

MUMFORD-TATE GROUPS AND ABELIAN VARIETIES

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1. INTRODUCTION

These are notes for a lecture in Elham Izadi's 2006 VIGRE seminar on the Hodge Conjecture.

Let us recall what we have done so far:

1. We have developed an algebraic formalism of \mathbb{Q} -Hodge structures. I like to think first of an \mathbb{R} -Hodge structure as simply being a finite-dimensional \mathbb{R} -vector space $V_{/\mathbb{R}}$ together with a representation h of the real algebraic group \mathbb{C}^\times (or equivalently, an “abstract Hodge decomposition” $V_{/\mathbb{C}} = \bigoplus_{(p,q) \in \mathbb{Z}^+} V^{p,q}$ with the additional property that $\overline{V^{(q,p)}} = V^{p,q}$); and second that a \mathbb{Q} -Hodge structure is just a finite-dimensional \mathbb{Q} -vector space V together with an \mathbb{R} -Hodge structure on $V \otimes_{\mathbb{Q}} \mathbb{R} = V_{/R}$.

We gave a complete description of the category of \mathbb{R} -Hodge structures: an \mathbb{R} -Hodge structure is completely determined by the dimensions $v^{p,q}$ of the (p,q) -subspaces $V^{(p,q)}$, and the only constraint is the obvious symmetry condition $v^{(q,p)} = v^{(p,q)}$.

2. We justified our consideration of \mathbb{Q} -Hodge structures in the sense that geometry gives us a large class of them – the cohomology groups $H^k(X, \mathbb{Q})$ of a compact Kähler manifold, and especially the case of a projective variety. In this case one can produce Hodge classes from algebraic cycles, and the main problem is to understand whether or not the image of this cycle class map spans, over \mathbb{Q} , all Hodge classes.

Maybe I should point out that it has not yet been made completely clear that the study of (abstract) \mathbb{Q} -Hodge structures is itself so useful in proving the Hodge conjecture. But the philosophy seems to be to take a maximally functorial perspective – i.e., to study the Hodge conjecture not just on one variety at a time but to exploit maps between varieties – and to do this one wants to put the Hodge structures themselves into a nice algebraic category.

In today's lecture I will first introduce the Mumford-Tate group $MT(V)$ of a \mathbb{Q} -Hodge structure. This is our first nontrivial *invariant* of V , and it is already enough to show that \mathbb{Q} -Hodge structures are much richer objects than \mathbb{R} -Hodge structures. The point is that the Hodge classes arise as invariants of the Mumford-Tate group, so when $MT(V)$ is a large group, there are relatively few Hodge classes, and the Hodge conjecture is easy to prove. This points out an important feature of the Hodge conjecture: morally speaking it does hold generically – even for compact

Kähler manifolds – because generically there are not enough Hodge classes to make the conjecture interesting. Thus, if we are interested in exploring new cases of the Hodge conjecture, an excellent start is constructing Hodge structures with interesting Mumford-Tate groups. This is, as I understand it, one of the main merits of the Kuga-Satake construction: it takes as input a weight two Hodge structure V of a certain type, but with no restrictions on $MT(V)$, and returns a weight one Hodge structure $KS(V)$ whose Mumford-Tate group is contained in an interesting subgroup of the largest possible group.

In fact Mumford-Tate groups seem to be most useful by far in the case of Hodge structures of type $(1,0) + (0,1)$, namely abelian varieties. Thus a major goal of this lecture is to introduce abelian varieties from the Hodge-theoretic point of view. The study of abelian varieties is certainly one of the oldest and richest branches of algebraic geometry (and is especially popular here at UGA), but in Hodge theory abelian varieties have a distinguished role to play – in some sense Hodge theory is a formal algebraic generalization of the theory of abelian varieties, and the miracle is that this “formal” generalization itself carries a lot of geometric content.

2. THE MUMFORD-TATE GROUP OF A POLARIZED \mathbb{Q} -HODGE STRUCTURE

To every polarized \mathbb{Q} -Hodge structure V we will associate a nontrivial “invariant,” the Mumford-Tate group, which is, in some sense, measuring the number of Hodge classes in V together with all of its tensor powers.

Let (V, h) be a \mathbb{Q} -Hodge structure, of weight k , with a polarization Ψ . In particular Ψ gives a bilinear form on the underlying \mathbb{Q} -vector space V , so defines a linear group

$$G(\Psi) = \{g \in GL(V) \mid \Psi(gv, gw) = \nu(g)\Psi(v, w)\};$$

here $\nu(g) \in \mathbb{G}_m$ is a scalar which is allowed to depend upon g (but not, of course, on v and w). $G(\Psi)$ is a \mathbb{Q} -linear algebraic group. If k is even, then Ψ is a quadratic form and $G(\Psi) = GO(\Psi)$ is (by definition) the associated general orthogonal group of Ψ – note that this group does in general depend upon the choice of Ψ . However, k is odd, Ψ is a symplectic form, and $G(\Psi) = GSp(V)$.

