(GC2) For all $x \in X$ and all $y \in Y$, $x \leq \Psi(y) \iff y \leq \Phi(x)$.

Remark 2.1: There is a pleasant symmetry between X and Y in the definition. That is, if (Φ, Ψ) is a Galois connection between X and Y, then (Ψ, Φ) is a Galois connection between Y and X.

Definition: If $(X, \leq)$ is a partially ordered set, then a mapping $f : X \to X$ is called a **closure operator** if it satisfies all of the following properties:

(C1) For all $x \in X$, $x \leq f(x)$.
(C2) For all $x_1, x_2 \in X$, $x_1 \leq x_2 \implies f(x_1) \leq f(x_2)$.
(C3) For all $x \in X$, $f(f(x)) = f(x)$.

Proposition 8. *The mapping $\Psi \circ \Phi$ is a closure operator on $(X, \leq)$ and the mapping $\Phi \circ \Psi$ is a closure operator on $(Y, \leq)$.*

Proof. By symmetry, it is enough to consider the mapping $x \mapsto \Psi(\Phi(x))$ on X.

If $x_1 \leq x_2$, then since both Φ and Ψ are antitone, we have $\Phi(x_1) \geq \Phi(x_2)$ and thus $\Psi(\Phi(x_1)) \leq \Psi(\Phi(x_1))$: (C2).[1]

For $x \in X$, $\Phi(x) \geq \Phi(x)$, and by (GC2) this implies $x \leq \Psi(\Phi(x))$: (C1).

Finally, for $x \in X$, applying (C1) to the element $\Psi(\Phi(x))$ of X gives
$$\Psi(\Phi(x)) \leq \Psi(\Phi(\Psi(\Phi(x)))).$$
Conversely, we have
$$\Psi(\Phi(x)) \leq \Psi(\Phi(x)),$$
so by (GC2)
$$\Phi(\Psi(\Phi(x)) \geq \Phi(x),$$
and applying the order-reversing map Ψ gives
$$\Psi(\Phi(\Psi(\Phi(x)))) \leq \Psi(\Phi(x)).$$

[1] In other words, the composition of two antitone maps is isotone.

Thus

$$\Psi(\Phi(x)) = \Psi(\Phi(\Psi(\Phi(x)))).$$

□

Corollary 9. *The following* ***tridempotence*** *properties are satisfied by* Φ *and* Ψ*:*
a) For all $x \in X$*,* $\Phi\Psi\Phi x = \Phi x$*.*
b) For all $y \in X$*,* $\Psi\Phi\Psi y = \Psi y$*.*

Proof. By symmetry, it suffices to prove a). Since $\Phi \circ \Psi$ is a closure operator, $\Phi\Psi\Phi x \geq \Phi x$. Moreover, since $\Psi \circ \Phi$ is a closure operator, $\Psi\Phi x \geq x$, and since Φ is antitone, $\Phi\Psi\Phi x \leq \Phi x$. So $\Phi\Psi\Phi x = \Phi x$. □

Proposition 10. *Let* (Φ, Ψ) *be a Galois connection between partially ordered sets* X *and* Y*. Let* $\overline{X} = \Psi(\Phi(X))$ *and* $\overline{Y} = \Psi(\Phi(Y))$*.*
a) $\overline{X}$ *and* $\overline{Y}$ *are precisely the subsets of closed elements of* X *and* Y *respectively.*
b) We have $\Phi(X) \subset \overline{Y}$ *and* $\Psi(Y) \subset \overline{X}$*.*
c) $\Phi : \overline{X} \to \overline{Y}$ *and* $\Psi : \overline{Y} \to \overline{X}$ *are mutually inverse bijections.*

Proof. a) If $x = \Psi(\Phi(x))$ then $x \in \overline{X}$. Conversely, if $x \in \overline{X}$, then $x = \Psi(\Phi(x'))$ for some $x' \in X$, so $\Psi(\Phi(x))) = \Psi(\Phi(\Psi(\Phi(x')))) = \Psi(\Phi(x')) = x$, so X is closed.
b) This is just a reformulation of Corollary 9.
c) If $x \in \overline{X}$ and $y \in \overline{Y}$, then $\Psi(\Phi(x)) = x$ and $\Psi(\Phi(y)) = y$. □

We speak of the mutually inverse antitone bijections $\Phi : \overline{X} \to \overline{Y}$ and $\Psi : \overline{Y} \to \overline{X}$ as the **Galois correspondence** induced by the Galois connection (Φ, Ψ).

Example 2.2: Let K/F be a field extension, and G a subgroup of $\mathrm{Aut}(K/F)$. Then there is a Galois connection between the set of subextensions of K/F and the set of subgroups of G, given by

$$\Phi : L \to G_L = \{\sigma \in G \mid \sigma x = x \forall x \in L\},$$

$$\Psi : H \to K^H = \{x \in K \mid \sigma x = x \forall \sigma \in H\}.$$

Taking $G = \mathrm{Aut}(K/F)$, we recover Kaplansky's connection of §1. Thus the material just exposed serves as a proof of Theorem 1. For a general Galois connection and especially for Kaplansky's connection, the content lies in determining which elements are closed.

Having established the basic results, we will now generally abbreviate the closure operators $\Psi \circ \Phi$ and $\Phi \circ \Psi$ to $x \mapsto \overline{x}$ and $y \mapsto \overline{y}$.

2.2. **Lattice Properties.**

Recall that a partially ordered set X is a **lattice** if for all $x_1, x_2 \in X$, there is a greatest lower bound $x_1 \wedge x_2$ and a least upper bound $x_1 \vee x_2$. A partially ordered set is a **complete lattice** if for every subset A of X, the greatest lower bound $\bigwedge A$ and the least upper bound $\bigvee A$ both exist.

Lemma 11. *Let* (X, Y, Φ, Ψ) *be a Galois connection.*
a) If X *and* Y *are both lattices, then for all* $x_1, x_2 \in X$*,*

$$\Phi(x_1 \wedge x_2) = \Phi(x_1) \vee \Phi(x_2),$$

$$\Phi(x_2 \vee x_2) = \Phi(x_1) \wedge \Phi(x_2).$$

b) If X and Y are both complete lattices, then for all subsets $A \subset X$,

$$\Phi(\bigwedge A) = \bigvee \Phi(A),$$

$$\Phi(\bigvee A) = \bigwedge \Phi(A).$$

Proof. ... □

Complete lattices also intervene in this subject in the following way.

Proposition 12. *Let A be a set and let $X = (2^A, \subset)$ be the power set of A, partially ordered by inclusion. Let $c : X \to X$ be a closure operator. Then the collection $c(X)$ of closed subsets of A forms a complete lattice, with $\bigwedge S = \bigcap_{B \in S} B$ and $\bigvee S = c(\bigcup_{B \in S} B)$.*

2.3. **Examples.**

Example 2.3 (Indiscretion): Let $(X, \leq)$ and $(Y, \leq)$ be posets with top elements T_X and T_Y respectively. Define $\Phi : X \to Y$, $x \mapsto T_Y$ and $\Psi : Y \to X$, $y \mapsto T_X$. Then (X, Y, Φ, Ψ) is a Galois connection. The induced closure operators are "indiscrete": they send every element of X (resp. Y) to the top element T_X (resp. T_Y).

Example 2.4 (Perfection): Let $(X, \leq)$ and $(Y, \leq)$ be **anti-isomorphic posets**, i.e., suppose that there exists a bijection $\Phi : X \to Y$ with $x_1 \leq x_2 \iff \Phi(x_2) \leq \Phi(x_1)$. Then the inverse map $\Psi : Y \to X$ satisfies $y_1 \leq y_2 \iff \Psi(y_2) \leq \Psi(y_1)$. Moreover, for $x \in X$, $y \in Y$, $x \leq \Psi(y) \iff y = \Psi(\Phi(y)) \leq \Phi(x)$, so (X, Y, Φ, Ψ) is a Galois connection. Then $\overline{X} = X$ and $\overline{Y} = Y$. As we saw above, the converse also holds: if $\overline{X} = X$ and $\overline{Y} = Y$ then Φ and Ψ are mutually inverse bijections. Such a Galois connection is called **perfect**.[2]

Example 2.5: Let R be a commutative ring. Let X be the set of all ideals of R and $Y = 2^{\operatorname{Spec} R}$ the power set of the set of prime ideals of R. For $I \in X$, put

$$\Phi(I) = V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subset \mathfrak{p}\}.$$

For $V \in Y$, put

$$\Psi(V) = \bigcap_{\mathfrak{p} \in V} \mathfrak{p}.$$

The maps Φ and Ψ are antitone, and for $I \in \mathcal{X}$, $V \in \mathcal{Y}$,

$$I \subset \Psi(V) \iff I \subset \bigcap_{\mathfrak{p} \in V} \mathfrak{p} \iff \forall \mathfrak{p} \in V, I \subset \mathfrak{p} \iff V \subset \Phi(I), \tag{1}$$

so (Φ, Ψ) is a Galois connection. Then $\overline{X}$ consists of all ideals which can be written as the intersection of a family of prime ideals. For all $I \in X$,

$$\overline{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} = \operatorname{rad} I = \{x \in R \mid \exists n \in \mathbb{Z}^+ \; x^n \in I\};$$

[2]There is a small paradox here: in purely order-theoretic terms this example is not any more interesting than the previous one. But in practice given two posets it is infinitely more useful to have a pair of mutually inverse antitone maps running between them than the trivial operators of the previous example: Galois theory is a shining example! The paradox already shows up in the distinction between indiscrete spaces and discrete spaces: although neither topology looks more interesting than the other, the discrete topology is natural and useful (as we shall see...) whereas the indiscrete topology entirely deserves its alternate name "trivial".

that is, the induced closure operation on X takes any ideal to its **radical** $r(I)$. In particular $\overline{X}$ consists precisely of the radical ideals.

It is not so easy to describe the closure operator on Y or even the subset $\overline{Y}$ explicitly, but there is still something nice to say. Since:

$$V((0)) = \operatorname{Spec} R,\ V(R) = \varnothing, \tag{2}$$

$$V(I_1) \cup V(I_2) = V(I_1 I_2), \tag{3}$$

$$\bigcap_{\alpha \in A} V(I_\alpha) = V(\sum_{\alpha \in A} I_\alpha), \tag{4}$$

the elements of $\overline{Y}$ are the closed subsets for a topology, the **Zariski topology**.

Example 2.6: Take R and X as above, but now let S be any set of ideals of R and put $Y = 2^S$. For $I \in X$, put

$$\Phi(I) = V(I) = \{\mathfrak{s} \in S \mid I \subset \mathfrak{s}\}$$

and for $V \in \mathcal{Y}$, put

$$\Psi(V) = \bigcap_{\mathfrak{s} \in V} \mathfrak{s}.$$

Once again Φ and Ψ are antitone maps and (1) holds, so we get a Galois connection. The associated closure operation on X is

$$I \mapsto \overline{I} = \bigcap_{\mathfrak{s} \in S} \mathfrak{s}.$$

The relation (4) holds for any S, and the relation (2) holds so long as $R \notin S$. The verification of (2) for $R = \operatorname{Spec} R$ uses the fact that a prime ideal $\mathfrak{p}$ contains $I_1 I_2$ iff it contains I_1 or I_2, so as long as $S \subset \operatorname{Spec} S$, $\overline{Y} = \{V(I) \mid I \in X\}$ are the closed subsets for a topology on S. This is indeed the topology S inherits as a subspace of $\operatorname{Spec} R$, so we call it the **(relative) Zariski topology**.

