ON THE HASSE PRINCIPLE FOR SHIMURA CURVES

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Abstract. Let C be an algebraic curve defined over a number field K, of positive genus and without K-rational points. We conjecture that there exists some extension field L over which C has points everywhere locally but not globally. We show that our conjecture holds for all but finitely many Shimura curves of the form $X_0^D(N)_{/\mathbb{Q}}$ or $X_1^D(N)_{/\mathbb{Q}}$, where $D > 1$ and N are coprime squarefree positive integers. The proof uses a variation on a theorem of Frey, a gonality bound of Abramovich, and an analysis of local points of small degree.

1. Introduction

1.1. **Statements of the main results.** Fix $D > 1$ a squarefree positive integer, and $N \geq 1$ a squarefree positive integer which is coprime to D. One may consider the Shimura curves $X_0^D(N)_{/\mathbb{Q}}$ (resp. $X_1^D(N)_{/\mathbb{Q}}$), which are (at least coarse) moduli spaces for abelian surfaces endowed with endomorphisms by a maximal order in the indefinite rational quaternion algebra B_D of discriminant D and $\Gamma_0(N)$ (resp. $\Gamma_1(N)$) level structure. There are "forgetful" modular maps $X_1^D(N) \rightarrow X_0^D(N) \rightarrow X^D = X_0^D(1)$.

Understanding the points on these curves rational over various number fields K and their completions is an important problem, analogous to the corresponding problem for the classical modular curves $X_1(N)$, $X_0(N)$ but even more interesting in at least one respect. Namely, although the classical modular curves have $\mathbb{Q}$-rational cusps, we have $X^D(\mathbb{R}) = \emptyset$. This raises the possibility that certain Shimura curves, when considered over suitable non-real number fields K, may violate the Hasse principle, i.e., may have points rational over every completion of K but not over K itself. And indeed, B.W. Jordan showed that the curve $X^{39}_{/\mathbb{Q}(\sqrt{-23})}$ violates the Hasse principle [13] (see also [21]). Later, Skorobogatov and Yafaev provided explicit conditions on D, N and $K = \mathbb{Q}(\sqrt{m})$ sufficient for $X_0^D(N)_{/K}$ to violate the Hasse principle [22]. Although it is plausible that their conditions can be met for infinitely many choices of (D, N, K), they include hypotheses about the class group of K, so proving that they hold infinitely often seems very difficult.

Using different methods, we shall show that almost all Shimura curves of squarefree level violate the Hasse principle. More precisely:

Theorem 1. *If $D > 546$, there is an integer m such that $X^D_{/\mathbb{Q}(\sqrt{m})}$ violates the Hasse principle.*

Theorem 2. *There exists a constant C such that if $D \cdot N > C$, then there exist number fields $K = K(D, N)$ and $L = L(D, N)$ such that $X_0^D(N)_{/K}$ and $X_1^D(N)_{/L}$ violate the Hasse principle.*

Theorem 3. *Maintain the notation of the previous theorem, and assume $D{\cdot}N > C$.*
a) We may choose K such that $[K : \mathbb{Q}] \mid 4$.
b) Let $\{N_i\}$ be a sequence of squarefree positive integers tending to infinity, and for each i, choose any *squarefree positive integer $D_i > 1$ which is prime to N_i and such that $D_i \cdot N_i > C$. For all i, choose any number field L_i such that $X_1^{D_i}(N_i)_{/L_i}$ violates the Hasse Principle. Then* $\lim_{i\to\infty}[L_i : \mathbb{Q}] = \infty$.

We choose to view these results as special cases of a more general conjecture on algebraic curves defined over number fields. Let V be a nonsingular, geometrically irreducible variety defined over a number field K. If there exists a number field L/K such that $V_{/L}$ has points rational over every completion of L but no L-rational points, we say that $V_{/K}$ is *a potential Hasse principle violation* (or for brevity, "$V_{/K}$ is PHPV.")

$V_{/K}$ can only be PHPV if it has no K-rational points. Moreover, restricting to the class of curves, the case of genus zero must be excluded, by Hasse-Minkowksi. No further restrictions spring to mind, and we conjecture that there are none:

Main Conjecture. *Let C be a curve defined over a number field K. Then* at least one *of the following holds:*
(i) *C has genus* 0*;*
(ii) *C has K-rational points.*
(iii) *$C_{/K}$ is a potential Hasse principle violation.*

A proof of this conjecture in the general case may not be within current reach. However, using work of Faltings and Frey we will derive a criterion for $C_{/K}$ to be PHPV (Theorem 6). We use this criterion, together with an analysis of local points on Shimura curves and a result of Abramovich, to prove Theorems 1 through 3.

A warning: our method **does not** give an effective procedure for finding the field extensions of Theorems 1-3. This is to be contrasted with the work of [13] and [22].

1.2. **Acknowledgments.** I showed that there exist infinitely many Shimura curves $X_0^D(N)$ violating the Hasse principle over suitable quadratic fields in my 2003 Harvard thesis [4, Main Theorem 5], but for no good reason the work was not written up in a timely manner. Nevertheless this result was cited in a recent paper of V. Rotger, A. Skorobogatov and A. Yafaev [20]. It was their interest which convinced me to return to this work then and there, rather than at some indefinite future time. While writing things up properly I was able to find some improvements, and the present results are stronger and in some ways simpler than what appeared in [4]. I would like to take this opportunity – better late than never – to acknowledge the support of Harvard University and of my thesis advisor, Barry Mazur.