Definition: Let G_1 be the algebraic subgroup of $G(\Psi)$ consisting of elements g which act as follows on Hodge classes $t \in B(V^{\otimes m})$: $g \cdot t = \omega(g)^m t$ for some $\omega(g) \in \mathbb{G}_m$.

Remark: It follows immediately from the definitions that $\mathbb{G}_m \subset G_1 \subset G(\Psi)$.

Let G_2 be the smallest algebraic subgroup of $GL(V)$ which is defined over \mathbb{Q} and satisfies $h(\mathbb{C}^\times) \subset G_2(\mathbb{R})$.¹

Theorem 1. *Let (V, h, Ψ) be a weight k polarized \mathbb{Q} -Hodge structure. Then*

- a) $G_1 = G_2$.*
- (b) $MT(V)$ is a reductive linear group.*

¹We remark that this definition makes no mention of the polarization, so could be used to define Mumford-Tate groups for nonpolarizable Hodge structures. But while the existence of a polarization is not required to define G_2 , it has important implications for its structure.

(c) The Hodge classes in $V^{\otimes m}$ are the (twisted) $MT(V)$ -invariants:

$$B(V^{\otimes m}) = \{w \in V^{\otimes m} \mid gw = \omega(g)^m w \quad \forall g \in MT(V)\}.$$

(Here $\omega(g)$ is an element of \mathbb{G}_m which is allowed to depend on G .) We write $MT(V)$ for $G_1 = G_2$ and call it the Mumford-Tate group of (V, h) .

For the proof, see [DMOS].

Remark: For a field K of characteristic 0, a reductive algebraic K -group is a linear algebraic K -group – i.e., a Zariski-closed subgroup of $GL_n(K)$ for some n – for which every finite-dimensional K -linear representation of G is semisimple (a.k.a. completely reducible – i.e., decomposes as a direct sum of irreducible representations).²

Better perhaps than trying to understand what $MT(V)$ “really is” is to consider what it means for $MT(V)$ to be larger or smaller as a linear algebraic group. The largest that $MT(V)$ could be is $G(\Psi)$, the generalized orthogonal group of the polarization. It is important to observe that with regard to the Hodge conjecture – i.e., if $V = H^k(X, \mathbb{Q})$ – this is essentially the *trivial* case: one can check that in this case the only nontrivial Hodge classes are the ones coming from the polarization – so, roughly, are all derived from the one codimension one cycle we must have on X in order to be able to embed it into projective space.³ So the interesting case is when $MT(V)$ is a proper subgroup of $G(\Psi)$. Morally, the smaller $MT(V)$ is, the more Hodge classes we have, and the more interesting (and difficult) it is to verify the Hodge conjecture.⁴

If V is a finite-dimensional \mathbb{Q} -vector space and $G \subset GL(V)$ is any subgroup (in other words, G is some group acting effectively on V by linear automorphisms), then by $\text{End}_G(V)$ we mean the set of endomorphisms α of V which commute with every element of g : formally, it is the set of all α such that for $g \in G$ and $v \in V$, $\alpha gv = g\alpha v$. $\text{End}_G(V)$ is easily seen to be a subalgebra, and the relationship between G and $\text{End}_G(V)$ is of basic importance in representation theory.

Corollary 2. *We have $\text{End}_H(V) \cong \text{End}_{MT}(V)$.*

Proof: Using Ψ to identify V and V^\vee , we have $\text{End}(V) = V^\vee \otimes V \cong V \otimes V$, so $\text{End}_H(V) \cong B(V^{\otimes 2})$, whereas $\text{End}_{MT}(V)$ is the space of $MT(V)$ -invariants in $V^{\otimes 2}$.

Already we come to the end of the general theory of Mumford-Tate groups. To say more we would like to specialize to the case of $V = H^1(A, \mathbb{Q})$ for A/\mathbb{C} an abelian variety. But to do this we should first say something about abelian varieties!

²More formally this is a nice *characterization* of reductive groups in characteristic 0; it fails in positive characteristic, and the correct definition in all characteristics is that $G \subset GL(V)$ admits no nontrivial, normal, connected subgroup of unipotent matrices (matrices having all eigenvalues equal to 1).

