# SUMMER 2010 COURSE ON MODEL THEORY AND ITS APPLICATIONS 

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8. Ultraproducts and ultrapowers in model theory

### 6.1. Filters and ultrafilters.

A filter $\mathcal{F}$ on a set $X$ is a nonempty family of nonempty subsets of $X$ satisfying the following properties:
(F1) $A_{1}, A_{2} \in \mathcal{F} \Longrightarrow A_{1} \cap A_{2} \in \mathcal{F}$, and
(F2) $A_{1} \in \mathcal{F}, A_{2} \supset A_{1} \Longrightarrow A_{2} \in \mathcal{F}$.
That is, a filter is a family of nonempty subsets that is stable under finite intersections and passage to supersets.

Example 6.1: For $\emptyset \neq Y \subset X$, define $\mathcal{F}_{Y}=\{A \subset X \mid Y \subset A\}$ to be the family of all subsets of $X$ containing the fixed nonempty subset $Y$. This is a filter. Such filters are called principal.

Example 6.2: Let $X$ be an infinite set. A subset $Y \subset X$ is said to be cofinite if $X \backslash Y$ is finite. The collection of all cofinite subsets of $X$ is a nonprincipal filter, the Fréchet filter.

A filter $\mathcal{F}$ on a set $X$ is free if $\bigcap_{A \in \mathcal{F}} A=\emptyset$.
Exercise 6.3: Let $\mathcal{F}$ be a free filter on $X$.
a) Show that $\mathcal{F}$ is not principal.
b) Show that $\mathcal{F}$ contains the Fréchet filter.

Exercise 6.4: a) Let $\left\{\mathcal{F}_{i}\right\}_{i \in I}$ be an indexed family of filters on $X$. Show that $\mathcal{F}=\bigcap_{i \in I} \mathcal{F}_{i}$ is a filter, indeed the largest filter which is contained in each $\mathcal{F}_{i}$.
b) Let $X$ be a set with at least two elements. Exhibit filters $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ on $X$ such that there is no filter $\mathcal{F}$ containing both $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$.

The collection of all filters on a set $X$ is partially ordered under containment. By Exercise 6.4a), this poset contains arbitrary joins - i.e., any collection of filters admits a greatest lower bound; on the other hand, Exercise 6.4 b ) shows that when $|X|>1$ the poset of filters on $X$ is not directed. If $\mathcal{F}_{1} \subset \mathcal{F}_{2}$ we say that $\mathcal{F}_{2}$ refines $\mathcal{F}_{1}$ or is a finer filter than $\mathcal{F}_{1}$.

Definition: An ultrafilter on $X$ is a maximal element in the poset of filters on $X$, i.e., a filter which is not properly contained in any other filter on $X$.

The following is probably the single most important property of ultrafilters.
Theorem 1. Let $\mathcal{F}$ be a filter on $X$.
a) $\mathcal{F}$ is an ultrafilter iff: for all $Y \subset X$, exactly one of $Y, X \backslash Y$ lies in $\mathcal{F}$.

Proof. Let $\mathcal{F}$ be an ultrafilter on $X$ and $Y \subset X$. Suppose first that for all $A \in \mathcal{F}$, $(A \cap Y) \neq \emptyset$. Let $F^{\prime}=\{A \cap Y \mid A \in \mathcal{F}\}$ and let $\mathcal{F}^{\prime}$ be the collection of all subset of $X$ containing at least one element of $F^{\prime}$. It is easy to see that $\mathcal{F}^{\prime}$ is a filter on $X$ which contains $\mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, we must have $\mathcal{F}=\mathcal{F}^{\prime}$ and thus $Y=X \cap Y \in \mathcal{F}^{\prime}=\mathcal{F}$. Now suppose that there exists $A \in \mathcal{F}$ such that $A \cap Y=\emptyset$. Equivalently, $A \subset X \backslash Y$ and since $A \in \mathcal{F}, X \backslash Y \in \mathcal{F}$.
Now suppose that $\mathcal{F}$ is a filter on $X$ which, given any subset of $X$, contains as an element either that subset or its complement. Suppose $\mathcal{F}^{\prime}$ is a filter properly containing $\mathcal{F}$, so that there exists some subset $Y \in \mathcal{F}^{\prime} \backslash \mathcal{F}$. But then $X \backslash Y \in \mathcal{F}^{\prime} \subset \mathcal{F}$ so that $\mathcal{F}^{\prime}$ contains both $Y$ and $X \backslash Y$ and thus contains their intersection, the empty set: contradiction.

Corollary 2. Let $\mathcal{F}$ be an ultrafilter on $X$, let $A \in \mathcal{F}$, and let $A_{1}, A_{2}$ be subsets of $X$ such that $A_{1} \cup A_{2}=A$. Then at least one of $A_{1}$ and $A_{2}$ lies in $\mathcal{F}$.
Proof. Assume not. Then by Theorem 1, both $X \backslash A_{1}$ and $X \backslash A_{2}$ lie in $\mathcal{F}$, and hence so does

$$
\left(X \backslash A_{1}\right) \cap\left(X \backslash A_{2}\right)=X \backslash\left(A_{1} \cup A_{2}\right)=X \backslash A
$$

Thus $\mathcal{F}$ contains both $A$ and its complement $X \backslash A$, contradiction.
Corollary 3. Let $\mathcal{F}$ be an ultrafilter on $X$. Then the following are equivalent:
(i) $\mathcal{F}$ is not free.
(ii) $\mathcal{F}$ is principal.
(iii) There exists $x \in X$ such that $\mathcal{F}$ is the collection of all subsets containing $x$.