1.3. **The organization of the paper.** There are three main tools needed for the proof. In §2 we derive, using theorems of Faltings and Frey, a general criterion for an algebraic curve to be PHPV: it suffices for it to have no rational points, large gonality and local points of small degree.

In §3 we show that semistable Shimura curves have points of degree dividing 4 everywhere locally. This result follows readily enough from a description of their integral canonical models. Unfortunately I know of no unique, complete reference for this material. I have myself written first (my 2003 Harvard thesis) and second (notes from a 2005 ISM course in Montréal) approximations of such a work, and in so doing I have come to respect the difficulty of this expository problem. In this paper I have chosen to spend a few pages, but no more, summarizing this important material and giving sufficient (if not always ideal) references. Readers who have relatively little background in this material will, I hope, get a useful orientation, whereas even these few words will be sufficient for the wise.

Finally we need to recall and manipulate some lower bounds on genera and gonalities, especially a striking inequality of Abramovich; these are handled in §4.

In §5 these pieces are put together to prove Theorems 1-3.

After reflecting on the properties of Shimura curves used to prove the main theorems, it occurred to me that, to derive the qualitative results, one could get away with less precise information. Section §6 contains a more general theorem about families of semistable curves. This general result is applied to the family of semistable Shimura curves over all totally real fields. In point of fact much of the finer theory of §3 seems to go through verbatim over a totally real field, but the setup is technically more complicated and there *are* some differences: for instance, some of these curves have rational points over the ground field! Thus in the final section the focus shifts from constructing PHPV curves to giving further confirmatory evidence for our Main Conjecture.

2. Criteria for a Curve to be PHPV

Let $X_{/K}$ be a variety over a field K. Define the m-**invariant** $m(X) = m(X_{/K})$ to be the minimum degree of a finite field extension L/K such that $X(L) \neq \emptyset$.

For most of the remainder of this paper K shall denote a number field, and we write Σ_K for the set of all places of K. In this case we can consider the "global m-invariant" $m(X_{/K})$ and also the "local m-invariants" of the base extensions of X to the various completions of K. Indeed, a key feature of our strategy is to exploit discrepancies between global and local m-invariants, so we introduce further notation to facilitate this. Namely, for a place $v \in \Sigma_K$, we put

$$m_v(X) := m(X_{/K_v})$$

and

$$m_{\text{loc}}(X) = \text{lcm}_{v \in \Sigma_K} m_v(X).$$

Remark 2.1: Applying Bertini's theorem and the Weil bound for curves over finite fields, one gets the (well known) fact that $m_v(X) = 1$ for all but finitely many v, so m_{loc} is well-defined.

The K-*gonality* of a curve $C_{/K}$, denoted $d = d_K(C)$, is the least positive integer n for which there exists a degree n morphism $\varphi_{/K} : C \to \mathbb{P}^1$.

Proposition 4. *For any curve $C_{/K}$ we have*

$$m(C) \leq d_K(C). \tag{1}$$

Proof: The preimages of $\mathbb{P}^1(K)$ under a degree $d = d_K(C)$ morphism $\varphi : C \to \mathbb{P}^1$ yield infinitely many points P on C of degree at most d.

Of course the simple bound (1) holds for curves defined over an arbitrary field.

When K is a number field, we have the following (very much deeper!) result, which is, morally, a sort of converse.

Theorem 5. *Let $C_{/K}$ be a curve over a number field, and, for $n \in \mathbb{Z}^+$, let $\mathcal{S}_n(C)$ be the set of points $P \in C(\overline{K})$ of degree dividing n. If $\mathcal{S}_n(C)$ is infinite, then $d_K(C) \leq 2n$.*

Proof: This is a small variation on a theorem of G. Frey [8, Prop. 2], which itself is a quick consequence of G. Faltings' spectacular theorem on rational points on subvarieties of abelian varieties. In Frey's theorem, instead of $\mathcal{S}_n(C)$ there appears the set $C^{(d)}(K)$ of points of degree *less than or equal to* d. However, there also appears the extra hypothesis that there exists $P_0 \in C(K)$, which must be removed for our applications. The existence of P_0 is used (only) to define a map from the d-fold symmetric product $C^{(d)}$ to the Jacobian $\mathrm{Jac}(C)$, namely

$$\Phi : P_1 + \ldots + P_d \mapsto [P_1 + \ldots + P_d - dP_0].$$

However, if $\mathcal{S}_n(C)$ is infinite, it is certainly nonempty, so that for some $m \mid n$ there exists an effective K-rational divisor D_m of degree m, hence indeed an effective K-rational divisor of degree n, namely $D_n = \left(\frac{n}{m}\right) D_m$. Then one can define the map

$$\Phi_D : P_1 + \ldots + P_d \mapsto [P_1 + \ldots + P_d - D_n],$$

and Frey's argument goes through verbatim with Φ_D in place of Φ.

The following is the main result of this section.

Theorem 6. *Let $C_{/K}$ be an algebraic curve defined over a number field. Suppose:*
a) $C(K) = \emptyset$.
b) $d_K(C) > 2m > 2$ for some multiple m of $m_{\mathrm{loc}}(C)$.
Then there exist infinitely many extensions L/K with $[L : K] = m$ such that $C_{/L}$ is a counterexample to the Hasse principle.

Proof: By Theorem 5, $\mathcal{S}_m(C)$ is a finite set. It follows that the field M defined as the compositum of $K(P)$ as P ranges through elements of $\mathcal{S}_m(C)$, is a number field. Let L/K be a number field which is linearly disjoint from M/K and such that $[L : K] = m$. Since $C(K) = \emptyset$, it follows that $C(L) = \emptyset$.