³I confess that I have in fact *not* checked this, although it seems to me that it must be true. I hope to get some confirmation from Elham.

⁴This philosophy is perhaps not quite correct in the case when the Hodge group is commutative; one says (with good reason!) that the Hodge structure is of CM-type. The Hodge Conjecture remains open even in this case, but it is presumably easier than the general case.

3. HODGE-THEORETIC INTRODUCTION TO ABELIAN VARIETIES

3.1. \mathbb{Z} -Hodge structures. To see abelian varieties as a special case of Hodge structures, let us introduce the notion of a \mathbb{Z} -Hodge structure. So:

Definition: A weight k \mathbb{Z} -Hodge structure is a finitely generated free \mathbb{Z} -module Λ together with a weight k representation of \mathbb{C}^\times on $V_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Note that we may naturally view Λ as a subgroup of $V_{\mathbb{R}}$; it is a full lattice (meaning e.g. that it is a discrete, cocompact subgroup).

For example, we get a \mathbb{Z} -Hodge structure by taking the integral cohomology group $H^k(X, \mathbb{Z})$ of a compact Kähler manifold modulo torsion.

A polarization on an \mathbb{Z} -Hodge structure is a morphism of \mathbb{Z} -Hodge structures $\Psi : \Lambda \times \Lambda \rightarrow \mathbb{Z}(-k)$, i.e., a homomorphism of abelian groups satisfying

$$\Psi(h(z)v, h(z)w) = (z\bar{z})^k \Psi(v, w)$$

such that the bilinear form $(v, w) \mapsto \Psi(v, h(i)w)$ on $\Lambda \otimes \mathbb{R}$ is symmetric and positive definite. We will say that a Hodge structure (integral or rational) is *polarizable* if it admits some polarization.

Proposition 3. (*Poincaré Complete Reducibility*) *The polarizable \mathbb{Q} -Hodge structures form a semisimple \mathbb{Q} -linear category.*

Explanation: The \mathbb{Q} -linear category just means that we have an underlying \mathbb{Q} -vector space structure, which we most certainly do. The semisimplicity means that every object in the category is a direct sum of simple (or irreducible) objects. As usual, this means that every sub-Hodge structure $W \subset V$ is a direct summand.

Sktech proof: The idea is a standard one: assuming the existence of a polarization Ψ on V allows us to take “perps” – i.e., taking W^\perp to be the set of all $v \in V$ such that $\Psi(v, w) = 0$ for all $w \in W$. Then one shows that $V = W \oplus W^\perp$ as \mathbb{Q} -Hodge structures.

Remark: Although we shall not digress to give an explicit example, the category of not-necessarily polarizable \mathbb{Q} -Hodge structures is not semisimple. This is one of many reasons to include polarizations.

Remark: Moreover, even the category of polarizable \mathbb{Z} -Hodge structures is not semisimple. Indeed, the special case of weight 0 is just the category of finite free \mathbb{Z} -modules, and it is an elementary observation that submodules of free abelian groups need not be direct summands: consider for instance $2\mathbb{Z} \subset \mathbb{Z}$!

3.2. Complex tori and abelian varieties. Let us define a Hodge structure (integral or rational) to be *of abelian type* if $V^{p,q} = 0$ unless $(p, q) = (1, 0)$ or $(0, 1)$, and to be *of generalized K3 type* if $V^{p,q} = 0$ unless $(p, q) = (2, 0)$, $(1, 1)$, or $(0, 2)$, and $\dim V^{0,2} = \dim V^{2,0} = 1$.⁵

⁵I literally just made these definitions up, although they are reasonable enough so that I would not be surprised if they appeared elsewhere. In case you are wondering, I would define K3 type to be generalized K3 type together with $\dim V^{1,1} = 20$: these are the Hodge numbers of a K3 surface.

Proposition 4. *We have natural equivalences of categories between:*

- a) \mathbb{Z} -Hodge structures of abelian type and complex tori.
- b) Polarizable \mathbb{Z} -Hodge structures of abelian type and abelian varieties.

The first statement is straightforward: recall that a complex torus is of the form \mathbb{C}^g/Λ where $\Lambda \cong \mathbb{Z}^{2g}$. An \mathbb{Z} -Hodge structure of abelian type, on the other hand, is a finite free \mathbb{Z} -module Λ – say of rank r – together with a certain homomorphism $h : \mathbb{C}^\times \rightarrow \text{Aut}(\Lambda \otimes \mathbb{R}) = GL(V_{\mathbb{R}})$. Restricting h to \mathbb{R}^\times we get the usual scalar multiplication – since the weight is equal to 1 – so that h is \mathbb{R} -linear and is nothing else than a *complex structure* on $V_{\mathbb{R}}$, so endows $V_{\mathbb{R}}$ with the structure of a \mathbb{C} -vector space, say of dimension g (so $r = 2g$). So there exists an isomorphism $\rho : V \cong \mathbb{C}^{2g}$, and under the isomorphism Λ goes to some full rank lattice in \mathbb{C}^{2g} , so that $\mathbb{C}^{2g}/\rho(\Lambda)$ is a complex torus. And clearly the construction can be reversed.