Various particular choices of $S \subset \operatorname{Spec} R$ have been considered. Of these the most important is certainly $S = \operatorname{MaxSpec} R$, the set of all maximal ideals of R. In this case, $\overline{X}$ consists of all ideals which can be written as the intersection of some family of maximal ideals. Such ideals are necessarily radical, but in a general ring not all radical ideals are obtained in this way. Observe that in a general ring every radical ideal is the intersection of the maximal ideals containing it iff every prime ideal is the intersection of maximal ideals containing it; a ring satisfying these equivalent conditions is called a **Jacobson ring**.

Example 2.7: Let k be a field and put $R = k[t_1, \ldots, t_n]$. Then R is a Jacobson ring. To prove this one needs as prerequisite knowledge **Zariski's Lemma** – for every $\mathfrak{m} \in \operatorname{MaxSpec} R$, the field extension $R/\mathfrak{m}/k$ is finite – and the proof uses a short but clever argument: the **Rabinowitsch trick**.

Suppose that k is algebraically closed. Then Zariski's Lemma assumes a stronger form: for all $\mathfrak{m} \in \operatorname{MaxSpec} R$, the k-algebra $R/\mathfrak{m}$ is equal to k. Let $q : R \to R/\mathfrak{m} = k$ be the quotient map, and for $1 \leq i \leq n$, put $x_i = q(t_i)$ and $x = (x_1, \ldots, x_n)$. It follows that $\mathfrak{m}$ contains the ideal $\mathfrak{m}_x = \langle t_1 - x_1, \ldots, t_n - x_n \rangle$, and since $\mathfrak{m}_x$ is maximal, $\mathfrak{m} = \mathfrak{m}_x$. This gives the following description of the Galois connection between the set X of ideals of R and $Y = 2^{\operatorname{MaxSpec} R}$, **Hilbert's Nullstellensatz**:

(i) Maximal ideals of R are canonically in bijection with n-tuples of points of k,

i.e., with points of **affine n-space** $\mathbb{A}^n_{/k}$.
(ii) The closure operation on ideals takes I to its radical ideal $\operatorname{rad} I$.
(iii) The closure operation on subsets of $\mathbb{A}^n$ coincides with topological closure with respect to the Zariski topology, i.e., the topology on $\mathbb{A}^n$ for which the closed subsets are the intersections of the zero sets of polynomial functions.

Example 2.8: Let K be a field, let $X = 2^K$, let $\operatorname{RSpec} K$ be the set of orderings on K, and let $Y = 2^{\operatorname{RSpec} K}$. Let $H : X \to Y$ by

$$S \mapsto H(S) = \{P \in \operatorname{RSpec} K \mid \forall x \in S \; x >_P 0\}.$$

Let $\Psi : Y \to X$ by

$$T \mapsto \Psi(T) = \{x \in \operatorname{RSpec} K \mid \forall P \in T \; x >_P 0\}.$$

Then (X, Y, H, Ψ) is a Galois connection.

The set $\operatorname{RSpec} K$ carries a natural topology. Namely, we may view any ordering P as an element of $\{\pm 1\}^{K^\times}$: $P : x \in K^\times \mapsto +1$ if $P(x) > 0$ and -1 is $P(x) < 0$. Giving $\{\pm 1\}$ the discrete topology and $\{\pm 1\}^{K^\times}$, it is a compact (by Tychonoff's Theorem) zero-dimensional space. It is easy to see that $\operatorname{RSpec} K$ embeds in $\{\pm 1\}^{K^\times}$ as a closed subspace, and therefore $\operatorname{RSpec} K$ is itself compact and zero-dimensional.

Example 2.9: Let $\mathcal{L}$ be a language, let X be the set of $\mathcal{L}$-theories, and let Y be the class of all classes $\mathcal{C}$ of $\mathcal{L}$-structures, partially ordered by inclusion.[3] For a theory $\mathcal{T}$, let $\Phi(\mathcal{T}) = \mathcal{C}_{\mathcal{T}}$ be the class of all models of $\mathcal{T}$, whereas for a class $\mathcal{C}$, we define $\Psi(\mathcal{C})$ to be the collection of all sentences φ such that for all $X \in \mathcal{C}$, $X \models \varphi$.

2.4. Galois Connections Decorticated (Relations).

Example 2.10: Let S and T be sets, and let $R \subset S \times T$ be a **relation** between S and T. As is traditional, we use the notation xRy for $(x, y) \in R$. For $A \subset S$ and $y \in T$, we let us write ARy if xRy for all $x \in A$; and dually, for $x \in S$ and $B \subset T$, let us write xRB if xRy for all $y \in B$. Finally, for $A \subset S$, $B \subset T$, let us write ARB if xRy for all $x \in A$ and all $y \in B$.

Let $X = (2^S, \subset)$, $Y = (2^T, \subset)$. For $A \subset S$ and $B \subset T$, we put

$$\Phi_R(A) = \{y \in T \mid ARy\},$$
$$\Psi_R(B) = \{x \in S \mid xRB\}.$$

We claim that $\mathcal{G}_R = (X, Y, \Phi_R, \Psi_R)$ is a Galois connection. Indeed, it is immediate that Φ_R and Ψ_R are both antitone maps; moreover, for all $A \subset S$, $B \subset T$ we have

$$A \subset \Psi_R(B) \iff ARB \iff B \subset \Phi_R(A).$$

Remarkably, this example includes most of the Galois connections above. Indeed:

- In Example 2.2, take X to be 2^K and $Y = 2^{\operatorname{Aut}(K/F)}$. The induced Galois connection is the one associated to the relation $gx = x$ on $K \times \operatorname{Aut}(K/F)$.
- In Example 2.5, take X to be 2^R. The induced Galois connection is the one associated to the relation $x \in \mathfrak{p}$ on $R \times \operatorname{Spec} R$. Similarly for Examples 2.7 and 2.8.
- The Galois connection of Example 2.8 is the one associated to the relation $x \in P$

[3]Here we are cheating a bit by taking instead of a partially ordered set, a *partially ordered class*. We leave it to the interested reader to devise a remedy.

on $K \times \mathrm{RSpec}\, K$.
• The Galois connection of Example 2.9 is the one associated to the relation $X \models \varphi$.

Theorem 13. *Let S and T be sets, let $X = (2^S, \subset)$, $Y = (2^S, \subset)$, and let $\mathcal{G} = (X, Y, \Phi, \Psi)$ be any Galois connection. Define a relation $R \subset S \times T$ by xRy if $y \in \Phi(\{x\})$. Then $\mathcal{G} = \mathcal{G}_R$.*

Proof. Note first that X and Y are complete lattices, so Lemma 11b) applies. Indeed, for $A \subset S$, $A = \bigcup_{x \in A}\{x\} = \bigvee_{x\in A}\{x\}$, so

$$\Phi(A) = \bigcap_{x \in A} \Phi(\{x\}) = \bigcap_{x\in A} \{y \in T \mid xRy\} = \{y \in T \mid ARy\} = \Phi_R(A).$$

Moreover, since $\mathcal{G}$ is a Galois connection we have $\{x\} \subset \Psi(\{y\}) \iff \{y\} \subset \Phi(\{x\}) \iff xRy$. Thus for $B \subset T$, $B = \bigcup_{y\in B}\{y\} = \bigvee_{y \in B}\{y\}$, so

$$\Psi(B) = \bigcap_{y \in B} \Psi(\{y\}) = \bigcap_{y\in A} \{x \in S \mid xRy\} = \{x \in S \mid xRB\} = \Psi_R(B).$$

□

For any partially ordered set $(X, \leq)$, a **downset** is a subset $Y \subset X$ such that for all $x_1, x_2 \in X$, if $x_2 \in Y$ and $x_1 \leq x_2$ then $x_1 \in Y$. Let $D(X)$ be the collection of all downsets of X, viewed as a subset of $(2^X, \subset)$. To each $x \in X$ we may associate the **principal downset** $d(x) = \{y \in X \mid y \leq x\}$. The map $d : X \to D(X)$ is an order embedding; composing this with the inclusion $D(X) \subset 2^X$ we see that every partially ordered set embeds into a power set lattice.

Let $\mathcal{G} = (X, Y, \Phi, \Psi)$ be a Galois connection with X and Y complete lattices. Then we may extend $\mathcal{G}$ to a Galois conection between 2^X and 2^Y as follows: for $A \subset X$, put $\Phi(A) = \bigwedge\{\Phi(x)\}_{x\in A}$, and simialrly for $B \subset Y$, put $\Psi(B) = \bigwedge\{\Psi(y)\}_{y \in B}$. Thus every Galois connection between complete lattices may be viewed as the Galois connection induced by a relation between sets.

2.5. **Indexed Galois Connections.**

Let $\omega^+ = \mathbb{Z}^+ \cup \{\infty\}$ be the extended positive integers, i.e., the usual positive integers together with a distinct element ∞, with the order structure, addition and multiplication extended as follows: $\forall x \in \omega^+$, $x \leq \infty$, $x + \infty = \infty$, $x \cdot \infty = \infty$.

Let $(X, \leq)$ be a poset. An **index structure** on X is the assignment to each pair $(x_1, x_2) \in X^2$ with $x_1 \leq x_2$ an **index** $[x_2 : x_1] \in \omega^+$ satisfying both of the following:

(IP1) (Multiplicativity) For all $x_1 \leq x_2 \leq x_3 \in X$ we have

$$[x_3 : x_1] = [x_3 : x_2][x_2 : x_1].$$

(IP2) (Equality by Degree) For all $x_1 \leq x_2 \in X$, we have

$$[x_2 : x_1] = 1 \iff x_1 = x_2.$$

An index structure is **finite** if it takes values in $\mathbb{Z}^+$.

Example 2.11: An index structure on $(\mathbb{Z}, \leq)$ is completely and freely determined by the assignment of a value in ω^+ to each $[n + 1 : n]$. In particular $(\mathbb{Z}, \leq)$ admits $\mathfrak{c} = 2^{\aleph_0}$ finite index structures.

Proposition 14. *Let $(X \leq)$ be a poset.*
a) The assignment $[x:x] = 1$, $[x_1 : x_2] = \infty$ for all $x_1 < x_2$ is an index structure on X, which we will denote by $\mathbb{I}_\infty$.
b) If $(X, \leq)$ is dense – i.e., for all $x_1 < x_2 \in X$, there exists x_3 with $x_1 < x_3 < x_2$ – it admits no index structure other than $\mathbb{I}_\infty$.
c) If $(X, \leq)$ has top and bottom elements and admits a finite index structure then it has finite height: the supremum of all lengths of chains in X is bounded.
d) If $(X, \leq)$ has finite height and is catenary *– i.e., for all $x \leq y$, any two maximal chains from x to y have the same length – then it admits a finite index structure, e.g. $[x : B] = 2^{\text{height}\, x}$.*
e) If $\iota : (X, \leq) \to (Y, \leq)$ is an isotone injection and $\mathbb{I}$ is an index structure on Y, then the $\mathbb{I}|_X$ is an index structure on X.
f) Every finite poset admits a finite index structure.