Let $S = \{v_1, \ldots, v_r\}$ be the places of K for which $m_{v_i}(C) > 1$. By definition of $m_{\mathrm{loc}}(C) = \mathrm{lcm}_v\, m_v(C)$, for each finite place $v_i \in S$, there exists a field extension L_i/K_{v_i} of degree (m_{loc}, and *a fortiori* of degree) m such that $C(L_i)$ is nonempty. At each Archimedean place $v_i \in S$ (if any), we take L_i to be the $\mathbb{R}$-algebra $\mathbb{C}^{\frac{m}{2}}$. Let v_0 be any finite place not in S and unramified in M, and let L_0/K_{v_0} be a totally ramified extension of degree m. For $0 \leq i \leq r$ let $f_i \in K_{v_i}[x]$ be a defining polynomial for L_i/K_{v_i}. By weak approximation, for any $\epsilon > 0$, there exists a degree m polynomial $f \in K[x]$ such that, for all i, each coefficient of $f - f_i$ has v_i-adic norm at most ϵ, so by Krasner's Lemma, for sufficiently small ϵ, $L = K[x]/(f)$ defines a degree m field extension with $L \otimes_K K_{v_i} \cong L_i$ for $0 \leq i \leq r$. By construction $m_{\mathrm{loc}}(C_{/L}) = 1$; moreover, L/K is disjoint from M/K, so $C(L) = \emptyset$. By varying the choice of v_0 we clearly get infinitely many distinct fields L.

For $1 \leq n \leq 3$ there is a complete classification of algebraic curves which have infinitely many points of degree at most n (Faltings for $n = 1$, [9] for $n = 2$, [2] for $n = 3$). The quadratic case leads to the following "supplement" to Theorem 6.

Theorem 7. *Suppose $C_{/K}$ is a curve over a number field with $m_{\text{loc}}(C) = 2$, $d_K(C) > 2$, and C does not admit a degree two morphism $\varphi : C \to E$, where $E_{/K}$ is an elliptic curve of positive rank. Then there exist infinitely many quadratic field extensions L/K such that $C_{/L}$ is a counterexample to the Hasse principle.*

Proof: By the main result of [9], the hypotheses imply that $\mathcal{S}_2(C)$ is finite, and the rest of the proof is the same as that of Theorem 6.

3. Local points on Shimura curves

We recall our notation: D is the discriminant of a nonsplit indefinite rational quaternion algebra B_D, and is thus a nontrivial product of an even number of primes; N is a squarefree positive integer prime to D.

The goal of this section is to prove the following result.

Theorem 8. *a) For all D, $m_{\text{loc}}(X^D) = 2$.*
b) For all D and all N, $m_{\text{loc}}(X_0^D(N))$ is either 2 or 4.

Remark 3.1: Of course, for part b) one would like to know which of the two alternatives obtains. It is possible to give a precise answer for this; more exactly, there are several nice sufficient conditions for $m_{\text{loc}}(X_0^D(N)) = 2$ – this holds, e.g., for fixed D and all sufficiently large prime numbers N – and if none of these sufficient conditions hold there is a straightforward finite computation (coming from the Eichler-Selberg trace formula) that will determine the answer for any given pair (D, N) (cf. [22, Prop. 2.1]). We hope to return to the study of quadratic points on Shimura curves in a future work.

3.1. Integral structures.

In order to study p-adic points on Shimura curves, we will continually make use of the following result.

Lemma 9. *(Hensel's Lemma) Let R be a complete DVR with quotient field K and residue field k. Let $X_{/K}$ be a smooth algebraic variety. Suppose that $\mathcal{X}_{/R}$ is a(ny) regular model of $X_{/K}$. The following are equivalent:*
a) X has a K-rational point.
b) The special fiber $\mathcal{X}_{/k}$ has a smooth k-rational point.

Proof: For instance, this is [14, Lemma 1.1].

Clearly $m_{\text{loc}}(X_0^D(N)_{/\mathbb{Q}}) = 2$ if $X_0^D(N)$ has genus zero. If $X_0^D(N)$ has positive genus, then there exists a unique *minimal* regular $\mathbb{Z}$-model [11, Thm. 9.3.21], which we will denote by $X_0^D(N)_{/\mathbb{Z}}$.

On the other hand, work of Morita and Drinfeld gives an extension of the moduli problem for $X_0^D(N)$ to the category of schemes over $\mathbb{Z}$ and a coarse solution, which we shall denote by $M_0^D(N)_{/\mathbb{Z}}$. $M_0^D(N)_{/\mathbb{Z}}$ is projective, flat and of relative dimension one, but not necessarily smooth or even regular. For general N the relationship between $M_0^D(N)_{/\mathbb{F}_p}$ and $X_0^D(N)_{/\mathbb{F}_p}$ is quite complicated when $p \mid N$. As we shall recall, our assumption that N is squarefree implies a simple relationship between the two special fibers. (Actually, the fact that they are not the same shall work out in our favor.)

3.2. **Case of good reduction.** If $(p, DN) = 1$, $M_0^D(N)_{/\mathbb{Z}_p}$ is smooth. It follows that $M_0^D(N)_{/\mathbb{Z}_p} = X_0^D(N)_{/\mathbb{Z}_p}$.