Proof. The imlications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) clearly hold (for arbitrary filters). Suppose that $\mathcal{F}$ is not free, i.e., there exists $x \in \bigcap_{A \in \mathcal{F}} A$. Then $X \backslash\{x\}$ is not an element of $\mathcal{F}$, so by Theorem 1 we have $\{x\} \in \mathcal{F}$, so that $\mathcal{F}$ is the principal filter on the singleton set $\{x\}$.

Remark: There exist (non ultra)filters which are neither free nor principal, for instance the filter $\{\{0\}, \mathbb{R}\}$ on $\mathbb{R}$. But no matter.

Proposition 4. a) For a family $\mathcal{A}$ of nonempty subsets of a set $X$, the following are equivalent:
(i) $I$ has the finite intersection property: if $A_{1}, \ldots, A_{n} \in I$, then $\bigcap_{i=1}^{n} A_{i} \neq \emptyset$.
(ii) There exists a filter $\mathcal{F} \supset \mathcal{A}$.

A family $F$ satisfying these equivalent conditions is called a filter subbase. ${ }^{1}$
b) For any filter subbase $\mathcal{A}$, there is a unique minimal filter $\mathcal{F}$ containing $\mathcal{A}$, called the filter generated by $\mathcal{A}$.

Proof. a) Certainly the finite intersection property (f.i.p., for short) is necessary for $\mathcal{A}$ to extend to a filter. Conversely, given a family of sets $\mathcal{A}$ satisfying fi.p., we build the filter it generates in much the same way that we build the topology generated by a subbase. Namely, let $F$ be the family of all finite intersections of elements of $\mathcal{A},{ }^{2}$ and let $\mathcal{F}$ be the family of all subsets of $X$ containing some element of $F$. It is easy to check that $\mathcal{F}$ is a filter.
b) Every filter $\mathcal{G}$ containing every element of $\mathcal{A}$ must contain all supersets of all finite intersections of elements of $\mathcal{A}$, so the filter $\mathcal{F}$ constructed in part a) above is the unique minimal filter containing $\mathcal{A}$.

Exercise 6.5: Let $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ be two filters on a set $X$. Show that the following are equivalent:
(i) For all $A \in \mathcal{F}_{1}$ and all $B \in \mathcal{F}_{2}, A \cap B \neq \varnothing$.
(ii) The set $\mathcal{F}_{1} \cup \mathcal{F}_{2}$ satisfies the finite intersection condition.
(iii) There exists a filter $\mathcal{F}$ containing both $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$.

When these equivalent conditions are satisfied, we say that the filters $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ are compatible. This should be thought of in analogy to the situation of ideals $I_{1}$ and $I_{2}$ such that the ideal $I_{1}+I_{2}$ is proper.

The next result collects some further properties of filters, indeed exactly those that we will need for our model-theoretic applications.

## Proposition 5.

a) Let $\mathcal{F}$ be a filter on $X$. Then there exists an ultrafilter containing $\mathcal{F}$.
b) Any infinite set admits a nonprincipal ultrafilter. Indeed, let $Y \subset X$ with $Y$ infinite. Then there exists a nonprincipal ultrafilter $\mathcal{F}$ on $X$ such that $Y \in \mathcal{F}$.

Proof. a) It is easy to see that the union of a chain of filters on $X$ is a filter on $X$. Therefore Zorn's Lemma applies to give a maximal element in the poset of filters containing a given filter $\mathcal{F}$, i.e., an ultrafilter containing $\mathcal{F}$.
b) Let $\mathcal{F}_{0}$ be the Fréchet filter (of cofinite subsets of $X$ ), and let $\mathcal{F}_{Y}=\{A \subset$ $X \mid A \supset Y\}$ be the principal filter on $Y$. Since $Y$ is infinite, if $B \subset X$ is any cofinite set, $Y \cap B \neq \varnothing$. It follows that the filters $\mathcal{F}_{0}$ and $\mathcal{F}_{Y}$ are compatible in the sense of Exercise 6.5, so there exists an ultrafilter $\mathcal{F}$ containing both of them. Since $\mathcal{F}$ contains the Fréchet filter, it is nonprincipal.

Exercise 6.6 (harder; not used later): Show that in fact, for any infinite set $X$, the number of nonprincipal ultrafilters on $X$ is $2^{2^{|X|}}$.

[^0]
### 6.2. Filters in Topology: An Advertisement.

The night before giving the lecture on ultrafilters and ultraproducts, it occurred to me that ultrafilters might not be part of the working vocabulary of my audience. So I sent out an email advising them to book up on them a little bit and providing a link to some notes on general topology. At the lecture itself, I found out that indeed most of my audience had not studied filters before. ${ }^{3}$

So here is a quick précis of the use of filters and ultrafilters in topology. For more details, please see [GT, Ch. II, §5].

Let $f: X \rightarrow Y$ be a function and $\mathcal{F}$ a filter on $X$. Then the family of subsets $\{f(A) \mid A \in \mathcal{F}\}$ of $Y$ satisfies the finite intersection condition, so is the subbase for a unique filter on $Y$, which we denote $f(\mathcal{F})$.

Let $X$ be a topological space and $x$ a point of $X$. Then the set $\mathcal{N}_{x}$ of neighborhoods of $x$, i.e., of subsets $N$ of $x$ such that $x$ lies in the interior of $N$, is a filter on $X$. It is the principal ultrafilter $\mathcal{F}_{x}$ iff $x$ is an isolated point of $X$.

A filter $\mathcal{F}$ on $X$ is said to converge to $\mathbf{x}$ if $\mathcal{F} \supset \mathcal{N}_{x}$, i.e., if every neighborhood of $x$ lies in $\mathcal{F}$. We write $\mathcal{F} \rightarrow x$. Again, for a trivial example, note that the principal ultrafilter $\mathcal{F}_{x}$ converges to $x$ no matter what the topology on $X$ is. We say that a filter converges if it converges to at least one point. (If $X$ is Hausdorff, a filter converges to at most one point.)