Proof. Parts a) through e) are left to the reader. f) Let $d : X \to (2^X, \subseteq)$, $x \mapsto d(x) = \{y \in X \mid y \leq x\}$. This is an embedding from X into a catenary poset of finite height, hence by parts d) and e) X admits a finite index structure. $\square$

Example 2.12: Let K/F be a field extension, and $\mathcal{L}$ be the lattice of subextensions of K/F. For $F \subset L_1 \subset L_2 \subset K$, defining $[L_2 : L_1]$ to be the dimension of L_2 as an L_1 vector space if this dimension is finite and ∞ if this dimension is infinite is an index structure on $\mathcal{L}$. It is a finite index structure iff K/F is a finite extension. (Note though that for finite L/K the lattice $\mathcal{L}$ need not be finite, although by the Primitive Element theorem $\mathcal{L}$ will be finite if L/K is finite separable.)

Example 2.13: Let G be a group, and let $\mathcal{H}$ be the lattice of subgroups of G. For $\{1\} \subset H_1 \subset H_2 \subset G$, let $[H_2 : H_1]$ be the number of cosets of H_1 in H_2 if this number is finite and ∞ if this number is infinite. This index structure is finite iff G is finite iff the lattice $\mathcal{H}$ is finite.

Let $\mathcal{G} = (X, Y, \Phi, \Psi)$ be a Galois connection. An **index structure** $\mathcal{I} = (\mathbb{I}_X, \mathbb{I}_Y)$ on $\mathcal{G}$ is a pair of index structures $\mathbb{I}_X$ on X and $\mathbb{I}_Y$ on Y satisfying

(IGCX) For all $x_1 \leq x_2 \in X$, $[\Phi(x_1) : \Phi(x_2)] \leq [x_2 : x_1]$, and
(IGCY) For all $y_1 \leq y_2 \in Y$, $[\Psi(y_1) : \Psi(y_2)] \leq [y_2 : y_1]$.

An **indexed Galois connection** $(\mathcal{G}, \mathbb{I})$ is a Galois connection endowed with an index structure. An indexed Galois connection is **finite** if both $\mathbb{I}_X$ and $\mathbb{I}_Y$ are finite. The notions of an indexed Galois connection and finite indexed Galois conncetion are symmetric in X and Y: if $(X, Y, \Phi, \Psi, \mathbb{I}_X, \mathbb{I}_Y$ is an indexed Galois connection (resp. finite indexed Galois connection) so is $(Y, X, \Psi, \Phi, \mathbb{I}_Y, \mathbb{I}_X)$.

Example 2.14: Let $\pi : Y \to X$ be a covering map. The associated Galois connection $\mathcal{G}$ has a natural index structure: if $H_1 \leq H_2 \leq \mathrm{Aut}(\pi)$, let $[H_2 : H_1]$ be the index of H_1 in H_2, and for subcovers $Y \to Z_2 \to Z_1 \to X$ we let $[Z_2 : Z_1]$ be the degree of $Z_2 \to Z_1$. Here $(\mathcal{G}, \mathbb{I})$ is finite iff X and Y are both finite.

Theorem 15. *(Kaplansky-Roman)*
Let $(\mathcal{G}, \mathbb{I})$ be an indexed Galois connection, and let $x_1 \leq x_2 \in X$.
a) If $x_1, x_2 \in \overline{X}$, then $[\Phi(x_1) : \Phi(x_2)] = [x_2 : x_1]$.

b) If $x_1 \in \overline{X}$ and $[x_2 : x_1] < \infty$, then $x_2 \in \overline{X}$.
c) Suppose $\overline{x_1} \leq x_2$, $[x_2 : \overline{x_1}] < \infty$ and $[x_2 : x_1] = [\Phi(x_1) : \Phi(x_2)]$. Then $x_1 = \overline{x_1}$.
d) In particular, if $[T : x] = [\overline{x} : B] < \infty$, then $x = \overline{x}$.

Proof. a) We have

$$[x_2 : x_1] \geq [\Phi(x_1) : \Phi(x_2)] \geq [\Psi(\Phi(x_2)) : \Psi(\Phi(x_1))] = [\overline{x_2} : \overline{x_1}] = [x_2 : x_1].$$

b) We have

$$[x_2 : x_1] \geq [\Phi(x_1) : \Phi(x_2)] \geq [\overline{x_2} : x_1] = [\overline{x_2} : x_2][x_2 : x_1].$$

Since $[x_2 : x_1] < \infty$ we may cancel, getting $[\overline{x_2} : x_2] = 1$. Now apply (IGX).
c) By part b), $\overline{x_2} = x_2$, so by tridempotence

$$[x_2 : \overline{x_1}] = [\Phi(x_1) : \Phi(x_2)] = [x_2 : x_1] = [x_2 : \overline{x_1}][\overline{x_1} : x_1].$$

Cancelling $[x_2 : \overline{x_1}]$ and applying (IGX) we get $x_1 = \overline{x_1}$.
d) If $[T : x] < \infty$, then $[T : \overline{x}] < \infty$, and we apply part c) with $x_1 = x$, $x_2 = T$. □

Corollary 16. *Let $(\mathcal{G}, \mathbb{I})$ be a finite indexed Galois connection possessing top and bottom elements. TFAE:*
(i) The Galois connection is perfect.
(ii) The bottom elements of X and Y are closed.

3. Galois Theory of Group Actions

3.1. **Basic Setup.**

Let X be a set and G be a group acting on X. The maps:

$$\Phi : Y \subset X \mapsto G_Y := \{\sigma \in G \mid \forall y \in Y \ \sigma y = y\},$$
$$\Psi : H \subset G \mapsto X^G := \{y \in X \mid \forall \sigma \in H \ \sigma y = y\}$$

give a Galois connection between the complete lattice 2^X of all subsets of X and the complete lattice of all subgroups of G. Indeed, extending its codomain to 2^G, it is the Galois connection associated to the "fixing relation" $\{(x, \sigma) \in X \times G \mid \sigma x = x\}$ between X and G.

3.2. **Normality and Stability.**

Proposition 17. *Let H be a subgroup of G, Y a subset of X and $\sigma \in G$. We have:*
a) $\sigma G_Y \sigma^{-1} = G_{\sigma Y}$.
b) $\sigma X^H = X^{\sigma H \sigma^{-1}}$.

Proof. We have $g \in G_{\sigma Y} \iff \forall y \in Y,\ g\sigma y = \sigma y \iff \forall y \in Y,\ \sigma^{-1} g \sigma y = y \iff \sigma^{-1} g \sigma \in G_Y \iff g \in \sigma G_Y \sigma^{-1}$. Similarly, $y \in \sigma X^H \iff \sigma^{-1} y \in X^H \iff \forall h \in h,\ h\sigma^{-1} y = \sigma^{-1} y \iff \forall h \in H,\ (\sigma h \sigma^{-1}) y = y \iff y \in \sigma H \sigma^{-1}$. □

Corollary 18.
a) If $Y \subset X$ is Galois-closed and $\sigma \in G$, then σY is also Galois-closed.
b) If $H \subset G$ is Galois-closed and $\sigma \in G$, then $\sigma H \sigma^{-1}$ is also Galois-closed.

Proof. a) By Proposition 17, $\sigma Y = \sigma X^{G_Y} = X^{\sigma G_Y \sigma^{-1}} = X^{G_{\sigma Y}}$.
b) By Proposition 17, $\sigma H \sigma^{-1} = \sigma G_{X^H} \sigma^{-1} = G_{\sigma X^H}$. □

We say a subset $Y \subset X$ is **stable** if for all $g \in G$, $gY \subset Y$. Since G acts by bijections, stability is equivalent to $gY = Y$ for all $g \in G$.

Corollary 19.
a) A Galois-closed subgroup H of G is normal iff X^H is stable.
b) A Galois-closed subset Y of X is stable iff G_Y is normal in G.
c) The Galois-closure of a normal subgroup is normal.
d) The Galois-closure of a stable subset is stable.

Proof. a) Put $Y = X^H$, so $H = G_Y$. Suppose H is normal. Then for all $g \in G$,

$$G_{X^H} = gG_{X^H}g^{-1} = G_{gX^H}.$$

By Corollary 18, gX^H is Galois-closed, so

$$X^H = X^{G_{X^H}} = X^{G_{gX^H}} = gX^H.$$

The remaining parts of a) and b) are quite similar and left to the reader.
c) Suppose H is a normal subgroup of G. Then for all $g \in G$,

$$gG_{X^H}g^{-1} = G_{gX^H} = G_{X^{gHg^{-1}}} = G_{X^H}.$$

d) Suppose Y is a stable subset of X. Then for all $g \in G$,

$$gX^{G_Y} = X^{gG_Yg^{-1}} = X^{G_{gY}} = X^{G_Y}.$$

□

3.3. **The $\mathcal{J}$-topology and the $\mathcal{K}$-topology.**

Let G be a group acting effectively on a set K. For $x \in K$, we define

$$G_x = \{\sigma \in G \mid \sigma x = x\}$$

and

$$U_x = G \setminus G_x = \{\sigma \in G \mid \sigma x \neq x\}.$$

The **$\mathcal{J}$-topology** on G is the topology with subbase of open sets given by all sets of the form $\sigma_1 U_x \sigma_2$ for $\sigma_1, \sigma_2 \in G$ and $x \in K$.

Proposition 20. *Let G be a group acting effectively on a set K, and consider the topological space $(G, \mathcal{J})$.*
a) For all $\sigma \in G$, the translation maps $L_\sigma : G \to G$ by $\tau \mapsto \sigma\tau$ and $R_\sigma : G \to G$ by $\tau \mapsto \tau\sigma$ are homeomophisms of G, as is the inversion map $\iota : G \to G$ by $\sigma \mapsto \sigma^{-1}$.
b) G is a separated (or "T_1") topological space: points are closed.
c) G has a subbase of open sets of the form $\{\sigma U_x\}_{\sigma \in G, x \in K}$.
d) Every Galois-closed subgroup H of G is $\mathcal{J}$-closed.

Proof. a) ...
b) For all $x \in K$, the set G_x is closed, hence so is $\bigcap_{x \in K} G_x = G_K = \{1\}$. Since translations are homeomorphisms, this implies that all points are closed.
c) As in [SV71]...
d) ... □

We define the **$\mathcal{K}$-topology** on G as follows: first, we endow K with the discrete topology, and then give the set K^K of all maps from K to K the product topology. Finally, we give $G \subset K^K$ the subspace topology. More explicitly, a subbase for the $\mathcal{K}$-topology on G is given by all sets of the form

$$V_{x,y} = \{\sigma \in G \mid \sigma(x) = y\}.$$

A topological space is **zero-dimensional** if it is Kolmogorov ("T_0") and admits a base of clopen sets; such spaces are necessarily Hausdorff.

Proposition 21. *Let G be a group acting effectively on a set K, and consider the topological space $(G, \mathcal{K})$.*
a) G is a topological group and a zero-dimensional space.
b) The $\mathcal{K}$-topology is finer than the $\mathcal{J}$-topology.
c) Every Galois-closed subgroup is $\mathcal{K}$-closed.
d) The $\mathcal{K}$-topology is the coarsest topology on G such that the action of G on K is continuous for the discrete topology on K.

Proof. a) This is routine, and we leave it to the reader.
b) For fixed x, y,

$$V_{x,y} = G \setminus \bigcup_{x' \neq x} V_{x',y},$$

so each $V_{x,y}$ is $\mathcal{K}$-closed as well as $\mathcal{K}$-open. In particular $V_{x,x} = G_x$ is $\mathcal{K}$-closed, so its complement U_x is $\mathcal{K}$-open. Since $(G, \mathcal{K})$ is a topological group, translations are homeomorphisms, hence for all $x \in K$ and all$\sigma_1, \sigma_2 \in G$, $\sigma_1 U_x \sigma_2$ is $\mathcal{K}$-open.
c) This follows from part b) and Proposition 20d).
d) If K is discrete, the action $G \times K \to K$ is continuous iff for all $\sigma \in G,\ x, y \in K$ such that $\sigma x = y$, there exists a neighborhood N of σ such that $Nx = y$ iff $V_{x,y}$ is a neighborhood of σ. □

A **Cantor cube** is a topological group of the form 2^A for some A, i.e., a product of copies of the discrete group of order 2.