What is more challenging is part b): this is a rather unlikely looking definition of an abelian variety. (One might imagine that we have ordered our abelian variety over the internet, and it has arrived via next-day shipping, but some assembly is required!) A more natural-looking definition of an abelian variety is:

(AV1) A connected, projective algebraic group $A_{/\mathbb{C}}$.

In fact, if we are given such an algebraic group, its \mathbb{C} -valued points $A(\mathbb{C})$ form a compact, complex manifold, which is moreover a group, and whose group law is given by holomorphic maps: that is:

(AV2) An abelian variety is a connected, compact complex Lie group which may be holomorphically embedded in $\mathbb{P}^N(\mathbb{C})$ for some N .⁶

Already in entertaining a connected, compact, complex Lie group we have accumulated quite a long list of nice properties, so it should not be too surprising that such a guy is necessarily a complex torus \mathbb{C}^g/Λ (the proof of this is nontrivial). The real subtlety begins when we ask which complex tori can be holomorphically embedded in projective space. Note that when $g = 1$ the answer is all of them – recall that every compact Riemann surface can be embedded in projective space (Riemann’s theorem). However, for $g > 1$ there are nontrivial compatibility relations between the lattice Λ and the complex structure h in order for the torus to embed in projective space (Riemann’s bilinear relations). The real miracle is that this embeddability condition is precisely equivalent to the existence of a polarization. This is quite a deep result, and I shall not say anything about it here.

One can in fact show that there is a natural topological space parameterizing isomorphism classes of g -dimensional complex tori, it has (real) dimension $2g^2$. On the other hand, there is another space parameterizing isomorphism classes of g -dimensional complex abelian varieties, and it has (real) dimension $g^2 + g$. So in dimension greater than one, most complex tori are not abelian varieties.

⁶Here we have snuck in the fact that a closed holomorphic submanifold of $\mathbb{P}^N(\mathbb{C})$ is necessarily the solution set of finitely many polynomial equations: Chow’s Theorem.

3.3. Endomorphism rings and endomorphism algebras. For a complex torus $T = \mathbb{C}^g/\Lambda$, an endomorphism $\alpha : T \rightarrow T$ will necessarily lift to give a \mathbb{C} -linear map $L(\alpha)$ on the universal cover \mathbb{C}^g , and the condition that such a linear map descend to T is just that it preserve the lattice: $L(\alpha)\Lambda \subset \Lambda$. On the other hand, by linearity α is determined by its action on Λ , so the endomorphism ring is a priori a subring of $\text{End}(\Lambda) \cong M_{2g}(\mathbb{Z})$. In particular, the underlying abelian group is a finite-free \mathbb{Z} -module, always containing at least the subring \mathbb{Z} (in which $n \in \mathbb{Z}$ acts as multiplication by n , as it does on any abelian group.)

It turns out to make life much easier to work not with $\text{End}(T)$ but with the *endomorphism algebra* $\text{End}^0(T) := \text{End}(T) \otimes \mathbb{Q}$, which by the above is some finite-dimensional \mathbb{Q} -algebra. In terms of Hodge theory, this means relaxing things to look only at the \mathbb{Q} -Hodge structures, and this means that we shall regard as equivalent two abelian varieties \mathbb{C}^g/Λ_1 and \mathbb{C}^g/Λ_2 where Λ_1 and Λ_2 are commensurate lattices – i.e., there is some lattice Λ_3 containing both Λ_1 and Λ_2 with finite index. This equivalence relation between abelian varieties is known as *isogeny*.

So abelian varieties up to isogeny correspond to polarizable \mathbb{Q} -Hodge structures of abelian type.