As above, the special fiber $X_0^D(N)_{/\mathbb{F}_p}$ is again a moduli space of abelian surfaces $A_{/k}$ (where k is a field of characteristic p) endowed with a quaternionic structure, i.e., an injection $\mathcal{O} \hookrightarrow \operatorname{End} A$. However, in contrast to characteristic zero, where the generic QM surface is geometrically simple, *all* QM surfaces defined over the algebraic closure of $\mathbb{F}_p$ are isogenous to $E \times E$, where $E_{/k}$ is an elliptic curve [16]. Thus the full endomorphism ring of A depends on $\operatorname{End}(E)$. Most often, $\operatorname{End}(E)$ is an order in an imaginary quadratic field – in which case we say A (and its corresponding point on the moduli space) is *ordinary* – but there is a finite non-empty set of points for which the endomorphism ring of E is an order in a definite rational quaternion algebra (of discriminant p). The union of such points determines the *supersingular* locus on $X^D_{/\mathbb{F}_p}$, and the supersingular locus on $X_0^D(N)_{/\mathbb{F}_p}$ is (by definition) its complete preimage under the modular "forgetful" map $X_0^D(N) \rightarrow X^D$.

Proposition 10. *There exists at least one supersingular point on $X_0^D(N)_{/\mathbb{F}_p}$. Moreover, every supersingular point is defined over $\mathbb{F}_{p^2}$.*

Proof: This is well-known; see e.g. [17].

Corollary 11. *For p prime to ND, $m_p(X_0^D(N)) \leq 2$.*

Proof: By Proposition 10, $m(X_0^D(N)_{/\mathbb{F}_p}) \leq 2$. Since $X_0^D(N)$ is smooth over $\mathbb{Z}_p$, Hensel's Lemma implies that the m-invariant of the generic fiber is at most the m-invariant of the special fiber, and the result follows.

3.3. **Case of Cerednik-Drinfeld reduction.** For a positive integer a, we write $\mathbb{Q}_{p^a}$ for the unramified extension of $\mathbb{Q}_p$ of degree a and $\mathbb{Z}_{p^a}$ for its valuation ring.

If $p \mid D$, then there exists a Mumford curve $C(D, N)_{/\mathbb{Q}_p}$ whose base change to $\mathbb{Q}_{p^2}$ is isomorphic to $X_0^D(N)_{/\mathbb{Q}_{p^2}}$.

Recall that a Mumford curve $C_{/\mathbb{Q}_p}$ is an algebraic curve which can be uniformized by the p-adic upper half-plane. Equivalently, the special fiber of the minimal model is a semistable curve which is $\mathbb{F}_p$-split and degenerate: every component is isomorphic to $\mathbb{P}^1$, with transverse intersections at $\mathbb{F}_p$-rational points. The special fiber is thus completely determined by its *dual graph*, a finite graph in which the vertices correspond to the irreducible components C_i and the edges correspond to intersection points of C_i and C_j. Moreover, the degree of each vertex is at most $p+1$ (and would be exactly $p+1$ in the absence of ramification in the uniformization map).

Proposition 12. *If $p \mid D$, then $m_p(X_0^D(N)) \leq 2$.*

Proof: By Hensel's Lemma, it is enough to find a smooth $\mathbb{F}_{p^2}$-rational point on $X_0^D(N) \cong_{\mathbb{Z}_{p^2}} C(D, N)$. But any irreducible component C_i on $C(D, N)_{/\mathbb{F}_p}$ has at most $p + 1 = \#\mathbb{P}^1(\mathbb{F}_p)$ singular points, hence at least $p^2 + 1 - (p + 1) > 0$ smooth $\mathbb{F}_{p^2}$-rational points. The result follows.

3.4. **Case of Deligne-Rapoport reduction.** Suppose $p \mid N$.

Theorem 13. *The arithmetic surface $M_0^D(N)_{/\mathbb{Z}_p}$ has the following structure:*
a) The special fiber $M_0^D(N)_{/\mathbb{F}_p}$ has two irreducible components, each isomorphic to

the smooth curve $M_0^D(\frac{N}{p})_{/\mathbb{F}_p}$.
b) The two irreducible components intersect transversely at the supersingular points, a corresponding point on the first copy of $M_0^D(\frac{N}{p})_{/\mathbb{F}_p}$ *being glued to its image under the Frobenius map.*
c) At each supersingular point $z \in M_0^D(N)_{/\mathbb{F}_{p^2}}$*, the complete local ring is isomorphic to* $\mathbb{Z}_{p^2}[[X,Y]]/(XY - p^{a_z})$ *for some positive integer* a_z*.*

Proof: The analogous statement for $D = 1$ (i.e., classical modular curves) is due to Deligne and Rapoport [7]. In that same work it was pointed out that their result would continue to hold in the quaternionic context. Apparently the first careful treatment of the quaternionic case is due to Buzzard [3], who worked however under the assumption of some additional rigidifying level structure. The case of $X_0^D(N)$ (which is not a fine moduli scheme) was worked out (independently) in the theses of the author [4, §0.9] and of David Helm [10, Appendix].

Proposition 14. *For any prime* $p \mid N$, $m_p(X_0^D(N)) \leq 4$.

Proof: Consider $M_0^D(N)_{\mathbb{F}_{p^2}}$. Let z be any point on this curve coming from a supersingular point of $M_0^D(\frac{N}{p})(\mathbb{F}_{p^2})$. The completed local ring at z is isomorphic to $\mathbb{Z}_{p^2}[[x,y]]/(xy - p^a)$ for some integer $a \geq 1$.

Suppose first that $a > 1$. Then in order to get the minimal regular model, one must blow up the point z $(a-1)$ times, getting a chain of $a-1$ rational curves defined over $\mathbb{F}_{p^2}$. Each of these curves has p^2+1-2 smooth $\mathbb{F}_{p^2}$-rational points, which lift to give points on $X_0^D(N)$ rational over $\mathbb{Q}_{p^2}$, the unramified quadratic extension.