Theorem 22.
a) Subspaces, products and inverse limits of zero-dimensional spaces are zero-dimensional.
b) A zero-dimensional space may be embedded in a Cantor cube.
c) A topological space is zero-dimensional compact iff it is homeomorphic to a closed subspace of some Cantor cube.
d) An infinite topological group is zero-dimensional compact iff it is homeomorphic to some Cantor cube.

Proof. a) Left to the reader.
b) [E, Thm. 6.2.16].
c) Since 2^A is compact, a subspace of 2^A is compact iff it is closed.
d) More generally: let G be a topological group with infinitely many connected components, and let G_0 be its identity component. By [HM, Thm. 10.40], G is homeomorphic to $G_0 \times 2^A$ for some index set A. □

The action of G on K is **locally finite** if for all $x \in K$, the G-orbit of x is finite; by the Orbit-Stabilizer Theorem, this condition is equivalent to all the point stabilizers G_x having finite index in G.

Theorem 23. *Let G be a group acting locally finitely on a set K.*
a) We have $\mathcal{J} = \mathcal{K}$.
b) A subgroup of G is $\mathcal{K}$-open iff it contains G_S for some finite subset $S \subset K$.
c) The following are equivalent:
(i) Every $\mathcal{K}$-open subroup is of the form G_S for some finite subset $S \subset K$.
(ii) A subgroup is $\mathcal{K}$-closed iff it is Galois-closed.

Proof. a) Suppose the action is locally finite, fix $x \in K$, and let

$$x_1 = x, x_2 = \sigma_2 x, \ldots, x_n = \sigma_n x$$

be the G-orbit of x. Then $V_{x,y} = \varnothing$ if $y \neq x_i$ for some i, and

$$V_{x,x_i} = \bigcap_{1 \leq j \leq n,\ j \neq i} \sigma_j U_x.$$

b) By definition of the $\mathcal{K}$-topology G_S is open for all finite S. Moreover a subgroup of any topological group containing an open subgroup is itself open. Conversely, every $\mathcal{K}$-subbasis element $V_{x,y}$ is a coset of a subgroup G_x. Let H be a $\mathcal{K}$-open subgroup of G. Then G is a union of finite intersections of subbase elements, so G contains at least one finite intersection of subbase elements containing $\{1\}$. But a finite intersection of subbase elements contains 1 iff every one of the subbase elements contains 1 iff every subbase element is of the form G_x for some $x \in K$, and thus H contains $\bigcap_{i=1}^n G_{x_i} = G_{\{x_1,\ldots,x_n\}} = G_S$.
c) ... □

Proposition 24. *If $(G, \mathcal{K})$ is compact, then the action is locally finite.*

Proof. Fix $x \in K$. Then $\{V_{x,y}\}_{y \in K}$ is a covering of G by pairwise disjoint $\mathcal{K}$-open subsets. By compactness, we must have $V_{x,y} = \varnothing$ for all but finitely many y. But $\{y \in K \mid V_{x,y} \neq \varnothing\}$ is the G-orbit of x. □

The converse of Proposition 24 does not hold in this level of generality. For instance, a countably infinite group admits no compact group topology but may well act locally finitely on a set. For a specific example, $\mathbb{Z}$ acts locally finitely on $\overline{\mathbb{F}_p}$ by $(n, x) \mapsto x^{p^n}$. The issue here is that $\mathbb{Z}$ is not the *full* automorphism group of $\overline{\mathbb{F}_p}/\mathbb{F}_p$. This suggests a remedy in our general context, as follows.

We say that an action of G on K is **finite-complete** if when given for each finite G-stable set $S \subset K$ a permutation s_S of S which is the restriction to S of the action of some $\sigma_S \in G$ compatibly with each other in the sense that when $S' \supset S$, $(s_{S'})_S = s_S$, there exists $\sigma \in G$ such that for all S, $\sigma|_S = s_S$.

Let $\mathcal{F}$ be the family of finite G-stable subsets of K. Put

$$\hat{G} = \varprojlim_{S \in \mathcal{F}} G/G_S$$

and let $\iota : G \to \hat{G}$ denote the natural map, called the **profinite completion of G**. (Note that this is the profinite completion of G with respect to the $\mathcal{K}$-topology, not necessarily its profinite completion as an abstract group.)

For any group G acting effectively on a set K, there is a unique maximal subset on which G acts locally finitely, namely the union of all finite G-orbits. We denote this subset by K_f. Thus the action of G on K is locally finite iff $K_f = K$.

Theorem 25. *Let G be a group acting effectively on a set K with profinite completion $\iota : G \to \hat{G}$.*
a) If the action is locally finite, ι is injective.
b) The action is finite-complete iff ι is surjective.
c) The following are equivalent:

(i) The map ι is an isomorphism of topological groups.
(ii) The action is locally finite and finite-complete.
(iii) $(G, \mathcal{K})$ is compact.

Proof. a) We have

$$\ker \iota = \bigcap_{S \in \mathcal{F}} G_S = G_{\bigcup_{S \in \mathcal{F}} S} = G_{K_f} = G_K = \{1\}.$$

b) This is immediate from the definition.
c) (i) $\implies$ (ii): If ι is an isomorphism, then the action is finite-complete. Moreover $(G, \mathcal{K})$ is compact, so the action is locally finite by Proposition XX.
(ii) $\implies$ (iii): By part a), ι is injective and thus an isomorphism, so $(G, \mathcal{K}) \cong \hat{G}$ is compact.
(iii) $\implies$ (i): $(G, \mathcal{K})$ is always zero-dimensional, so if it is compact it is profinite and then ι is an isomorphism. □

Remark: There are non-locally finite actions with injective ι. Indeed, let G be any infinite group acting locally finitely on a set K, so that $G_{K_f} = G_K = \{1\}$. Now extend the action to $K \coprod G$ by letting G act on itself by translation.

Corollary 26. *Let $\iota : G \to \hat{G}$ be the profinite completion of $(G, \mathcal{K})$, and put $N = \ker \iota$. Then $\operatorname{Aut}_G K_f = \hat{G}$, and G/N acts effectively on K_f as a dense subgroup of $\hat{G}$.*

Example: Suppose that G acts *freely* on a set K, i.e., $G_x = \{1\}$ for all $x \in K$. Then every $\mathcal{J}$-closed set is a (possibly empty) intersection of finite unions of translates of the identity subgroup, so the proper $\mathcal{J}$-closed sets are precisely the finite sets: $\mathcal{J}$ is the cofinite topology on G (the coarsest separated topology). On the other hand G_x is $\mathcal{K}$-open, so the $\mathcal{K}$-topology on G is discrete. Thus $\mathcal{J} = \mathcal{K}$ iff G is finite iff (by the Orbit-Stabilizer Theorem) the action is locally finite. When G is infinite, $(G, \mathcal{J})$ is separated but not Hausdorff so is not a group topology.

The associated Galois-closure on subgroups is as follows: $\overline{\{1\}} = G_{K^{\{1\}}} = G_K = \{1\}$, and for any nontrivial subgroup H, $G_{K^H} = G_\varnothing = G$. For no nontrivial G does this coincide with the $\mathcal{K}$-closure; it coincides with the $\mathcal{J}$-closure iff G has no nontrivial elements of finite order.

Finally, note that when G is finite and noncyclic, the equivalent conditions of Theorem 23c) *do not* hold.

4. Return to the Galois Correspondence for Field Extensions

4.1. The Artinian Perspective.

Artin's approach to Galois theory is to start not with a field extension K/F but with a field K and a group G of field automorphisms of K. Then we can "recover" F as $F = K^G$, but something is gained: K/F is now necessarily Galois.

In this section we generalize the Artinian perspective to the context of a topological group G acting effectively by automorphisms on a field K.

One further piece of terminology will be helpful. Throughout this paper the only topology we will ever consider on a field is the discrete topology. So let us agree that when we say "the action of a topological group (G,τ) on a field K is topological," we mean that the action is continuous when K is given the discrete topology.

Theorem 27. *Let K be a field, and let (G,τ) be a topological group which acts effectively on K by field automorphisms, with fixed field F. Consider the induced embedding $\iota : G \hookrightarrow \operatorname{Aut}(K/F)$.*
a) The following are equivalent:
(i) The action of G on K is locally finite.
(ii) The extension K/F is algebraic.
b) The following are equivalent:
(i) The map $\iota : (G,\tau) \to (\operatorname{Aut}(K/F),\mathcal{K})$ is continuous.
(ii) The action of (G,τ) on K is topological.
c) Suppose (G,τ) is compact. TFAE:
(i) The action of (G,τ) on K is topological.
(ii) K/F is algebraic and ι is an isomorphism of topological groups.

Proof. a) (i) $\implies$ (ii): Since the action if locally finite, the point stabilizers G_x have finite index in G. For any finite subset $S \subset K$, put $G_S = \bigcap_{x\in S} G_x$, and let $N_S = \bigcap_{g\in G} gH_Sg^{-1}$ be the normal core of G_S, so that N_S is a finite index normal subgroup of G and every element of N_S leaves S pointwise fixed. Let L_S be the subextension of K/F obtained by adjoining to F all elements gx for $g \in G$ and $x \in S$. Then N_S is the subgroup of G which acts trivially on L_S, so the finite group G/N_S acts faithfully on L_S. By the finite Artin Theorem, L_S/F is a finite Galois extension with Galois group G/N_S. But $K = \varinjlim_S L_S$, so K/F is algebraic.
(ii) $\implies$ (i): For $x \in K$, let $p(t) \in K[t]$ be the minimal polynomial for x. Elements of the G-orbit of x are roots of $p(t)$, so the orbit is finite.
b) The map τ is continuous iff (G,τ) is finer than $(\iota(G),\mathcal{K})$. The equivalence of (i) and (ii) now follows from Proposition 21d).
c) (i) $\implies$ (ii): Since (G,τ) is compact, the open subgroups G_x have finite index, and therefore the action is locally finite. By part a), K/K^F is algebraic. Moreover, by part b) the homomorphism $\iota : (G,\tau) \to (\iota(G),\mathcal{K})$ is a continuous bijection from a compact space to a Hausdorff space, so it is an isomorphism of topological groups.
(ii) $\implies$ (i): If K/K^G is algebraic, then by part a) G acts locally finitely on K, so by Theorem XX $(G,\tau) = (\iota(G),\mathcal{K})$ is profinite and acts topologically on K. $\square$

We immediately deduce the following result of W.C. Waterhouse [Wa74, Thm. 1].

Corollary 28. *Let K be a field, and let (G,τ) be a profinite group acting effectively on K by field automorphisms, and such that for all $x \in K$ the point stabilizer G_x is open in (G,τ). Then K/K^G is algebraic Galois, and the canonical map $\iota : G \to \operatorname{Aut}(K/K^G)$ is an isomorphism of topological groups.*

Theorem 29. *(Leptin* [Le55]*, Waterhouse* [Wa74]*) For every profinite group G there is an algebraic Galois extension K/F and an isomorphism of topological groups $G \cong \operatorname{Aut}(K/F)$.*

Proof. ([Wa74]) Let T be the disjoint union of the coset spaces G/H as H ranges over the open subgroups of G. Let k be any field. View T as a set of independent indeterminates and put $K = k(T)$. There is a unique G-action on K extending the action on T and leaving k pointwise fixed. It is faithful, locally finite, with open stabilizers, and G is compact, so by Corollary 28, $G \cong \operatorname{Aut}(K/K^G)$. $\square$

4.2. **The Index Calculus.**

Theorem 30.
Let K/F be a field extension, and let G be a subgroup of $\mathrm{Aut}(K/F)$. The Galois connection between the indexed poset of subextensions L of K/F and the indexed poset of subgroups of G is an indexed Galois connection.