A point $x \in X$ is said to be a limit point of a filter $\mathcal{F}$ if the filters $\mathcal{N}_{x}$ and $\mathcal{F}$ are compatible, i.e., are simultaneously contained in some filter. In other words, $x$ is a limit point of $\mathcal{F}$ if every neighborhood of $x$ meets every element $A \in \mathcal{F}$.

With these definitions, get a theory of convergence via filters paralleling that of sequences in a metrizable (or first countable) space. Here some of the most important tenets of this theory.

## Theorem 6.

a) Let $X$ be a topological space. The closure of a subset $A$ of $X$ is the set of all $x \in X$ such that there exists a filter $\mathcal{F}$ on $X$ with $A \in \mathcal{F}$ and $\mathcal{F} \rightarrow x$.
b) Let $X$ and $Y$ be topological spaces and $f: X \rightarrow Y$ be a map of sets. Then $f$ is continuous iff: for all $x \in X$ and all filters $\mathcal{F}$ on $X, \mathcal{F} \rightarrow x$ iff $f(\mathcal{F}) \rightarrow f(x)$.
c) Let $X=\prod_{i} X_{i}$ be a product of spaces and $\pi_{i}: X \rightarrow X_{i}$ be the projection map, $\mathcal{F}$ a filter on $X$ and $x=\left(x_{i}\right) \in X$. Then $\mathcal{F} \rightarrow x$ iff for all $i \in I, \pi_{i}(\mathcal{F}) \rightarrow x_{i}$.
d) A space $X$ is quasi-compact iff every ultrafilter on $X$ converges.

Each of these statements is straightforward to prove. And they have a nonitrivial consequence.

Exercise 6.7: Deduce from Theorem 6 Tychonoff's theorem, that a product $X=$

[^1]$\prod_{i} X_{i}$ of nonempty spaces if quasi-compact iff each factor $X_{i}$ is quasi-compact.
There are of course many proofs of Tychonoff's theorem, but this one has the merit of making the result look completely evident and natural.

### 6.3. Ultraproducts and Los' Theorem.

The notion of a product of structures is a fundamental one in mathematics. For instance, one has the product of sets, groups, rings, topological spaces, schemes... For many (but not all) of these products, the unifying theme is a certain universal mapping property.

Suppose we have a family $\left\{X_{i}\right\}_{i \in I}$ of models of a theory $\mathcal{T}$. It would be nice, wouldn't it, to be able to define some kind of product model $X=\prod_{i} X_{i}$ ? (This is not much in the way of motivation, but we will soon see just how nice it would be!) Unfortunately, this only works halfway: we may define a product of $\mathcal{L}$-structures, but the product of models of a theory $\mathcal{T}$ need not be a model of $\mathcal{T}$.

Indeed, let $\mathcal{L}$ be a language and $\left\{X_{i}\right\}$ a family of $\mathcal{L}$-structures. Put $X=\prod_{i} X_{i}$, the Cartesian product. We may endow $X$ with an $\mathcal{L}$-structure, as follows: for every constant symbol $c \in \mathcal{L}$, we put $c_{X}=\prod_{i} c_{X_{i}}$. For every $n$-ary function symbol $f \in \mathcal{L}$, we define $f_{X}$ to be the evident function from $\left(\prod_{i} X_{i}\right)^{n} \rightarrow \prod_{i} X_{i}$, i.e., the one whose $i$-coordinate is $f_{X_{i}}$. Similarly, for every $n$-ary relation symbol $R \in \mathcal{L}$, we define $R_{X}$ as the product relation, i.e., $\prod_{i} R_{X_{i}} \subset \prod_{i} X_{i}^{n}=\left(\prod_{i} X_{i}\right)^{n}$.

Exercise 6.8: If you know and care about such things, show that the product we have defined satisfies the universal mapping property in the sense of category theory.

Thus for instance, if $\mathcal{L}$ is the language of rings, we may take a product of rings. For example, take $I$ to be the set of prime numbers and for $p \in I$, put $R_{i}=\mathbb{F}_{p}$. Then the product $\prod_{i} \mathbb{F}_{p}$ is again an $\mathcal{L}$-structure (and even a ring). However, suppose $\mathcal{T}$ is the theory of fields. Then each $\mathbb{F}_{p}$ is a model of $\mathcal{T}$ but the product certainly is not: it is not even a domain.

All this is remedied by passing to a certain quotient of the direct product. To do this, we need an extra ingredient - the crazy part. Namely, we "choose" an ultrafilter $\mathcal{F}$ on the index set $I$. Then, we define the relation $\sim_{\mathcal{F}}$ on the Cartesian product $X=\prod_{i} X_{i}$ by $\left\{x_{i}\right\} \sim_{\mathcal{F}}\left\{y_{i}\right\}$ iff the set of indices $i \in I$ such that $x_{i}=y_{i}$ is an element of $\mathcal{F}$. We define the ultraproduct $X=\prod_{\mathcal{F}} X_{i}$ to be the quotient $\tilde{X} / \mathcal{F}$.

Exercise 6.9: Check that $\sim_{\mathcal{F}}$ is indeed an equivalence relation and that the ultraporudct $\prod_{\mathcal{F}} X_{i}$ is indeed an $\mathcal{L}$-structure in a natural way.

So what is going on here? Magic, I say! Actually, there is one case in which the magic isn't real: there is a little man behind the curtain.

Proposition 7. Let $I$ be an index set, $i_{0} \in I$, and let $\mathcal{F}_{i_{0}}$ be the principal ultrafilter at $i_{0}$. Then the ultraproduct $\prod_{\mathcal{F}_{i_{0}}} X_{i}$ is isomorphic to $X_{i_{0}}$.