Now assume that we have $a = 1$. Let $K = \mathbb{Q}_{p^2}(\sqrt{p})$, with valuation ring R and uniformizing element $\pi = \sqrt{p}$. Then the completed local ring at z of $M_0^D(N)_{/R}$ is isomorphic to $R[[x,y]]/(xy - \pi^2)$. We see that this local ring is no longer regular and must be blown up once to give a rational curve on the regular model. Thus, as in the previous case, we get a K-rational point on $X_0^D(N)$, completing the proof.

Proof of Theorem 8: From Propositions 11, 12 and 14, we get $m_{\text{loc}}(X_0^D(N)) \mid 4$. Sinc $X_0^D(N)(\mathbb{R}) = \emptyset$, $m_{\text{loc}}(X_0^D(N)) > 1$, and the result follows.

4. Lower bounds on the gonality and genus

4.1. **A result of Abramovich.** Let $\mathcal{O}^1$ be the group of norm 1 units in a maximal $\mathbb{Z}$-order $\mathcal{O}$ of B_D. By restricting any embedding $B \hookrightarrow B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$ to $\mathcal{O}^1$, one gets a realization of $\mathcal{O}^1$ as a discrete cocompact subgroup of $SL_2(\mathbb{R})$, well-determined up to conjugacy. Then $\mathcal{O}^1 \backslash \mathcal{H}$ is a compact Riemann surface isomorphic to $X^D(\mathbb{C})$. For any finite index subgroup $\Gamma \subset \mathcal{O}^1$, we get a covering $X^D(\Gamma) \to X^D$. Recall that Γ is said to be a *congruence subgroup* if it is of the form $G(\mathbb{Q}) \cap U$, where $G = B^\times$ viewed as a linear algebraic group over $\mathbb{Q}$ and U is a compact open subgroup of $G(\hat{\mathbb{Z}} \otimes \mathbb{Q})$. Equivalently, for any positive integer N one defines as in the classical modular case the principal congruence subgroup $\Gamma^D(N)$, and the congruence subgroups are those containing $\Gamma^D(N)$ for some N. In particular $\Gamma_0^D(N)$ and $\Gamma_1^D(N)$ are congruence subgroups.

We are now ready for the last key ingredient.

Theorem 15. *(Abramovich) Suppose that $\Gamma \subset \mathcal{O}^1$ is a congruence subgroup. Then*

$$\frac{21}{200}(g(X^D(\Gamma)) - 1) \leq d_{\mathbb{C}}(X^D(\Gamma)). \tag{2}$$

Proof: This is Theorem 1.1 of [1].

4.2. **Genus estimates.** In this section, all asymptotics are as $\max(D, N) \to \infty$.

For coprime squarefree positive integers A and N, define

$$e_2(D, N) = \prod_{p \mid D}(1 - (-4/p)) \prod_{q \mid N}(1 + (-4/q)),$$

$$e_3(D, N) = \prod_{p \mid D}(1 - (-3/p)) \prod_{q \mid N}(1 + (-3/q)).$$

Let φ be Euler's function, i.e., the multiplicative function such that $\varphi(p^k) = p^k - p^{k-1}$. Let ψ be the multiplicative function such that $\psi(p^k) = p^k + p^{k-1}$.

Then we have [17, p. 301]:

$$g(X_0^D(N)) = 1 + \frac{1}{12}\varphi(D)\psi(N) - \frac{e_2(D, N)}{4} - \frac{e_3(D, N)}{3} \sim \frac{\varphi(D)\psi(N)}{12}. \tag{3}$$

In the case of $X_1^D(N)$ we will content ourselves with a lower bound for the genus. Indeed, we will work instead with the Shimura covering $X_2^D(N)$, which is by definition the largest intermediate covering

$$X_1^D(N) \to Y \to X_0^D(N)$$

such that $Y \to X_0^D(N)$ is unramified abelian, say with Galois group $\Sigma(D, N)$.[1] From [15, Corollary 1], we get

$$\#\Sigma(D, N) \geq \frac{\varphi(N)}{2 \cdot \prod_{q|N} 6}. \tag{4}$$

Because $X_2^D(N) \to X_0^D(N)$ is unramified, we have

$$g(X_2^D(N)) - 1 = \#\Sigma(D, N)(g(X_0^D(N)) - 1), \tag{5}$$

and combining equations (2) through (4), we get

$$g(X_1^D(N)) - 1 \geq g(X_2^D(N)) \gg \varphi(N) \cdot \frac{\varphi(D)}{24} \cdot \prod_{q \mid N} \frac{q+1}{6}.$$

5. Proofs of the main theorems

Consider first the case of $D = 1$. According to Theorem 8, $m_{\mathrm{loc}}(X^D) = 2$. Moreover, by [19], X^D has only finitely many quadratic points when $D > 546$. Thus Theorem 7 applies to show that for such D, there exist infinitely many quadratic fields K such that $X^D_{/K}$ violates the Hasse principle.

Next consider the case of $X_0^D(N)$. By Theorem 8 we have $2 \mid m_{\mathrm{loc}}(X_0^D(N)) \mid 4$; in particular, $X_0^D(N)(\mathbb{Q}) = \emptyset$. Applying Theorem 6 with $m = 4$, we get that whenever the gonality $d_{\mathbb{Q}}(X_0^D(N)) > 8$, there exists infinitely many quartic fields $K/\mathbb{Q}$ such

[1]We introduce the Shimura covering only to save a couple of lines of messy calculation with the Riemann-Hurwitz formula.

that $X_0^D(N)_{/K}$ violates the Hasse principle. But by (1) and (2), $d_{\mathbb{Q}}(X_0^D(N)) \geq d_{\mathbb{C}}(X_0^D(N)) \gg \varphi(D)\psi(N)$. Since the latter quantity approaches infinity with $\min(D,N)$, the gonality condition holds for all but finitely many pairs (D,N).