Proof. a) Let $F \subset M_1 \subset M_2 \subset K$ with $[M_2 : M_1] = n < \infty$. We must show $[G_{M_1} : G_{M_2}] \leq n$.

We go by induction on n, the case $n = 1$ being trivial. Suppose that $n > 1$ and that the result holds for all pairs of subextensions of smaller degree. If there exists $M_1 \subsetneq M_3 \subsetneq M_2$ then degree multiplicativity and induction gives us the result. Thus we may assume that M_2/M_1 has no proper subextensions, so in particular it is monogenic: $M_2 = M_1[u]$. Let $f \in M_1[t]$ be the minimal polynomial of u over M_1, so $\deg f = n$. Let $C = gG_{M_2}$ be a left coset of G_{M_2} in G_{M_1}. Since every $\sigma \in G_{M_2}$ fixes u, for every $\sigma \in C$ we have $\sigma(u) = g(u)$. We claim that if $C' = g'G_{M_2}$ is a coset distinct from C, then $g'(u) \neq g(u)$. Indeed, if $g'(u) = g(u)$ then $(g^{-1}g')(u) = u$, and then $g^{-1}g$ fixes M_2 pointwise, so $(g^{-1}g') \in G_{M_2}$ and $C = C'$. Therefore the number of left cosets of G_{M_2} in G_{M_1} is at most the size of the G-orbit $G(u)$, but since every element of $G(u)$ is a root of f, $\#G(u) \leq \deg f = n$.
b) Let $H_1 \subset H_2 \subset G \subset \mathrm{Aut}(K/F)$ with $[H_2 : H_1] = n < \infty$. We must show that $[K^{H_1} : K^{H_2}] \leq n$. Let $C = gH_1$ be a left coset of H_1 in H_2, and let $x \in K^{H_1}$. As above, the action of $\sigma \in H_2$ on x depends only on σH_1, so we may write $Cx = gx$ and speak of applying C to any element of K^{H_1}.

Seeking a contradiction we suppose that $[K^{H_1} : K^{H_2}] > n$ and choose $x_1, \ldots, x_{n+1} \in K^{H_1}$ which are K^{H_2}-linearly independent. Let $C_1, \ldots, C_n$ be the distinct cosets of H_1 in H_2. Now consider the linear system

$$a_1(C_1x_1) + a_2(C_1x_2) + \ldots + a_{n+1}(C_1x_{n+1}) = 0,$$

$$a_1(C_2x_1) + a_2(C_2x_2) + \ldots + a_{n+1}(C_2x_{n+1}) = 0,$$

$$\ldots$$

$$a_1(C_nx_1) + a_2(C_nx_n) + \ldots + a_{n+1}(C_nx_{n+1}) = 0,$$

where we regard $a_1, \ldots, a_{n+1} \in K$ as being unknowns. Because we have more unknowns than equations, there exists a nonzero solution $(a_1, \ldots, a_{n+1}) \in K^{n+1}$. Among all nonzero solutions we choose one with as many of the coordinates equal to zero as possible, and after permuting the variables we may assume this solution is of the form $v = (a_1, \ldots, a_r, 0, \ldots, 0)$ with $a_1, \ldots, a_r \neq 0$. Since the solutions form a linear space, we may also assume $a_1 = 1$. It is not possible for $a_1, \ldots, a_r$ to all lie in K^{H_2}, for one of the cosets, say C_i, is equal to H_1 so the ith equation reads

$$a_1x_1 + \ldots + a_{n+1}x_{n+1} = 0,$$

giving a nontrivial linear dependence relation for the x_i's over K^{H_2}. Let J be such that $a_J \notin K^{H_2}$, so there exists $\sigma \in H_2$ such that $\sigma a_J \neq a_J$. Applying σ to the above linear system, we get $\sum(\sigma a_i)(\sigma C_j x_i) = 0$ for all $1 \leq j \neq n+1$. But $\sigma_1 C_1, \ldots, \sigma_n C_n$ is simply a permutation of $C_1, \ldots, C_n$, so the new system is simply a permutation of the old system and thus $w = (1, \sigma a_2, \ldots, \sigma a_r, 0, \ldots, 0)$ is also a solution of the system. Then $v - w$ is a solution with at least one more of the coordinates equal to zero than v but with Jth coordinate nonzero, contradiction. $\square$

4.3. **Normality and Stability...and Normality.**

Let K/F be a field extension with $G = \mathrm{Aut}(K/F$, and let L be a subextension. As in §3.2, we say that L is **stable** if $\sigma(L) \subset L$ for all $\sigma \in \mathrm{Aut}(K/F)$. The results established there in the context of general G-actions specialize as follows [K, Thm. 12, Cor. 13].

Proposition 31.
a) If L is a stable subextension, then $\mathrm{Aut}(K/L)$ is normal in G.
b) If H is a normal subgroup of G, then K^H is a stable subextension.
c) The Galois-closure of a normal subgroup of G is normal.
d) The Galois-closure of a stable subextension of K/F is stable.

Theorem 32. *Suppose K/F is Galois and L is stable. Then L/F is Galois.*

Proof. For $x \in L \setminus F$, choose $\sigma \in \mathrm{Aut}(K/F)$ such that $\sigma x \neq x$. Since L is stable, σ restricts to an automorphism of L. □

Proposition 33. *Let $K/L/F$ be a tower of fields.*
a) If L/F is normal algebraic, then L is stable.
b) If K/F is Galois and L is stable then L/F is normal.

Proof. a) Let $\sigma \in \mathrm{Aut}(K/F)$ and $x \in L$. Then $\sigma(x)$ satisfies the minimal polynomial of x over F, so $\sigma(x) \in L$. Thus $\sigma(L) \subset L$.
b) Let $x \in L$, and let $x_1, \ldots, x_d$ be the roots of the minimal polynomial of x over F. For each $1 \leq i \leq n$, there is a unique F-algebra automorphism $\sigma_i : F(x) \to K$ such that $\sigma_i(x) = x_i$. By Lemma 41, σ_i extends to an F-algebra automorphism σ_i of K. Since L is stable, $x_i = \sigma_i(x) \in L$. □

Theorem 34. *Suppose L is a stable subextension of K/F. Then $\mathrm{Aut}(K/F)/\mathrm{Aut}(K/L)$ is canonically isomorphic to the subgroup of $\mathrm{Aut}(L/F)$ consisting of automorphisms which can be extended to K.*

Proof. By Proposition 31, $\mathrm{Aut}(K/L)$ is normal in $\mathrm{Aut}(K/F)$ so the quotient group exists. Since L is stable, restriction to L gives a homomorphism $\Phi : \mathrm{Aut}(K/F) \to \mathrm{Aut}(L/F)$. Moreover, $\mathrm{Ker}\,\Phi = \mathrm{Aut}(K/L)$ and by construction, $\Phi(\mathrm{Aut}(K/F))$ is the subgroup of automorphisms which can be extended to K. □

Example (**GIVE AN EXAMPLE WHERE Φ IS NOT SURJECTIVE**).

4.4. **Finite Galois Extensions.**

Theorem 35. *Let K/F be a field extension of finite degree, and put $G = \mathrm{Aut}(K/F)$. For L a subextension of K/F, put $G_L = \mathrm{Aut}(K/L)$.*
a) The following are equivalent:
(i) K/F is Dedekind: for all subextensions L of K/F, $K^{G_L} = L$.
(ii) K/F is Galois: $K^G = F$.
(iii) $\#G = [K : F]$.
(iv) K/F is normal and separable.

Proof. (i) $\implies$ (ii) is immediate.
(ii) $\implies$ (i) follows from Theorem 30 and Corollary 16.
(ii) $\implies$ (iii): in our finite indexed Galois connection G is closed, so by Theorem 15 $\#G = [G : \{1\}] = [K^{\{1\}} : K^G] = [K : F]$.

(iii) $\Longrightarrow$ (ii): Apply Theorem 30c) with $x_1 = F$, $x_2 = K$.
(iii) $\Longleftrightarrow$ (iv): By Lemma 2 K/F is separable, hence monogenic: $K = F[u] = F[t]/(p(t))$ with $\deg p = [K : F] = n$, say. So $\# \operatorname{Aut}(K/F)$ is equal to the number of distinct roots of $p(t)$ in K. Therefore, if this number is equal to $[K : F] = \deg p$, K is the splitting field of p and is normal. Conversely, if K/F is separable and normal it is of the form $K[t]/(f(t))$ for an irreducible polynomial which splits into $[K : F]$ distinct roots in K, so $\operatorname{Aut}(K/F) = [K : F]$. □

Corollary 36. *(Artin) Let G be a finite group of automorphisms of a field K, and put $F = K^G$. Then K/F is a finite Dedekind extension.*

Proof. We consider the Galois connection between subextensions of K/F and subgroups of G. By Theorem 30 this is an indexed Galois connection, and by hypothesis F is Galois-closed. Therefore we may apply Theorem 15a) we get

$$[K : F] = [G_F : G_K] = [G : 1] = \#G < \infty.$$

Thus K/F is finite Galois, so by Theorem 42 it is finite Dedekind. □

4.5. **Algebraic Galois Extensions.**

Theorem 37. *Let K/F be a field extension, and put $G = \operatorname{Aut}(K/F)$. TFAE:*
(i) G is compact in the $\mathcal{K}$-topology.
(ii) K/F is algebraic.

Proof. (i) $\Longrightarrow$ (ii): If $(G, \mathcal{K})$ is compact, then by XX the action of G on K is locally finite, so by XX K/F is algebraic.
(ii) $\Longrightarrow$ (i): Since K/F is algebraic, by Theorem 27 the action of G on K is locally finite. Moreover it is finite-complete: that is, an automorphism of K is determined by a compatible sequence of automorphisms on finite G-stable subsets of K. Indeed, observe first that finite subextension L of K/F is of the form $L = F(S)$ for a finite subset S of K and the action of G on S determines the action on L. Next, observe that since K/F is algebraic, K is the direct limit of its finite normal subextensions, so giving an automorphism on each finite normal subextension in a compatible way determines a unique automorphism of K. By Theorem 25c), G is $\mathcal{K}$-compact. □

Theorem 38. *Let K/F be algebraic, and put $G = \operatorname{Aut}(K/F)$. For a subgroup $H \subset G$, TFAE:*
(i) H is $\mathcal{K}$-compact.
(ii) H is $\mathcal{K}$-closed.
(iii) H is Galois-closed, i.e., $H = G_L$ for some subextension L of K/F.