Exercise 6.10: Prove Proposition 7.
However, when we restrict to nonprincipal ultrafilters, the magic is quite real.
Example 6.11: Let $\mathcal{L}$ be the language of rings, $I$ an index set, $\mathcal{F}$ an ultrafilter on $I$, for each $i \in I$, let $R_{i}$ be an integral domain. Then the ultraproduct $R=\prod_{\mathcal{F}} X_{i}$ is a domain. Indeed, let $x$ and $y$ be elements of $R$ such that $x y=0$. We must show that $x=0$ or $y=0$. Represent $x$ by a sequence $\left\{x_{i}\right\}$ and $y$ by a sequence $\left\{y_{i}\right\}$. Then, to say that $x y=0$ is to say that the set of indices $i$ such that $x_{i} y_{i}=0$ lies in the filter $\mathcal{F}$ : let us call this set $A$. Let $A_{1}$ be the set of indices $i$ such that $x_{i}=0$ and let $A_{2}$ be the set of indices $i$ such that $y_{i}=0$. Since each $R_{i}$ is a domain, we have $A=A_{1} \cup A_{2}$. By Corollary 2, we have either $A_{1} \in \mathcal{F}$ or $A_{2} \in \mathcal{F}$, that is, $x=0$ or $y=0$ : qed.

Now let us show that the ultraproduct $K=\prod_{\mathcal{F}} K_{i}$ of fields is again a field. So, let $0 \neq x \in K$. We need to show that there exists $y \in K$ such that $x y=1$. Let $\left\{x_{i}\right\} \in \prod K_{i}$ be any element representing $x$, and let $A \subset I$ be the set of indices such that $x_{i} \neq 0$. Define $y$ to be the element whose $i$ coordinate is: $x_{i}^{-1}$ if $i \in A$ (so $x_{i}$ is nonzero in the field $K_{i}$ and thus has an inverse) and 0 otherwise. Then $x_{\bullet} y_{\bullet}$ has $i$ coordinate 1 for all $i \in A$ and 0 otherwise. Hence it is equal to the constant element 1 on a set of indices which lies in $\mathcal{F}$, so $x y=1$ in the quotient. (Note that we have a lot of leeway in the definition of $y_{\text {bullet }}$ - it does not matter at all how we define it at coordinates not lying in $A$ - but all of these elements become equal in the quotient.)

Here is a more interesting example. Let $F$ be the ultraproduct of the finite field $\mathbb{F}_{p}$. By the above, this is a field. So it has a characteristic - what is it?!?

Case 1: Despite what I said above, it's instructive to consider the case of a principal ultrafilter based at a particular prime $p_{0}$. In this case, the ultraproduct is just $\mathbb{F}_{p_{0}}$, so of course the characteristic is $p_{0}$.
Case 2: If $\mathcal{F}$ is nonprincipal, we claim that $F$ has characteristic 0 . It suffices to show that for any prime $\ell, 1+\ldots+1$ ( $\ell$ times) is not zero. Well, consider the diagonal elements $x_{\bullet}=\ell$ and $y_{\bullet}=0$. What does it mean for $x_{\bullet}$ and $y_{\bullet}$ to be equal in the ultraproduct? It means that the set $A$ of primes $p$ such that $\ell=0$ in $\mathbb{F}_{p}$ lies in the filter $\mathcal{F}$. But $A=\{p\}$, a finite set, which is not an element of any nonprincipal ultrafilter. Done!

The following result is a vast generalization of these observations. It is often called the Fundamental Theorem of Ultraproducts. Nor is the proof difficult; rather it is almost as easy as a proof which proceeds by induction on the complexity of a formula can be. Since we have, somewhat disreputably, not given such a proof thus far, ${ }^{4}$ we present the proof of Los' Theorem in all its gory detail.
Theorem 8. (Los) Let $I$ be an index set, $\mathcal{F}$ an ultrafilter on $I$, $\left\{X_{i}\right\}_{i \in I}$ an indexed family of $\mathcal{L}$-structures, and put $X=\prod_{\mathcal{F}} X_{i}$. For any formula $\phi$ in $n$ unbound variables and $\bar{x} \in X^{n}$,

$$
\left.X \models \phi(\bar{x})) \Longleftrightarrow\left\{i: X_{i} \models \phi\left(\overline{x_{i}}\right)\right)\right\} \in \mathcal{F} .
$$

[^2]Proof. We prove this by an induction on the complexity of the formulas. First recall that a term is the set of $\mathcal{L}$-terms is the smallest set containing the constant symbols of $\mathcal{L}$, the variable names $\left\{x_{i}\right\}_{i=1}^{\infty}$, and for each $n$-ary function, all expressions of the form $f\left(t_{1}, \ldots, t_{n}\right)$, where the $t_{i}$ 's are terms.

Step 1: Suppose $\varphi$ is of the form $t_{1}=t_{2}$, where $t_{1}$ and $t_{2}$ are terms involving $n$ variables $x_{1}, \ldots, x_{n}$. For $j=1,2$, put

$$
g_{j}(i)=t_{j}\left(x_{1}(i), \ldots, x_{n}(i)\right)
$$

Then $t_{1}\left(x_{1}, \ldots, x_{n}\right)=t_{2}\left(x_{1}, \ldots, x_{n}\right)$ as elements of $X$ iff the set of $i \in I$ such that $t_{1}\left(x_{1}(i), \ldots, x_{n}(i)\right)=t_{2}\left(x_{1}(i), \ldots, x_{n}(i)\right)$ is an element of $\mathcal{F}$. This is Los' Theorem in this case!