To deduce the $X_1^D(N)$ case, we need the following easy result.

Lemma 16. *Let K be a number field and $f_{/K} : X_1 \rightarrow X_2$ be a degree d Galois covering of curves. Then $m_{\text{loc}}(X_1) \mid d \cdot m_{\text{loc}}(X_2)$.*

Proof: First suppose that $f : X_2 \rightarrow X_1$ is a Galois covering of curves defined over any field K, and let $P \in X_1(L)$ with $[L : K] = m(X_1)$. Recall that the transitivity of the action of $\text{Gal}(L(X_2)/L(X_1))$ on the points $\{Q_1, \ldots, Q_g\}$ of X_2 lying over P implies the relation $efg = d$, where e is the relative ramification index of Q_i over P and $f = [L(Q_i) : L]$. Thus $[L(Q_i) : K] = fm(X_1) \mid dm(X_1)$. The result follows easily by applying this observation at every place of K.

Proposition 17. $m_{\text{loc}}(X_1^D(N)) \mid 2\varphi(N)$.

Proof: The natural map $X_1^D(N) \rightarrow X_0^D(N)$ is a Galois covering with group $(\mathbb{Z}/N\mathbb{Z})^\times/(\pm 1)$. Thus the result follows from Theorem 8 and Lemma 16.

Now

$$d_{\mathbb{Q}}(X_1^D(N)) \geq d_{\mathbb{C}}(X_1^D(N)) \geq \frac{21}{200}(g(X_1^D(N) - 1) \geq \frac{21}{200}(g(X_2^D(N)) - 1)$$
$$\gg \varphi(N)\frac{\varphi(D)}{24}\prod_{q\ \mid N}\frac{q+1}{6},$$

so that with $m = 2\varphi(N)$, we have

$$\frac{d_{\mathbb{Q}}(X_1^D(N))}{m} \gg \varphi(D)\prod_{q\ \mid\ N}\frac{q+1}{6}.$$

This last quantity approaches ∞ with $\max(D,N)$, so exceping finitely many (D,N), there is a field L of degree $2\varphi(N)$ such that $X_1^D(N)_{/L}$ violates the Hasse principle.

Finally, Theorem 3b) is an immediate consequence of the following:

Theorem 18. *For each prime number p and positive integer $d \geq 1$, there exists a constant $N_0 = N_0(p,d)$ with the following property: for any p-adic field $K/\mathbb{Q}_p$ with $[K : \mathbb{Q}_p] \leq d$ and any integer $N \geq N_0$, $X_1^D(N)(K) = \emptyset$.*

Proof: This is [6, Theorem 1].

Indeed, it follows that $m_2(X_1^D(N)) \rightarrow \infty$ with N, uniformly in D.

6. Complements

6.1. **Some remarks.**

Remark 6.1: The constant C in Theorem 1 can be made explicit; we leave it to the interested reader to do so.

Remark 6.2: More interesting (to me) than whittling down the set of excluded pairs

(D, N) to the optimal list which violate the inequality $d_{\mathbb{Q}}(X_0^D(N)) > 2m_{\mathrm{loc}}(X_0^D(N))$ is to investigate the cases of our Main Conjecture among Shimura curves of low genus. For instance, suppose $X_0^D(N)$ has genus one. All such curves can be given in the form $y^2 = P(x)$, where $P \in \mathbb{Q}[x]$ has degree 4. For example, X^{14} is given by the equation $(x^2 - 13)^2 + 7^3 + 2y^2 = 0$. It is not hard to see that $X^{14}_{/\mathbb{Q}(\sqrt{m})}$ has points everywhere locally if and only if m is negative and prime to 7, so that the set of such m has density $\frac{3}{7}$ (as a subset of the set of all squarefree integers). What can be said about the set of m for which $X^{14}(\mathbb{Q}(\sqrt{m}))$ is nonempty? Or even about its density?

Remark 6.3: Lower bounds on the $\mathbb{Q}$-gonality of modular curves and Shimura curves $X_0^D(N)$ were derived earlier by A. Ogg [18] by counting supersingular points, a method which is much closer in spirit to the arguments given here. His bounds are of the form $\gg g^{\frac{1}{2}}$ rather than $\gg g$. Moreover these arguments are made in the context of $\Gamma_0(N)$-level structure, but according to M. Baker (private communication) they can be adapted to the general case.

6.2. **A partial generalization of the main theorems.** One needs to know relatively little about the Shimura curves $X_0^D(N)$ beyond their semistability to show that almost all of them satisfy our Main Conjecture. Indeed, we have the following "abstract" version of our Main Theorem.

Notation: Denote by R_v the ring of integers of the completion of a number field K at a finite place v.

Theorem 19. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of curves over a number field K, each of genus $g(X_n) > 1$. Suppose:*
(i) Each X_n has everywhere semistable reduction.
(ii) $\lim_{n\to\infty} \frac{d_K(X_n)}{\log g(X_n)} = \infty$.
(iii) There exists a fixed positive integer A such that for all places v and all n, the Galois action on the irreducible components of the special fiber of the R_v-minimal model of X_n trivializes over an extension of degree A.
Then X_n satisfies the Main Conjecture for all sufficiently large n.