Proof. (i) $\Longleftrightarrow$ (ii): By Theorem 37, G is $\mathcal{K}$-compact. It follows that a subgroup H of G is $\mathcal{K}$-compact iff it is $\mathcal{K}$-closed.
(ii) $\Longleftrightarrow$ (iii): by Theorem 23c), it suffices to show that every $\mathcal{K}$-open subgroup is of the form G_S for some finite subset S of K, so let H be a $\mathcal{K}$-open subgroup of G. By Theorem 23b), H contains a subgroup $G_S = G_{F(S)}$ for some finite subset $S \subset K$. So H has a finite-index closed subgroup and is therefore itself closed by Theorem 15b). □

Corollary 39. *(Shimura* [S, 6.11]*) Let K/F be a field extension with $G = \operatorname{Aut}(K/F)$. The Galois correspondence induces a bijection between $\mathcal{K}$-compact subgroups H of G and subextensions L of K/F such that K/L is algebraic Galois.*

Proof. Let H be a $\mathcal{K}$-compact subgroup of G. Then by Theorem 37 K/K^H is algebraic Galois, and by Theorem 38, H is Galois-closed.

Conversely, if K/L is algebraic Galois then by Theorem 37 $\mathrm{Aut}(K/L)$ is $\mathcal{K}$-compact, and tautologically L is Galois-closed. $\square$

Theorem 40. *Let K/F be a field extension. Let L be a subextension of K/F such that $L \supset K^F$ and L/K^F is algebraic. Then K/L is Galois.*

Proof. It is no loss of generality to replace F with K^F and thus assume that K/F is Galois. Let us do so. The field L is the direct limit of the finite subextensions M of L/F. By XXX, since F is Galois closed, so is each M: $M = K^{G_M}$. Then

$$K^{\bigcap_M G_M} = \bigvee_M K^{G_M} = \bigvee_M M = L.$$

$\square$

Lemma 41. *(Extension of Automorphisms I) Let K/F be normal algebraic. Then every homomorphism $\iota : F \to K$ extends to an automorphism of K.*

Proof. A standard Zorn's Lemma argument shows that there is a maximal subextension L of K/F such that ι extends to $\iota_L : L \to K$. If $L \neq K$, let $x \in K \setminus L$. Suppose $x_1, \ldots, x_d$ are the distinct roots of the minimal polynomial for x over F. Since K/F is normal, $x_1, \ldots, x_d \in K$. If all of $x_1, \ldots, x_d$ lie in $\iota_L(L)$, then L itself contains all d distinct conjugates of x so contains x itself, contradiction. So there must exist at least one i such that $x_i \notin \iota_L(L)$, and then we can extend ι_L to a homomorphism from $L(x) \to K$ by mapping x to x_i. This contradicts the maximality of L, so we must have $L = K$ and thus an extension of ι to a homomorphism $\iota : K \to K$. Moreover, for any $x \in K$, ι is an injection from the finite set of F-conjugates of x to itself, so every F-conjugate of x lies in $\iota(K)$. $\square$

Theorem 42. *Let K/F be an algebraic field extension, and put $G = \mathrm{Aut}(K/F)$. For L a subextension of K/F, put $G_L = \mathrm{Aut}(K/L)$.*
a) The following are equivalent:
(i) K/F is Dedekind: for all subextensions L of K/F, $K^{G_L} = L$.
(ii) K/F is Galois: $K^G = F$.
(iii) K/F is normal and separable.
b) Suppose K/F is Galois. The following are equivalent:
(i) L is a stable subextension of K/F.
(ii) L/F is normal.
(iii) $\mathrm{Aut}(K/L)$ *is a normal subgroup of* $\mathrm{Aut}(K/F)$.
c) Under the equivalent conditions of part b), we have a canonical isomorphism $\mathrm{Aut}(K/F)/\mathrm{Aut}(K/L) = \mathrm{Aut}(L/F)$.

Proof. a) (i) $\implies$ (ii) is immediate.
(ii) $\implies$ (i) follows from Theorem 40.
(ii) $\implies$ (iii): By Lemma 2 K/F is separable. Moreover, every finite subextension L of K/F is normal, so K/F is the splitting field of a set of polynomials, so is normal.
(iii) $\implies$ (ii): Since K/F is algebraic, normal and separable, by Theorem 42 it is a direct limit of finite Galois extensions. So if $x \in K \setminus F$, there exists a finite Galois subextension L of K/F such that $x \in L \setminus F$, and thus there exists $\sigma \in G$ such that $\sigma x \neq x$. It follows that $K^G = F$.

b) (i) $\iff$ (ii) by Proposition 33.
(i) $\implies$ (iii) by Proposition 31a).
(iii) $\implies$ (i) by Proposition 31b).
c) This is immediate from Theorem 34 and Lemma 41. $\square$

Corollary 43. *(Shimura* [S, 6.13]*) Let K/F be Galois with $G = \mathrm{Aut}(K/F)$.*
a) The extension $\mathrm{Cl}_K(F)/F$ is algebraic Galois.
b) The subgroup $H = \mathrm{Aut}(K/\mathrm{Cl}_K(F))$ is normal in G, and G/H is naturally isomorphic to a $\mathcal{K}$-dense subgroup of $\mathrm{Aut}(\mathrm{Cl}_K(F)/F)$.

Proof. a) Since $\mathrm{Cl}_K(F)$ is a G-stable subextension of the normal extension K/F, it is itself a normal extension. Separability is clear from Lemma X.X. So $\mathrm{Cl}_K(F)$ is algebraic, normal and separable and thus algebraic Galois.
b) Since $\mathrm{Cl}_K(F)$ is G-stable, H is normal in G. The density follows from the fact that $F = (\mathrm{Cl}_K(F))^{G/H}$. $\square$

Theorem 44. *(Excess Cardinality I) Let K/F be infinite algebraic Galois with group $G = \mathrm{Aut}(K/F)$. Let $\mathcal{F}$ be the set of all finite normal subextensions L of K/F and put $\kappa = \#\mathcal{F}$.*
a) $[K : F] \leq \kappa < 2^\kappa = \#G$.
b) There exists a subgroup $H \subset G$ with $\#H = \kappa$ such that $K^H = F$.

Proof. a) Step 1: Since K/F is algebraic Galois, $K = \varinjlim L$ is the direct limit of its finite nomal subextensions. Choose for each $L \in \mathcal{F}$ a finite spanning set S_L for L/F. Then $\bigcup_{L\in\mathcal{F}} S_L$ is a spanning set for K/F of cardinality at most $\aleph_0 \cdot \kappa = \kappa$. Since every spanning set has a basis as a subset, this shows $[K : F] \leq \kappa$. (Of course $\kappa < 2^\kappa$ holds for all cardinals, by a famous theorem of Cantor.)
Step 2: The Galois correspondence puts $\mathcal{F}$ in bijection with the set of all open subgroups of G, so there are κ open subgroups. By [T, Thm. 4.9], $\#G = 2^\kappa$.
b) By part a) and Theorem 22d), G is homeomorphic as a topological space to the Cantor cube 2^κ. The subspace of 2^κ consisting of all finitely nonzero sequences is dense of cardinality κ, so G admits a dense subset S of cardinality κ, and the subgroup generated by S is a dense subgroup of cardinality κ. $\square$

(Remark: It seems likely that $[K : F] = \alpha$ in all cases. Try a little harder to prove this!)

Theorem 45. *(*[So70, Thm. 19]*) Let K/F be algebraic Galois, and put $G = \mathrm{Aut}(K/F)$. Then the Krull topology on G is:*
a) The coarsest group topology τ such that Galois-closed subgroups are τ-closed.
b) The unique quasi-compact group topology τ such that Galois-closed subggroups are τ-closed.
c) The unique compact group topology τ such that τ-closed subgroups are Galois-closed.
d) The unique locally compact group topology τ on G such that the Galois correpondence induces a bijection between all subextensions of K/F and all τ-closed subgroups of G.

Theorem 46. *(*[So70, Thm. 3]*) Let K/F be a Dedekind extension with $G = \mathrm{Aut}(K/F)$. Suppose that there exists a topology τ on G such that:*
(i) (G, τ) is a topological group and

(ii) A subgroup of G is Galois-closed iff it is τ-closed.
Then K/F is algebraic.

Proof. The trivial subgroup $\{1\}$ of G is Galois-closed, hence τ-closed. Thus (G, τ) is a separated space, and – since it is assumed to be a topological group – a Hausdorff space. But by [So70], this implies K/F is algebraic. $\square$

4.6. **The $\mathcal{J}$-topology.**

Lemma 47. *(*[SV71, Prop. 2.7]*) For any field extension K/F, the $\mathcal{J}$-topology on $G = \mathrm{Aut}(K/F)$ is compatible with the Galois connection and $(G, \mathcal{J})$ is a semi-topological group with continuous inversion. A subbase for the $\mathcal{J}$-open subsets of G is given by $\{\sigma U(x)\}_{\sigma \in G, x \in K}$.*

4.7. **The $\mathcal{K}$-topology.**

Theorem 48. *(Shimura* [S, 6.12]*) Let K/F be a Galois extension with $G = \mathrm{Aut}(K/F)$. Suppose there exists a subextension L_0 of K/F such that K/L_0 is algebraic Galois and L_0/F is finitely generated. Then:*
a) G is locally compact in the $\mathcal{K}$-topology.
b) The Galois correspondence induces a bijection from the set of $\mathcal{K}$-compact open subgroups H of G to the subextensions L such that K/L is algebraic Galois and L/F is finitely generated.

Proof. Step 1: Let $L = F(x_1, \ldots, x_n)$ be a finitely generated extension of F with K/L is algebraic Galois. By Theorem 39, $G_L = \mathrm{Aut}(K/L)$ is $\mathcal{K}$-compact. But also

$$G_L = \{\sigma \in G \mid \sigma(x_1) = x_1, \ldots, \ldots, \sigma(x_n) = x_n\}$$

is a finite intersection of $\mathcal{K}$-subbasic sets, so G_L is $\mathcal{K}$-open.
Step 2: In particular, it follows that G_{L_0} is a $\mathcal{K}$-compact open subgroup of the topological group G, so G is locally compact in the $\mathcal{K}$-topology.
Step 3: Let L be a subextension of K/F such that G_L is $\mathcal{K}$-compact open. Then $G_{LL_0} = G_L \cap G_{L_0}$ is a $\mathcal{K}$-compact open subgroup of the compact subgroup G_{L_0}, so

$$\aleph_0 > [G_{L_0} : G_{LL_0}] = [K^{G_{LL_0}} : K^{G_{L_0}}] = [LL_0 : L_0].$$

Since L_0/F is finitely generated and LL_0/L_0 is finite, LL_0/F is finitely generated, hence so is the subextension L_0. $\square$

Corollary 49. *If K/F is finitely generated, $(\mathrm{Aut}(K/F), \mathcal{K})$ is discrete.*

4.8. **When K is algebraically closed.**

Lemma 50. *(Extension of Automorphisms II) Let $K/L/F$ be a tower of field extensions, with K algebraically closed. Let $\iota : L \to L$ be an F-algebra automorphism of L. Then there is an extension of ι to an F-algebra automorphism of K.*

Proof. Let $T = \{t_i\}_{i \in I}$ be a transcendence basis for K/L. There is a unique F-algebra automorphism of $L(T)$ extending ι and mapping each t_i to itself. We are now reduced to extending $\iota : L(T) \to L(T)$ over the algebraic extension $K/L(T)$, and this is possible by Lemma 41. $\square$

Theorem 51. *Let K be an algebraically closed field. Then $\# \mathrm{Aut}(K) = 2^{\# K}$.*