Step 2: Suppose $\varphi$ is a relation $R\left(t_{1}, \ldots, t_{n}\right)=R(t)$. Then $R\left(t_{1}(\bar{x}), \ldots, t_{n}(\bar{x})\right)$ holds in $X$ iff the tuple $\left(t_{1}(x), \ldots, t_{n}(x)\right)$ lies in $R_{X^{n}} \subset X^{n}$ iff there exists $\bar{y}=$ $\left(y_{1}, \ldots, y_{n}\right) \in R_{X^{n}}$ such that $\left(t_{1}(x), \ldots, t_{n}(x)\right)=\left(y_{1}, \ldots, y_{n}\right)$ iff for a set of indices $I$ which lies in $\mathcal{F}$ we have $t\left(x_{i}\right)=y(i)$ iff for a set o findices $I$ which lies in $\mathcal{F}$, $R\left(t\left(x_{i}\right)\right) \in R_{X_{i}}$.

Step 3: Suppose Los' Theorem holds for $\alpha$ and $\beta$ and $\varphi=\alpha \wedge \beta$. Then $\varphi(x)=$ $\alpha(x) \wedge \beta(x)$ holds in $X$ iff both $\alpha(x)$ and $\beta(x)$ hold in $X$, iff the sets $A_{1}$ (resp. $A_{2}$ ) of indices $i$ such that $\alpha\left(x_{i}\right)$ (resp. $\beta\left(x_{i}\right)$ ) holds in $X_{i}$ lie in $\mathcal{F}$ iff (since $\mathcal{F}$ is a filter) the set $A=A_{1} \cap A_{2}$ of indices $i$ such that both $\alpha\left(x_{i}\right)$ and $\beta\left(x_{i}\right)$ hold lies in $\mathcal{F}$ iff the set of indices $i$ such that $\alpha\left(x_{i}\right) \wedge \beta\left(x_{i}\right)=\varphi\left(x_{i}\right)$ holds lies in $\mathcal{F}$.

Step 4: Suppose that Los' Theorem holds for $\varphi(x)$. Then it also holds for $\neg \varphi(x)$. Indeed, $\neg \varphi(x)$ holds in $X$ iff $\varphi(x)$ does not hold in $X$ iff the set $A$ of indices $i$ for which $\varphi\left(x_{i}\right)$ holds in $X_{i}$ is not in $\mathcal{F}$. But the set $A^{\prime}$ of indices $i$ for which $\neg \varphi\left(x_{i}\right)$ holds in $X_{i}$ is of course $I \backslash A$, and since $\mathcal{F}$ is an ultrafilter and $A$ is not in $\mathcal{F}, A^{\prime}$ must be in $\mathcal{F} .{ }^{5}$

Step 5: Write $x=\left(x_{1}, \ldots, x_{n}\right)$ and $y=\left(x_{2}, \ldots, x_{n}\right)$, so $x=\left(x_{1}, y\right)$. Suppose Los' Theorem holds $\psi(x)$; we show that it also holds for $\exists v \psi(v, y)$.
This time we handle the two implications separately. First suppose that $\exists v \psi(v, y)$ holds in $X$. Then for some $x=\left(x_{1}, y\right) \in X^{n}, \psi(x)$ holds in $X$. It follows that the set $A$ of indices $i$ such that $\psi(x(i))$ holds in $X_{i}$ lies in $\mathcal{F}$. Now the set $A^{\prime}$ of indices $i$ such that $\exists v \psi(v, y(i))$ holds in $X_{i}$ contains $A$, so $A^{\prime}$ lies in $\mathcal{F}$.

Conversely, suppose that the set $A$ of indices $i$ such that $\exists v \psi(v, y(i))$ lies in $\mathcal{F}$. For each such $i$, choose $x_{1}(i) \in X_{i}$ such that $\psi\left(x_{1}(i), y(i)\right)$ holds in $X_{i}$; for all other indices $i$, define $x_{1}(i)$ arbitrarily. There is then an induced element $x_{1}=\prod_{\mathcal{F}} x_{1}(i)$ in the ultraproduct, and then $\varphi\left(x_{1}, y\right)$ holds in $X$ hence so does $\exists v \psi(v, y)$.

Corollary 9. a) In the setup of Los' Theorem, let $\mathcal{T}$ be an $\mathcal{L}$-theory, and suppose that each $X_{i}$ is a model of $\mathcal{T}$. Then $X$ is a model of $\mathcal{T}$.
b) In particular, if for all $i, j \in I, X_{i} \equiv X_{j}$, then $X \equiv X_{i}$ for all $i$.

Proof. a) This is a very special case of Theorem 8: for each sentence $\varphi \in \mathcal{T}$, the set of indices $i$ such that $\varphi_{i}$ holds in $X_{i}$ is the entire index set $I$, so is certainly an

[^3]element of $\mathcal{F}$. Thus by Los' Theorem, $\varphi$ holds in $X$. Part b) follows immediately.

One way to enforce $X_{i} \equiv X_{j}$ for all indices is simply to choose a single model $X$ of $\mathcal{T}$ and take $X_{i}=X$ for all $i$. In this case, we abbreviate $\prod_{\mathcal{F}} X$ to $X^{\mathcal{F}}$, and we say that $X^{\mathcal{F}}$ is an ultrapower of $X$. Thus $X \equiv X^{\mathcal{F}}$, but $X^{\mathcal{F}}$ is guaranteed to be a "sufficiently rich" model of $\mathcal{T}$ in a sense that we will not have time to make precise. But, for example, if $X$ is an algebraically closed field, then any nontrivial ultrapower of $X$ is an algebraically closed field of infinite transcendence degree.

Exercise 6.12: Let $X$ be an $\mathcal{L}$-structure and $X^{\mathcal{F}}$ an ultrapower. Show that there is a natural embedding of $\mathcal{L}$-structures $\iota: X \hookrightarrow X^{\mathcal{F}}$ and this embedding is elementary.