Now the endomorphism algebra $\text{End}^0(T)$ is an invariant of the \mathbb{Q} -rational Hodge structure whereas $\text{End}(T)$ depends upon the integral structure, and is for many purposes is unnecessarily subtle.⁷ As above the combination of working “over \mathbb{Q} ” and using a polarization makes things much nicer:

Proposition 5. *a) Every complex abelian variety A is isogenous to a product of simple abelian varieties $\prod_{i=1}^n B_i$, the B_i 's being unique up to isogeny.
b) The endomorphism algebra $\text{End}^0(A)$ is a semisimple \mathbb{Q} -algebra.
c) In contrast, any finite-dimensional \mathbb{Q} -algebra is the endomorphism algebra of some complex torus.*

Proof: Part a) is the Poincaré Complete Reducibility Theorem: we are just decomposing our Hodge structure into a direct sum of simple guys. (Just to be sure, a simple complex torus is one without nonzero, proper subtori.) A basic algebraic fact – Schur’s Lemma – is that the endomorphism algebra of a simple module is a division algebra, and from this it follows rather easily that the endomorphism algebra of a general abelian variety is a direct sum of matrix algebras over division algebras over \mathbb{Q} , that is, a semisimple \mathbb{Q} -algebra. Part c) is given only to make you glad we are working with abelian varieties and not with arbitrary complex tori; a proof can be found in the book *Complex Tori* by Birkenhake and Lange.

From now on we will work only with \mathbb{Q} -Hodge structures.

Proposition 6. *For a \mathbb{Q} -Hodge structure V of abelian type, with corresponding abelian variety (up to isogeny) A , we have $\text{End}^0(A) = \text{End}_H(V)$.*

Proof: In other words, the endomorphism algebra of the abelian variety is precisely the \mathbb{Q} -algebra of endomorphisms of V preserving the Hodge structure. But

⁷As someone who has worked with abelian varieties with interesting endomorphism algebras before, let me say that the structure of the endomorphism *ring* is quite complicated and scary, and in practice one tries to avoid dealing with it directly.

preserving the Hodge structure here means equivariant with respect to the \mathbb{C}^\times -representation, i.e., preserving the complex structure on V/\mathbb{R} , and we have seen above that this is exactly what the endomorphism algebra is.

4. MUMFORD-TATE GROUPS OF ABELIAN VARIETIES

For A an abelian variety, we abbreviate $MT(A)$ for the Mumford-Tate group of the associated weight 1 Hodge structure.

Corollary 7. *For an abelian variety A ,*

$$\text{End}^0(A) \cong \text{End}_{MT(A)}(H^1(A, \mathbb{Q})).$$

Proof: Indeed, this follows immediately from Corollary 2 and Proposition 6.

Again, the significance of this result is that the endomorphism algebra of A controls $MT(A)$ and thus also the Hodge classes on A to a large extent. In particular, a very general abelian variety will have $\text{End}^0(A) = \mathbb{Q}$, so that $\text{End}_{MT(A)}(H^1(A, \mathbb{Q})) = \mathbb{Q}$, meaning that the Mumford-Tate group is quite large. For example, suppose that A is an elliptic curve with $\text{End}(A) = \mathbb{Z}$ (one says that E is without complex multiplication). Then $MT(A)$ is some subgroup G of $GL_2(\mathbb{Q})$ such that $\text{End}_G(\mathbb{Q}^2) = \mathbb{Q}$. It is not too hard to see that in this case the only two such (connected) groups are $G = SL_2$ and $G = GL_2$. Since scalar matrices are always in the Mumford-Tate group, we conclude that $MT(A) = GL_2(\mathbb{Q}) = G(\Psi)$. Thus it is not surprising that we will often be considering abelian varieties with larger endomorphism algebras.

In the case of abelian varieties, we can then use the endomorphism algebra to refine our upper bound on $MT(V)$, namely $MT(V)$ is always contained in $G(\Psi) \cap Z(\text{End}^0(A))$. Let me denote this possibly larger group by $\text{Lf}(A)$. An important fact is the following:

Theorem 8. *Suppose $\text{End}^0(A)$ is a field and $MT(A) = \text{Lf}(A)$. Then the Hodge conjecture holds for the abelian variety A in a particularly simple way: namely, every Hodge class in $B_p(A) = H^{2p}(A, \mathbb{Q}) \cap H^{p,p}(A, \mathbb{C})$ is the image of an intersection of $p-1$ divisors.*

In particular, in order to derive new cases of the Hodge conjecture for abelian varieties, one needs to look at cases in which the endomorphism algebra is not a field or the Mumford-Tate group is strictly smaller than $\text{Lf}(A)$. In the case of the Kuga-Satake construction, one starts with a weight 2 Hodge structure V in which we may well have $MT(V) = G(\Psi)$ – so that the polarization gives a quadratic form – and we will produce a Hodge structure $KS(V)$ of type $(0, 1) \cup (1, 0)$ and such that $MT(KS(V))$ is contained in a proper subgroup, $CSpin(\Psi)$ of $G(\Psi)$.