Proof. Step 0: Being algebraically closed, K is infinite, so the family of all set maps from K to K has cardinality $\#K^{\#K} = 2^{\#K}$. Thus it suffices to exhibit a subgroup of $\operatorname{Aut}(K)$ of cardinality $2^{\#K}$.
Step 1: Suppose K is algebraic over its prime subfield K_0. Then it follows from Theorem 44 that $\#\operatorname{Aut}(K/K_0) \geq 2^{[K:K_0]} \geq 2^{\aleph_0}$.[4]
Step 2: Suppose K is countable, and let $\overline{K_0}$ be the algebraic closure of the prime subfield. Then by Step 1, $\operatorname{Aut}\overline{K_0} = 2^{\aleph_0}$, and by Lemma 50 each of these automorphisms extends to an automorphism of K, so also $\operatorname{Aut} K \geq 2^{\aleph_0}$.
Step 3: Suppose K is uncountable, so $\#K = \operatorname{trdeg} K/K_0$. Let T be a transcendence basis for K over K_0. Then every permutation of T gives rise to an automorphism of $K(T)/K_0$, and the set of permutations of T has cardinality $2^{\#T} = 2^{\#K}$. By Lemma 50, each of these automorphisms extends to an automorphism of K. □

Theorem 52. *Suppose K/F is an extension with K algebraically closed. Then $K^{\operatorname{Aut}(K/F)} = F^{p^{-\infty}}$, the perfect closure of F.*

Proof. Let x be an element of K which is not purely inseparable algebraic over x. We claim there is an automorphism σ of K such that $\sigma(x) \neq x$.
Case 1: x is algebraic over F. Then the minimal polynomial $P(t)$ of x over F has at least one distinct root x' in K. The map $x \mapsto x'$ determines a unique F-algebra isomorphism $\sigma : F[x] \to F[x']$, which can be extended to an F-algebra automorphism σ of $\overline{F}$, the algebraic closure of F in K. By Lemma 50, σ extends to an automorphism of K.
Case 2: x is transcendental over F. Then, for instance, the map $x \mapsto x+1$ induces an F-algebra automorphism of the pure transcendental extension $F(x)$, which by Lemma 50 extends to an automorphism of K. □

Corollary 53. *Let K/F be a field extension with K algebraically closed.*
a) The extension K/F is quasi-Dedekind.
b) The extension K/F is Galois iff F is perfect.
c) The extension K/F is Dedekind iff either:
(i) F has characteristic zero or
(ii) F is perfect of characteristic $p > 0$ and K/F is algebraic.

Theorem 54. *(Excess Cardinality II) Let K/F be an extension of infinite degree, with K algebraically closed of characteristic* 0.
a) $\#\operatorname{Aut}(K/F) \geq 2^{[K:F]}$.
b) There exists $H \subset \operatorname{Aut}(K/F)$ with $\#H = [K:F]$ such that $K^H = F$.

Proof. a) Let $\overline{F}$ be the algebraic closure of F in K.
Case 1: If K/F is algebraic, the result is immediate from Corollary 53 and 44.
Case 2: Suppose K/F is transcendental and $\#K = \#F$. Let $t \in K$ be transcendental over F. By Lemma 50, we have

$$\#\operatorname{Aut}(K/F) \geq \#\operatorname{Aut}(\overline{F(t)}/F) \geq \#\operatorname{Aut}(\overline{F(t)}/F(t)) \geq \operatorname{Aut}(\overline{F(t)}/\overline{F}(t)).$$

Moreover, by Theorem 44, $\#\operatorname{Aut}(\overline{F(t)}/\overline{F}(t)) = 2^{\kappa}$, where κ is the number of finite normal subextensions of $\overline{F(t)}/\overline{F}(t)$. Clearly $\kappa = \#F = \#K \geq [K : F]$.

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[4]Perhaps the appeal to Theorem 44 is overkill. The topological groups $(\operatorname{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathcal{K})$, $(\operatorname{Aut}(\overline{\mathbb{F}_p}/\mathbb{F}_p), \mathcal{K})$ are infinite, compact and second countable, so are completely metrizable spaces without isolated points. They must therefore have at least continuum cardinality.

Case 3: Suppose K/F is transcendental and $\#K > \#F$. Then $\#K = \mathrm{trdeg}(K/F)$. Let T be a (necessarily infinite) transcendence basis for K/F. Then by Lemma 50,

$$\# \mathrm{Aut}(K/F) \geq \# \mathrm{Aut}(F(T)/F),$$

whereas as in the proof of Theorem 51, considering all permutations of T gives

$$\# \mathrm{Aut}(F(T)/F) = 2^{\#T} = 2^{\#K} \geq 2^{[K:F]}.$$

b) ... $\square$

Theorem 55. *If K/F is an extension with K algebraically closed, then a subextension L is stable iff $L = K$ or L is algebraic and normal over F.*

Proof. Let $\overline{F}$ be the algebraic closure of F in K – it is, in particular, an algebraic closure of F. Clearly for any $x \in K$ and $\sigma \in \mathrm{Aut}(K/F)$, x is algebraic over F iff $\sigma(x)$ is algebraic over F, so that $\overline{F}$ is stable. Moreover, as is well known, if L is a normal algebraic extension of F, it is the splitting field of a set S of polynomials with coefficients in F, and then for any $\sigma \in \mathrm{Aut}(K/F)$, $\sigma(L)$ is the splitting field of $\sigma(S) = S$, so $\sigma(L) = L$.

Conversely, of course if L/F is algebraic but not normal, then there exists an irreducible polynomial $P \in F[t]$ with a root $x \in L$ and another root $x' \in \overline{F} \setminus L$, and then we can build an F-automorphism of K which sends x to x', so L is not stable. Suppose now that $x \in L$ is transcendental over L. By the theory of automorphisms of algebraically closed fields, we know that the orbit of x under $\mathrm{Aut}(K/F)$ is the set of all elements of K which are transcendental over F, so if L is stable it must contain all such elements. Moreover, if $y \in K$ is algebraic over F then $x + y$ is transcendental over F, so $x + y \in L$ and therefore $x \in L$. That is, $L = K$. $\square$

We immediately deduce the following result.

Corollary 56. *Let K/F be an extension of algebraically closed fields. Then the only Galois-closed normal subgroups of G are $\{1\}$ and G.*

Remark: In the situation of Corollary 56 one wonders whether $G = \mathrm{Aut}(K/F)$ is simple as an abstract group (the result leaves open the possibility that there is a proper, nontrivial normal subgroup H of G with Galois closure G). In fact D. Lascar has shown that the abstract group $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ is simple [La97]. So far as I know the general case remains open and may not have received enough attention.

5. Three Flavors Revisited

5.1. **Galois Extensions.**

Theorem 57. *Let K be a field and $T = \{t_i\}_{i \in I}$ be a nonempty set of independent indeterminates over K, so that $K(T)/T$ is a purely transcendental extension. The following are equivalent:*
(i) $K(T)/K$ is Galois.
(ii) K is infinite or $\#T \geq 2$.

Proof. Step 0 (Background): Let K be any field and $t_1, \ldots, t_n$ independent indeterminates over K. Then $\mathrm{PGL}_{n+1}(K)$ acts effectively on $K(t_1, \ldots, t_n)$ by field automorphisms. This action comes from the natural action of $\mathrm{GL}_{n+1}(K)$ on $\mathbb{A}^{n+1}(K)$, which induces an effective action of $\mathrm{PGL}_{n+1}(K)$ on $\mathbb{P}^n_{/K}$ by birational automorphisms, hence automorphisms of the function field $K(\mathbb{P}^n) = K(t_1, \ldots, t_n)$. It is a

basic fact that for all n the group of *biregular* automorphisms of $\mathbb{P}^n$ is $\mathrm{PGL}_{n+1}(K)$ [H, Example II.7.1.1]. Since every rational map from a smooth curve into a projective variety is everywhere regular, when $n = 1$ we have $\mathrm{Aut}(K(t)/K) = \mathrm{Aut}_K(\mathbb{P}^1_{/K}) = \mathrm{PGL}_2(K)$.
Step 1: It follows from Step 0 that $\mathrm{Aut}(K(t)/K)$ is infinite iff K is infinite. So if K is finite, then by Corollary 36, $K(t)/K(t)^{\mathrm{Aut}(K(t)/K)}$ is finite, hence $K(t)^{\mathrm{Aut}(K(t)/K)}$ has transcendence degree 1 over K so is not equal to K.
Step 2: Suppose K is infinite and $T = \{t\}$. Let $f \in K(t) \setminus K$. The set of $x \in K$ such that $f(x) = f(0)$ is finite and nonempty, so since K is infinite there exists x with $f(x) \neq f(0)$, which means that f is not fixed by the automorphism $t \mapsto t + x$. It follows that $K(t)^{\mathrm{Aut}(K(t)/K)} = K$.
Step 3: Suppose K is arbitrary and $\#T \geq 2$. Write $T = \{t_i\}_{i \in I}$, and for each $i \in I$, put $K_i = K(\{t_j\}_{j \neq i}$ and $H_i = \mathrm{Aut}(K/K_i) = \mathrm{Aut}(K_i(t_i)/K_i)$. Since $\#I \geq 2$, each K_i is infinite, so by Step 2, $K(T)^{H_i} = K_i(t_i)^{\mathrm{Aut}(K_i(t_i)/K_i)} = K_i$. Therefore

$$K(T)^{\mathrm{Aut}(K(T)/K)} \subset K(T)^{\bigvee_{i \in I} H_i} = \bigcap_{i \in I} K(T)^{H_i} = \bigcap_{i \in I} K_i = K.$$

□

Theorem 58. *A nontrivial finitely generated regular extension K/F of* general type *has* $\mathrm{Aut}(K/F)$ *finite and is therefore* not *Galois.*

Proof. A variety $V_{/K}$ of general type has finite birational automorphism group: e.g. [Sz96]. As in the proof of Theorem 57 above, this immediately implies that the field extension $K(V)/K$ cannot be Galois, since $K(V)^{\mathrm{Aut}(K(V)/K)}/K$ has positive transcendence degree. □

More precise results are available for one-dimensional function fields.

Lemma 59. *A regular extension $k(C)/k$ is Galois iff $G = \mathrm{Aut}(k(C)/k)$ is infinite.*

Proof. We have $k \hookrightarrow k(C)^G \hookrightarrow k(C)$. Clearly $[k(C) : k] = \infty$, so if G is finite, $[k(C) : k(C)^G]$ is finite and then $[k(C)^G : k]$ is infinite, so $k \neq k(C)^G$. Conversely, if $[k(C) : k(C)^G]$ is infinite, then $k(C)/k(C)^G$ is transcendental, and that implies $k(C)^G = k$. □

Theorem 60. *For an absolutely integral algebraic curve C/k, the extension $k(C)/k$ is Galois iff k is infinite and one of the following conditions hold:*
(i) C has genus 0*;*
(ii) C has genus 1 *and its Jacobian* $\mathrm{Pic}^0(C)$ *has infinitely many k-rational points.*

Proof. The automorphism group of any curve C/k is an algebraic group over k, so if k is finite, the automorphism group is finite. So suppose k is infinite. If C has genus at least 2, then again it is known that the automorphism group of C/k is finite. If C has genus 1, then $\mathrm{Aut}(C/k)$ contains, as a finite index subgroup, the group $\mathrm{Pic}^0(C)(k)$ of k-rational points on the Jacobian. Therefore $k(C)/k)$ is Galois iff the Jacobian has infinitely many k-rational points. If C has genus 0, its automorphism group is $\mathrm{PGL}_2(k)$ if $C \cong \mathbb{P}^1$; otherwise C corresponds to a division quaternion algebra B/k and the automorphism group is $B^\times/k^\times$. Both of these groups are infinite when k is infinite. □

This result has the following consequence, which shows that transcendental Galois extensions behave quite differently from algebaic ones.