### 6.4. Proof of Compactness Via Ultraproducts.

Let $\mathcal{L}$ be a language and $\mathcal{T}$ be a theory such that every finite subset of $\mathcal{T}$ has a model. We wish to show that $\mathcal{T}$ has a model. Formerly, we deduced this as an immediate corollary of Gödel's Completeness Theorem and the finite character of syntactic implication. But, aside from using a proof-theoretic result that we are not going to prove (and is generally regarded as being fundamentally "un-modeltheoretic" in nature), this was a proof by contradiction. Much more impressive would be the following head-on attack: for each finite subtheory $\mathcal{T}^{\prime} \subset \mathcal{T}$, let $X_{\mathcal{T}^{\prime}}$ be a model of $\mathcal{T}$ '. Then using the $X_{\mathcal{T}}$ 's as data, we construct a model $X$ of $\mathcal{T}$.

Prepare to be impressed!
We may of course assume that $\mathcal{T}$ is infinite; otherwise there is nothing to prove. Let $I$ be the set of finite subtheories of $\mathcal{T}$. For $\varphi \in \overline{\mathcal{T}}$, let

$$
A(\varphi)=\left\{\mathcal{T}^{\prime} \in I \mid \varphi \in \overline{\mathcal{T}^{\prime}}\right\}
$$

and let $A=\{A(\varphi)\}_{\varphi \in \mathcal{T}}$. Evidently $A$ is a nonempty family of nonempty subsets of $I$. I claim that moreover $A$ satisfies the finite intersection condition: indeed, for any $\varphi_{1}, \ldots, \varphi_{n} \in \mathcal{T}$,

$$
\bigcap_{i=1}^{n} A(\varphi)=A\left(\bigwedge_{i=1}^{n} \varphi_{i}\right) \neq \emptyset .
$$

Thus, in the terminology of Proposition $4, A$ is a filter subbase on $I$. In other words, there is some filter containing $A$ and hence some ultrafilter $\mathcal{F}$ containing I. By hypothesis, for each finite $\mathcal{T}^{\prime} \subset \mathcal{T}$, there exists at least one $\mathcal{L}$-structure modelling $\mathcal{T}^{\prime}$ : choose one, and call it $X_{\mathcal{T}^{\prime}}$. Thus $X_{\mathcal{T}^{\prime}}$ is a family of $\mathcal{L}$-structures indexed by the elements of $I$, and $\mathcal{F}$ is an ultrafilter on $I$. So we may form the ultraproduct:

$$
X=\prod_{\mathcal{F}} X_{\mathcal{T}^{\prime}}
$$

We claim that $X$ is a model of $\mathcal{T}$. Indeed, for any $\varphi \in \mathcal{T}$, consider the set $J$ of finite subtheories $\mathcal{T}^{\prime}$ of $\mathcal{T}$ such that $X_{\mathcal{T}}$, is a model of $\varphi$. It is hard to say exactly what $J$ is (since we chose the models $X_{\mathcal{T}}$, "at random"), but certainly $J$ contains each finite subtheory $\mathcal{T}^{\prime}$ such that $\varphi \in \overline{\mathcal{T}^{\prime}}$, since then $\varphi$ holds in every model of $\mathcal{T}^{\prime}$. That is, $J \supset A(\varphi)$; since $A(\varphi) \in \mathcal{F}$ and $\mathcal{F}$ is a filter, $J \in \mathcal{F}$. We are done by Los' theorem.

So the use of ultraproducts gives a quick proof of the Compactness Theorem which, recall, was originally deduced from Gödel's Completeness Theorem and the finite character of syntactic implication. We used the Completeness Theorem and the finite character of syntactic implication at one other key juncture, namely in the proof of Ax's Transfer Principle (Theorem 3.10). We urge every reader to do the following exercise.

Exercise 6.13: In the proof of Ax's Transfer Principle, replace all appeals to syntactic considerations by an ultraproduct argument. (Suggestion: use Proposition $5 b)$. That's what it's there for!)

This is a typical phenomenon. Indeed, to the best of my knowledge, in the study of model theory one never needs to use Gödel's Completeness Theorem but can always make do with evident ultraproduct-theoretic analogues.

### 6.5. Characterization theorems involving ultraproducts.

First a result which we could have proven long ago, but is especially appropriate now that we have proved the Compactness Theorem.

## Proposition 10.

a) Let $\mathcal{L}$ be a language and $\mathcal{T}_{1}, \mathcal{T}_{2}$ be two $\mathcal{L}$-theories. Suppose that for an $\mathcal{L}$ structure $X, X$ is a model of $\mathcal{T}_{1}$ iff $X$ is not a model of $\mathcal{T}_{2}$. Then $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are finitely axiomatizable.
b) In particular, a class $\mathcal{C}$ is finitely axiomatizable iff both $\mathcal{C}$ and its negation are elementary.