Remark 6.4: If we replace (ii) with the stronger condition

(ii') $\lim_{n\to\infty} \frac{d(X_n)}{\log g(X_n)} = \infty$,

then all the hypotheses of the theorem are stable under finite base change. Recall that (ii') is satisfied for Shimura curves by the main result of [1].

Proof of Theorem 19: In verifying our Main conjecture, we may, and shall, assume that $X_n(K) = \emptyset$ for all n. At the heart of the matter is the following estimate.

Lemma 20. *Let C be a smooth, projective curve of positive genus g defined over a p-adic field F, with **split** semistable reduction. Suppose that $d \geq 2\log_2(4g + 2)$ is a positive integer. Then there exists an unramified degree d extension F_d of F with $C(F_d) \neq \emptyset$.*

Let us first verify the sufficiency of the lemma, which is easy. First, using weak approximation / Krasner's Lemma as usual, (iii) implies that there exists a field extension L/K of degree A such that for all n, $(X_n)_{/L}$ has *split* semistable reduction.

Using Lemma 20 and another Krasner's Lemma argument, we get that there is a field extension M/L of degree $\lceil 2\log_2(4g(X_n)+2)\rceil$ such that $(X_n)_{/M}$ has points everywhere locally. So overall X_n acquires points everywhere locally over a field extension of degree

$$m_n := A\lceil 2\log_2(4g(X_n)+2)\rceil.$$

Moreover, hypothesis (ii) implies that

$$d_K(X_n) > 2m_n$$

for all sufficiently large n, so using Theorem 5 as in the proof of Theorem 6, we can construct infinitely many degree m_n field extensions of K over which X_n violates the Hasse principle.

Proof of Lemma 20: Let $\mathbb{F}_q$ be the residue field of F. By Hensel's Lemma (Lemma 9), if $C_{/\mathbb{F}_q}$ has a nonsingular point of degree d, then so does C/F. We claim that there is a nonsingular point whose degree is logarithmically small in $g := g(C)$.

To gain confidence, let us first consider the case where $C_{/\mathbb{F}_q}$ is smooth. Then, applying Weil's bounds, for all $d \geq 1$,

$$\#C(\mathbb{F}_{q^d}) \geq q^d + 1 - 2gq^{\frac{d}{2}} > q^{\frac{d}{2}}(q^{\frac{d}{2}} - 2g).$$

The latter quantity is non-negative if $\frac{d}{2} \geq \log_q(2g)$, so certainly if $d \geq 2\log_2(4g+2)$.

Next suppose C is completely degenerate, i.e., every irreducible component has genus zero. Then g can be computed purely in terms of the dual graph: if $v = \#V(C)$, $e = \#E(C)$, then $g = e - v + 1$. The average vertex degree of the graph is $k = \frac{2\#E}{\#V}$. Certainly there is at least one vertex of degree at most k, meaning that there is at least one component C_0 of C which has at most k geometric singular points. Since $\#C_0(\mathbb{F}_{q^d}) = q^d + 1$, C_0 has a smooth point of degree d as soon as $q^d + 1 > k$. But if $d \geq \log_2(4g)$, then

$$q^d + 1 \geq 2^d + 1 \geq 4g + 1 > 2g + 2 > \frac{2g}{v} + 2 - \frac{2}{v} = \frac{2g + 2v - 2}{v} = \frac{2e}{v} = k.$$

Now for the general case, in which the special fiber consists of several irreducible components whose normalizations may have arbitrary genus. We still have a dual graph $\mathcal{G}$, with v vertices and e edges, and if we compare our curve $C_{/\mathbb{F}_q}$ to the degenerate curve C' with corresponding dual graph $\mathcal{G}$, we have [11, Lemma 10.3.18]

$$g' := g(C') \leq g = g(C).$$

So as above there is at least one component, say C_0, of C with at most $k = \frac{2e}{v}$ singular geometric points; moreover, $g(C_0) \leq g(C)$ [11, ibid]. So we can put together the above arguments, as follows: we want to select d such that

$$N_d := \#C_0(\mathbb{F}_{q^d}) > k = \frac{2e}{v}.$$

Since

$$k < 2g' + 2 \leq 2g + 2,$$

it will suffice to take $N_d \geq 2g + 2$; and since $q^{\frac{d}{2}} \geq 1$, for this suffices to have

$$q^{\frac{d}{2}} - 2g \geq 2g + 2,$$

which in turn will be satisfied if

$$d \geq 2\log_2(4g + 2).$$

Remark 6.5: Although we were apparently not very economical in our estimates, nevertheless $\log g$ is the correct order of magnitude in the result: for fixed q and $N \geq 4$, the Shimura curve $X_1^D(N)_{/\mathbb{F}_q}$ is a fine moduli space for certain order N torsion points on abelian surfaces $A_{/\mathbb{F}_q}$ with $A \sim E^2$ (cf. §3.2). Since $\#A(\mathbb{F}_{q^d}) = \#E(\mathbb{F}_{q^d})^2 \leq (q^d + 1 + 2\sqrt{q^d})^2 \leq 9q^{2d}$, say, if $X_1^D(N)(\mathbb{F}_{q^d}) \neq \emptyset$ we have $d \geq \frac{\log_q(N/9)}{2}$. Moreover, the genus of $X_1^D(N)$ grows quadratically as $N \to \infty$ through prime values. In other words, for each fixed finite field $\mathbb{F}_q$, there is a sequence of curves $(C_n)_{/\mathbb{F}_q}$ with genera tending to infinity and $m(C_n) \gg \log g(X_n)$.[2]