Corollary 61. *Let F be a field, and let K/F be a Galois extension which is finitely generated, regular and transcendence degree one. Then for any subextension L of K/F, L/F is a Galois extension.*

Proof. Translating the algebraic conditions on K/F into geometric language, we may write $K = F(C)$ for a nice curve $C_{/F}$. By Theorem 60, C is either a curve of genus zero or is of genus one with $\#C(F) \geq \aleph_0$. Now any subextension L of K/F is of the form $L = F(C')$ for a nice curve $C'_{/F}$ and the finite F-algebra extension L/K corresponds to a finite morphism of F-varieties $\varphi : C \rightarrow C'$. By Riemann-Hurwitz, the genus of C' is at most the genus of C, so either C' is itself a genus one curve – in which case by Theorem 60 L/F is Galois – or C is of genus one with infinitely many F-rational points, C' has genus one and φ is an unamified covering map. Choose any F-rational point O_C, and put $O_{C'} = \varphi(O_C)$. Then C and C' become elliptic curves and $\varphi : C \rightarrow C'$ is an isogeny. The induced homomorphism $\varphi_K : C(F) \rightarrow C'(F)$ has finite kernel (over $\overline{F}$, hence *a fortiori* over F) and hence $\varphi_K(C(F))$ is an infinite subgroup of $C'(F)$. Applying Theorem 60 once more we conclude that L/F is Galois. $\square$

5.2. **Dedekind Extensions.**

Lemma 62. *Let K/F be transcendental Dedekind. Then F has characteristic zero.*

Proof. We go by contraposition: suppose that F has characteristic $p > 0$. Let $x \in K$ be transcendental over F. Then every automorphism of K which fixes x^p also fixes x, so $F(x^p) \subsetneq F(x) \subseteq K^{\mathrm{Aut}(K/F(x^p))}$. $\square$

Theorem 63. *(*[So70, Thm. 23]*) Let K/F be a Dedekind extension with $G = \mathrm{Aut}(K/F)$. Suppose that τ is a topology on G such that (G, τ) is a quasi-compact topological group and such that a subgroup of G is Galois-closed iff it is τ-closed. Then K/F is algebraic and τ is the Krull topology.*

Theorem 64. *(*[SV71, Thm. 2.9]*) For K/F a Dedekind extension with $G = \mathrm{Aut}(K/F)$, TFAE:*
(i) The topologies $\mathcal{J}$ and $\mathcal{K}$ on G coincide.
(ii) The topology $\mathcal{J}$ is Hausdorff.
(iii) K/F is algebraic.

Theorem 65. *(*[SV71, Thm. 2.8]*) Let K/F be a Dedekind extension with $G = \mathrm{Aut}(K/F)$. Then the Galois correspondence induces a bijection between all subextensions L of K/F and all $\mathcal{J}$-closed subgroups of G.*

Proof. Since a subgroup is $\mathcal{J}$-closed iff it is Galois-closed, this is immediate from the definition of a Dedekind extension. $\square$

Theorem 66. *(*[SV71, Thm. 2.10]*) Let K/F be a Dedekind extension with $G = \mathrm{Aut}(K/F)$. If the transcendence degree of K/F is finite, then $(G, \mathcal{J})$ is quasi-compact.*

There is also a(n at least!) partial converse.

Theorem 67. *(*[SV71, Thm. 2.11]*) Let K/F be a field extension with $G = \mathrm{Aut}(K/F)$. Suppose K is algebraically closed and the transcendence degree of K/F is infinite. Then $(G, \mathcal{J})$ is not quasi-compact.*

Theorem 68. *([SV71, Thm. 2.12]) Let K/F be a field extension with $G = \mathrm{Aut}(K/F)$. Suppose K is algebraically closed and the transcendence degree of K/F is infinite. Then there is no quasi-compact quasi-group topology τ on G such that a subgroup of G is Galois-closed iff it is τ-closed.*

Theorem 69. *Let K/F be a transcendental Dedekind extension, and let T be a transcendence basis for K/F. Then $K/F(T)$ is infinite.*

Proof. Step 1: Write $T = T' \coprod \{t\}$. Since K/F is Dedekind, so is $K/F(T')$. We may thus replace F by $F(T')$ and reduce to showing: if $K/F(t)$ is finite, then K/F is not Dedekind.
Step 2: As in Step 1, we may replace F by $\mathrm{Cl}_K(F)$ and thus assume that K/F is a regular function field in one variable. Thus $K = F(C)$ for a nice algebraic curve $C_{/F}$ and the finite extension $K/F(t)$ corresponds to a finite morphism $\varphi : C \to \mathbb{P}^1$. If $K = F(t)$ then replace t by t^2; thus we may assume that $[K : F(t)] = \deg \varphi > 1$ and in particular that the map φ is ramified.
Step 3: If φ is not a Galois cover, we are done already, so assume it is. Our strategy is to compose with a finite map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ to get composite map $\psi \circ \varphi : C \to \mathbb{P}^1$ which is not a Galois covering. This corresponds to a non-Galois F-algebra extension $K/F(t)$ which may be viewed as a subextension of K/F.

To construct ψ, let $P \in \mathbb{P}^1(\overline{F})$ be a branch point of φ, and let $Q \in \mathbb{P}^1(\overline{F})$ such that every point in the Galois orbit of Q is a non-branch point. Let $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ be an F-morphism which is unramified at every point in the Galois orbit of P and Q and such that $\psi(P) = \psi(Q) = 0$. Then in the composite map $\psi \circ \varphi$ there is at least one preimage of 0 which is ramified and at least one preimage of 0 which is unramified, so the covering $\psi \circ \varphi$ is not Galois. □

Theorem 70. *(Barbilian-Krull) a) For a field extension K/F, TFAE:*
(i) K/F is Dedekind.
(ii) For every subextension M of K/F, $\mathrm{Cl}_K(M)/M$ is algebraic Galois.
b) For a field extension K/F, TFAE:
(i) K/F is quasi-Dedekind.
(ii) For every subextension M of K/F, $\mathrm{Cl}_K(M)/M$ is algebraic quasi-Galois (or equivalently, algebraic and normal).

Proof. a) (i) $\Longrightarrow$ (ii) is immediate from Theorem 43.
(ii) $\Longrightarrow$ (i): ...
b) ... □

5.3. **Perfectly Galois Extensions.**

Theorem 71. *For a field extension K/F, TFAE:*
(i) K/F is perfectly Galois.
(ii) K/F is Galois and finite.

Proof. (i) $\Longrightarrow$ (ii): Suppose K/F is perfectly Galois. Then so is K/L for any subextension L of K/F. Choose a transcendence base T for K/F, so the algebraic extension $K/F(T)$ is perfectly Galois. If $K/F(T)$ is infinite Galois, then its automorphism group is an infinite compact group, hence possesses a non-closed subgroup H (e.g. the subgroup generated by any countably infinite subset), so $K/F(T)$ is not perfectly Galois. Thus $K/F(T)$ is finite.

If $T = \varnothing$, then we are done. Otherwise K/F is a transcdendental Dedekind extension with $K/F(T)$ finite, contradicting Theorem 69. □

6. Notes

That there is an at least formal Galois correspondence for any field extension K/F is conspicuously missing from the "standard" (i.e., Artinian) exposition on Galois theory and is at best hinted at in most texts. A notable exception to this is Kaplansky's text [K], which sets things up just as we have at the beginning of §1 (or rather, the other way around, of course). Kaplansky writes (p. 11) "We shall mostly deal with the case where M is finite-dimensional over K, but it is interesting and enlightening to push as far as possible the general case where K and M are absolutely arbitrary." I agree most heartily.

Kaplansky's main technical contribution is the **index calculus** of §4.2, which establishes finite Galois theory in a relative setting. It is also valuable in that it sieves a nugget of field-theoretic content (Theorem 30) from a stream of results of a much more formal character. In this paper we have strived to continue this sieving process, attempting to discern which results crucially involve the rich structure and theory of fields, and which can be derived in a simpler context. To this end we have included an axiomatized version of the conclusion of Kaplansky's theorems on indices – namely the notion of an **indexed Galois connection**, which we have taken from a text of S. Roman [R, §6.1]. To the best of my knowledge, indexed Galois connections do not appear elsewhere in the literature. The material on **stability** for arbitrary field extensions also apparently first appeared in [K].

Our presentation of the Galois theory of group actions in §3 seems to be new. The role played by the two topologies $\mathcal{J}$ and $\mathcal{K}$ is inspired by [SV71] and also by M. Fried's MathSciNet review of it, in which he observes that the authors are studying the interplay of two topologies but with the second topology not made explicit. In fact the name **$\mathcal{K}$-topology** comes from Fried's review. Surely the 'K' stands for Krull, but the authors of [SV71] only use the term "Krull topology" in the case of an algebraic extension. This seems insightful, as the $\mathcal{K}$-topology has only a minor role to play for non-algebraic extensions. Let me also thank the authors of [SV71] for introducing the name **$\mathcal{J}$-topology**, as I would otherwise have felt it necessary to call it the Soundararajan-Venkatachaliengar topology!

The main theorem of §4.1 was inspired by the result of Waterhouse which appears as a corollary. Our presentation of Kaplansky's Index Calculus is a slightly generalized version of what appears in [K] in that we start with an arbitrary subgroup G of $\mathrm{Aut}(K/F)$. However the more general result is useful, since we get an easy proof of Corollary 36. We claim this as one of the merits of the present approach: Corollary 36 is a basic and important result – finiteness of invariants – which in standard treatments of Galois theory is often proved only under the additional hypothesis of finiteness of K/F. One should compare the present treatment with that of [L, Thm. VI.1.8], which *does* prove Corollary 36 and exerts some ingenuity to do so.

I have not found the Excess Cardinality Theorem in the literature.

The results on the size and structure of automorphism groups of algebraically closed fields given in §$X.X$ are easy consequences of the theory of transcendence bases developed by Steinitz a century ago [St10]. Nevertheless this seems to have been a source of consternation to many over the years. In his award-winning 1966 article [Ya66], P. Yale describes the existence of automorphisms of $\mathbb{C}$ other than the identity and complex conjugation as "[o]ne of the best known bits of mathematical folklore" and carefully discusses their construction using a special case of Lemma 50. His paper ends as follows: "As the final comment I mention an additional bit of mathematical folklore. In [1][5] it is claimed, without proof or reference to the proof, that the cardinality of the set of automorphisms of $\mathbb{C}$ is $2^{2^{\aleph_0}}$. I have heard this from other sources and am convinced that it is true although I don't know where the proof may be found." And indeed it seems that the first published proof of Theorem 51 is [Ch70].

The material of §5.1 concerning function fields may be new; it provided an opportunity to digress into some arithmetic geometry. For a field k and $n \in \mathbb{Z}^+$, the **Cremona group** $\mathrm{Cr}_n(k) = \mathrm{Aut}(k(t_1, \ldots, t_n)/k)$ of birational automorphisms of projective space has been the topic of intense algebraic geometric study for almost 150 years. As above, when $n = 1$ it is simply $\mathrm{PGL}_2(k)$, but for $n \geq 2$ $\mathrm{Cr}_n(K)$ strictly contains $\mathrm{PGL}_{n+1}(K)$. Broadly speaking, the discrepancy between these two groups is well understood when $n = 2$ and very poorly understood when $n \geq 3$, notwithstanding a lot of interesting work in the area: see e.g. [Se10] for a survey of some of the recent results in the field.

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