Proof. a) We give two proofs, the first using the Compactness Theorem and the second using ultraproducts as in the proof of the Compactness Theorem. Note that either way, by symmetry it suffices to prove that $\mathcal{T}_{1}$ is finitely axiomatizable.
First proof: Suppose that $\mathcal{T}_{1}$ is not finitely axiomatizable. In other words, for every finite subtheory $\mathcal{T}^{\prime}$ of $\mathcal{T}_{1}$, there exists an $\mathcal{L}$-structure which is a model of $\mathcal{T}^{\prime}$ but not of $\mathcal{T}_{1}$. By hypothesis, this means that $X_{\mathcal{T}^{\prime}}$ is a model of $\mathcal{T}^{\prime} \cup \mathcal{T}_{2}$. But every finite subset of $\mathcal{T}:=\mathcal{T}_{1} \cup \mathcal{T}_{2}$ is contained in some $\mathcal{T}^{\prime} \cup \mathcal{T}_{2}$ for $\mathcal{T}^{\prime}$ a finite subset of $\mathcal{T}_{1}$. Thus the theory $\mathcal{T}$ is finitely satisfiable, hence satisfiable by the Compactness Thorem. But this means that there is a structure $X$ which models both $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$, contradiction.
Second proof: Again, suppose $\mathcal{T}_{1}$ is not finitely axiomatizable, so that for every finite subtheory $\mathcal{T}^{\prime}$ of $\mathcal{T}_{1}$ there is a model $X_{\mathcal{T}^{\prime}}$ of $\mathcal{T}^{\prime}$ but not of $\mathcal{T}_{1}$. Again, by our hypothesis $X_{\mathcal{T}^{\prime}}$ is a model of $\mathcal{T}_{2}$. Letting $I$ be the set of finite subtheories of $\mathcal{T}_{1}$, as in the proof of the compactness theorem, there exists an ultrafilter $\mathcal{F}$ on $I$ such that $X=\prod_{\mathcal{F}} X_{\mathcal{T}}$, is a model of $\mathcal{T}$. On the other hand, each $X_{\mathcal{T}}$, is a model of $\mathcal{T}_{2}$, so by Los's Theorem $X$ is also a model of $\mathcal{T}_{2}$ : contradiction. Thus $\mathcal{T}_{1}$ is finitely axiomatizable. Of course, interchanging the roles of $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ we get that $\mathcal{T}_{2}$ is finitely axiomatizable.
b) If $\mathcal{C}$ is the class of all models of a finite theory, then certainly it is finitely axiomatizable and indeed is the class of all models of a single sentence $\varphi$. But then its negation is the class of all models of $\neg \varphi$ so is also finitely axiomatizable, hence elementary. The converse follows immediately from part a).

Theorem 11. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures.
a) $\mathcal{C}$ is elementary iff it is closed under ultraproducts and elementary equivalence. b) $\mathcal{C}$ is finitely axiomatizble iff both $\mathcal{C}$ and its negation are closed under ultraproducts and elementary equivalence.
c) The elementary closure of $\mathcal{C}$ - i.e., the least elementary class containing $\mathcal{C}$ - is the class of all $\mathcal{L}$-structures which are elementarily equivalent to some ultraproduct of elements of $\mathcal{C}$.

Proof. a) It is clear from the definition that an elementary class - i.e., the class of all models of some $\mathcal{L}$-theory $\mathcal{T}$ is closed under elementary equivalence; moreover that an elementary class is closed under passage to ultraproducts is Corollary 9a). Conversely, suppose that $\mathcal{C}$ is a class which is closed under elementary equivalence and passage to ultraproducts. We wish to show that $\mathcal{C}$ is an elementary class. Clearly the only candidate theory is the complete theory of $\mathcal{C}$, i.e., the set of all $\mathcal{L}$-sentences which hold in every element of $\mathcal{C}$. Let $X$ be a model of $\mathcal{T}$. What we need to show is that $X \in \mathcal{C}$. Let $\Sigma$ be the complete theory of $X$ - so $\Sigma \supset \mathcal{T}$ - and as in the proof of the compactness theorem, let $I$ be the family of all finite subsets of $\Sigma$. For each $\mathcal{T}^{\prime}=\left\{\varphi_{1}, \ldots, \varphi_{n}\right\} \in I$, there exists $X_{\mathcal{T}^{\prime}} \in \mathcal{C}$ which is a model of $\mathcal{T}^{\prime}$, for otherwise the sentence $\neg\left(\varphi_{1} \wedge \ldots \wedge \varphi_{n}\right)$ would belong to $\mathcal{T} \backslash \Sigma$, a contradiction. Just as in the proof of the compactness theorem, there exists an ultrafilter $\mathcal{F}$ on $I$ such that the ultraproduct $X^{\prime}=\prod_{\mathcal{F}} X_{i}$ is a model of $\mathcal{T}$. By hypothesis, $X^{\prime} \in \mathcal{C}$. Moreover, since $\mathcal{T}$ is the complete theory of $X$, this means $X \equiv X^{\prime}$, and thus by hypothesis $X \in \mathcal{C}$.
Part b) follows immediately from part a) together with Proposition 11. The proof of part c) is similar and left to the reader.

Theorem 12. (Keisler-Shelah) Let $X$ and $Y$ be $\mathcal{L}$-structures. TFAE:
(i) $X \equiv Y$.
(ii) There exists an index set $I$ and an ultrafilter $\mathcal{F}$ on I such that the ultrapowers $X^{\mathcal{F}}$ and $Y^{\mathcal{F}}$ are isomorphic.

This theorem involves delicate set-theoretic considerations. Indeed, it was first proved by H.J. Keisler in 1961 under the assumption of the Generalized Continuum Hypothesis (GCH) and then unconditionally by S. Shelah in 1972. See e.g. [CK90, Thm. 6.1.15] for a proof.

Exercise 6.14: By considering a nontrivial ultraproduct of cyclic groups of prime order, show that the class of simple groups is not an elementary class.

Exercise 6.15 (harder): ${ }^{6}$ For all $n \in \mathbb{Z}^{+}$, we may view $S_{n}$ as a subgroup of $\operatorname{Aut}\left(\mathbb{Z}^{+}\right)$ by viewing it as the subgroup of permutations of $\mathbb{Z}^{+}$which pointwise fix every integer greater than $n$. With this convention, define the infinite alternating group $A_{\infty}=\bigcup_{n=1}^{\infty} A_{n}$ as a subgroup of $\operatorname{Aut}\left(\mathbb{Z}^{+}\right)$.
a) Show that $A_{\infty}$ is a simple group.
b) Show that no nontrivial ultrapower of $A_{\infty}$ is simple.
c) Deduce that the class of simple groups is not closed under elementary equivalence.