The hypotheses of Theorem 19 are satisfied, with $A = 2$, for the family of all semistable Shimura curves over a fixed totally real field F. (For general F we shall regard these curves as being defined over the maximal abelian extension of F which is unramified at every finite place. It is sometimes possible to define the curves over a smaller field than this, but by Remark 6.4 any "extra" base extension is at any rate harmless.) In fact, [12, §4] shows that in the countable collection of *all* quaternionic Shimura curves – i.e., taken over all totally real fields at once! – the genera tend to infinity. We deduce:

Corollary 21. *The set of counterexamples to the Main Conjecture among all semistable Shimura curves over all totally real fields is finite.*

Remark 6.6: For fixed F and D, say with $X^D(1)(F) = \emptyset$, I believe that the proof of Theorem 3 should go through, showing that all but finitely many curves of the form $X_1^D(\mathcal{N})_{/F}$, with squarefree level $\mathcal{N}$, are PHPV. Some further ideas seem necessary to show that the Main Conjecture holds for all but (possibly) a finite set of all Shimura curves associated to arithmetic Fuchsian groups of congruence type, but the problem seems to be a tractable one: it might make a reasonable thesis problem.

Remark 6.7: Every curve achieves semistable reduction over a suitable finite degree field extension. So in place of hypothesis (i) it suffices to require only that the curves X_n attain semistable reduction over a sequence of field extensions of uniformly bounded degree. This still seems like a rather strong condition, i.e., unlikely to be satisfied by a "typical" sequence of curves.

Remark 6.8: In contrast, hypothesis (ii) looks quite weak: generically on the moduli space $\mathcal{M}_g$ of genus g curves even the geometric gonality is on the order of g. The final hypothesis, (iii), is the most technical. When this paper was first written, I was optimistic that this condition might be superfluous: I did not know any family of semistable curves that violated it. I have since learned better: in [5] I construct, for every genus $g \neq 1$, a genus g curve over a local field with (the largest possible) m-invariant $|2g-2|$ by lifting curves over the residue field which are degenerate, with $2g-2$ rational components which are *transitively* permuted by Galois. This shows the importance of the word "split" in Lemma 20.

Acknowledgement: I am grateful to D. Lorenzini for a careful reading of this final section (written more than a year after the rest of the paper) and to J. Voight for making me aware of [12], which was not published until after this paper was first submitted.

[2] This observation was made jointly with Noam Elkies in 2002.

References

[1] D. Abramovich, *A linear lower bound on the gonality of modular curves*, Internat. Math. Res. Notices 1996, 1005-1011.

[2] D. Abramovich and J. Harris, *Abelian varieties and curves in $W_d(C)$*, Compositio Math. 78 (1991), 227-238.

[3] K. Buzzard, Integral models of certain Shimura curves, *Duke Math Journal*, Duke Math. J. 87 (1997), 591-612.

[4] P.L. Clark, *Rational points on Atkin-Lehner quotients of Shimura curves*, 2003 Harvard thesis.

[5] P.L. Clark, *On the index of curves over local fields*, submitted for publication.

[6] P.L. Clark and X. Xarles, *Local bounds for torsion points on abelian varieties*, to appear in the Canadian J. of Math.

[7] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques,* in: A. Dold and B. Eckmann (eds.) Modular Functions of One Variable II, Lecture Notes in Math. 349, Springer-Verlag, New York, (1973), 143-316.

[8] G. Frey, *Curves with infinitely many points of fixed degree*, Israel J. Math. 85 (1994), 79-83.

[9] J. Harris and J. Silverman, *Bielliptic curves and symmetric products*, Proc. Amer. Math. Soc. 112 (1991), 347-356.

[10] D. Helm, *On maps between modular Jacobians and Jacobians of Shimura curves*, submitted for publication.

[11] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics 6, 2002.

[12] D. Long, C. Maclachlan and A. Reid, *Arithmetic Fuchsian Groups of Genus Zero*, Applied Math. Quaterly (Special Issue: In honor of John H. Coates, Part 2 of 2), 459-489, 2006.

[13] B.W. Jordan, *Points on Shimura curves rational over number fields*, J. Reine Angew. Math. 371 (1986), 92-114.

[14] B.W. Jordan and R. Livné, *Local Diophantine properties of Shimura curves*, Math. Ann. 270 (1985), 235-248.

[15] S. Ling, *Shimura subgroups of Jacobians of Shimura curves*, Proc. Amer. Math. Soc. 118 (1993), 385-390.

[16] J. Milne, *Points on Shimura varieties mod p*. Proc. Symp. Pure Math. XXXIII (1979), Part 2, 165-184.

[17] A. Ogg, *Real points on Shimura curves*, Progr. Math. 35 (1983), Vol. I, 277-307.

[18] A. Ogg, *Hyperelliptic modular curves*, Bull. Math. Soc. France 102 (1974), 449-462.

[19] V. Rotger, *On the group of automorphisms of Shimura curves and applications*, Compositio Math. 132 (2002), 229-241.

[20] V. Rotger, A. Skorobogatov and A. Yafaev, *Failure of the Hasse principle for Atkin-Lehner quotients of Shimura curves over* $\mathbb{Q}$, Moscow Math. J. 5 (2005).

[21] A. Skorobogatov, *Shimura coverings of Shimura curves and the Manin obstruction*, Math. Research Letters 12 (2005), 779-788.

[22] A. Skorobogatov and A. Yafaev, *Descent on certain Shimura curves*, Israel J. Math. 140 (2004), 319-332.

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