[^4]
## 7. A Glimpse of the Ax-Kochen Theorem

Let $d \in \mathbb{Z}^{+}$and $i \in \mathbb{R}^{\geq 0}$. We say that a field $K$ has property $C_{i}(d)$ if every degree $d$ homogeneous polynomial in at least $d^{i}+1$ variables has a nontrivial zero. It is clear that this property is equivalent to a sentence $\varphi_{d}$ in the language of fields, so the class $C_{i}(d)$ of fields is finitely axiomatizable. We also define a field to be $C_{i}$ if it is $C_{i}(d)$ for every positive number $d$. This is the conjunction of the infinitely many sentences $\varphi_{d}$, so $C_{i}$ is an elementary class.

Some relatively elementary facts:
a) a field is $C_{i}$ for some $i<1$ iff it is $C_{0}$ iff it is algebraically closed.
b) A finite field is $C_{1}$ (Chevalley).
c) If $K$ is $C_{i}$ and $L / K$ has transcendence degree $j$, then $L$ is $C_{j}$ (Tsen-Lang).
d) A complete discretely valued field with algebraically closed residue field is $C_{1}$ (Lang).
e) The field $\mathbb{F}_{q}((t))$ is $C_{2}$ (Lang).
f) If $k$ is $C_{i}$, then $k((t))$ is $C_{i+1}$ (Greenberg).

In particular, combining Chevalley and Greenberg, we find that the locally compact fields of positive characteristic, namely $\mathbb{F}_{q}((t))$, are $C_{2}$.

In view of Greenberg's theorem, it is natural to speculate that a complete discretely valued field with $C_{i}$ residue field is $C_{i+1}$. The simplest case of this which is left open by Lang's theorem is that of $p$-adic fields. Indeed, it was conjectured by E. Artin that a $p$-adic field is $C_{2} .^{7}$

Lang's seminal paper [Lan52] contains the sentence "If the residue field of [the CDVF] $F$ is finite, it has been conjectured that $F$ is $C_{2}$. We can prove this only in the case of power series fields, leaving the question open in the case of $p$-adic fields." This was part of Lang's thesis work; I can only imagine his consternation at not being able to prove the $p$-adic case. Lang and many others tried to prove this throughout the 50 's and the first half of the 60 's, without success. What was known is that $p$-adic fields are $C_{2}(2)$; in other words, a quadratic form over a $p$-adic field in at least 5 variables is isotropic. This is part of the classical theory of quadratic forms over local fields (and is discussed e.g. in the 8410 course notes). It was also known relatively early on that a cubic form in at least 10 variables has a nontrivial zero (due, I believe, to Davenport). And that was that!

Quite dramatically, in 1966 Guy Terjanian exhibited an anisotropic (i.e.., without nontrivial zero) quartic form over $\mathbb{Q}_{2}$ in 17 variables [Ter66]. Less well-known is a 1980 theorem of Terjanian [Ter80]: let $d>2$. Then for all primes $p$ with $p(p-1) \mid d$, there exists an anisotropic degree $d$ form in $d^{2}+1$ variables over $\mathbb{Q}_{p}$. In particular, for no prime $p$ is $\mathbb{Q}_{p} C_{2}$ !

On the other hand, James Ax and Simon Kochen proved in 1965 that p-adic fields are "almost $C_{2}$ ". More precisely:

[^5]Theorem 13. (Ax-Kochen Diophantine Theorem) For every positive integer d, there exists a constant $P(d)$ such that for all primes $p>d, \mathbb{Q}_{p}$ is $C_{2}(d)$.

Note that their proof gives precisely zero information about the constant $P(d)$, but Terjanian's work gives some lower bounds on it. To the best of my knowledge, for $d \geq 4$ no explicit upper bounds on $P(d)$ are known.

But this theorem reeks of model theory, and in particular of Ax's Transfer Principle. Here is what they actually proved:

Theorem 14. (Ax-Kochen Transfer Principle) Let $\mathcal{F}$ be a nonprincipal ultrafilter on the set $\mathcal{P}$ of prime numbers. Then the fields $\prod_{\mathcal{F}} \mathbb{Q}_{p}$ and $\prod_{\mathcal{F}} \mathbb{F}_{p}((t))$ are elementarily equivalent.

Exercise 6.16: Deduce Theorem 13 from Theorem 14. (Again, use Proposition 5b).)
The proof of Theorem comes from a penetrating analysis of the model theory of Henselian valued fields which is interesting and useful in its own right. (Sample result: the embedding from a Henselian valued field of characteristic 0 to its completion is an elementary embedding.) This would be a nice topic for a second half-course on applied model theory!

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[^0]:    ${ }^{1}$ One also has the notion of a filter base. But we won't use it, so let's skip the definition.
    ${ }^{2}$ This would be a filter base, had we defined such a thing. (Sorry!)

[^1]:    ${ }^{3}$ I had thought that they were covered in a standard undergraduate topology course. In retrospect, I think they were not covered in $m y$ undergraduate topology course (which used Munkres' book, as many such courses do) and indeed I may have learned about them for the first time when I started studying model theory in late 2002.

[^2]:    ${ }^{4}$ The fact that truth of quantifier-free formulas is preserved by embeddings of structures was given a somewhat handwavy proof earlier in these notes. What is required to formalize it is precisely an induction on formula complexity

[^3]:    ${ }^{5}$ Note that this is the only place in the proof where we use that $\mathcal{F}$ is an ultrafilter!

[^4]:    ${ }^{6}$ The material for this exercise was furnished by Simon Thomas as an answer to a question on Math Overflow. Thanks very much to him.

[^5]:    ${ }^{7}$ Or so people say; I am not sure if Artin's conjecture appears in written form.

