INDUCTION AND COMPLETENESS IN ORDERED SETS

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ABSTRACT. We define an **inductive subset** of an ordered set and show that an ordered set X is Dedekind complete iff the only inductive subset of X is X itself. This result is used to give short proofs of the characterizations of connectedness and compactness in order topologies.

Many mathematicians have found that certain results in elementary real analysis can be snappily proved by arguments reminiscent of proofs by mathematical induction. The various enunciations have usually been superficially different – c.f. [Kh23], [Pe26], [Kh49], [Du57], [Fo57], [MR68], [Sh72], [Be82], [Le82], [Sa84], [Ka07] – but recently D. Hathaway [Ha11] and the author both gave the following one.

Proposition 1. *(Principle of Real Induction) Let $S \subset [a,b]$, and suppose:*
(i) $a \in S$,
(ii) for all $x \in S$, if $x \neq b$ there exists $y > x$ such that $[x,y] \subset S$.
(iii) For all $x \in \mathbb{R}$, if $[a,x) \in S$, then $x \in S$.
Then $S = [a,b]$.

In this note we give an inductive characterization of Dedekind completeness in linearly ordered sets, and apply it to prove three topological characterizations of completeness which generalize familiar results from elementary analysis.

If a is an element of a linearly ordered set X, we define the intervals

$$(-\infty, a) = \{x \in S \mid x < a\}, \ (-\infty, a] = \{x \in S \mid x \leq a\},$$

$$(a, \infty) = \{x \in S \mid a < x\}, \ [a, \infty) = \{x \in S \mid a \leq x\}.$$

For a subset S of a linearly ordered set X, a **supremum** $\sup S$ of S is a least element which is greater than or equal to every element of S, and an **infimum** $\inf S$ of S is a greatest element which is less than or equal to every element of S. X is **complete** if every subset has a supremum; equivalently, if every subset has an infimum. If X is complete, it has a least element $0 = \sup \varnothing$ and a greatest element $1 = \inf \varnothing$. X is **Dedekind complete** if every nonempty bounded above subset has a supremum; equivalently, if every nonempty bounded below subset has an infimum. X is complete iff it is Dedekind complete and has 0 and 1.

We say a subset $S \subset X$ is **inductive** if it satisfies all of the following:
(IS1) There exists $a \in X$ such that $(-\infty, a] \subset S$.
(IS2) For all $x \in S$, either $x = 1$ or there exists $y > x$ such that $[x, y] \subset S$.
(IS3) For all $x \in X$, if $(-\infty, x) \subset S$, then $x \in S$.

Theorem 2. *(Principle of Ordered Induction) For a linearly ordered set X, TFAE:*
(i) X is Dedekind complete.
(ii) The only inductive subset of X is X itself.

Proof. (i) $\implies$ (ii): Let $S \subset X$ be inductive. Seeking a contradiction, we suppose $S' = X \setminus S$ is nonempty. Fix $a \in X$ satisfying (IS1). Then a is a lower bound for S', so by hypothesis S' has an infimum, say y. Any element less than y is strictly less than every element of S', so $(-\infty, y) \subset S$. By (IS3), $y \in S$. If $y = 1$, then $S' = \{1\}$ or $S' = \varnothing$: both are contradictions. So $y < 1$, and then by (IS2) there exists $z > y$ such that $[y, z] \subset S$ and thus $(-\infty, z] \subset S$. Thus z is a lower bound for S' which is strictly larger than y, contradiction.

(ii) $\implies$ (i): Let $T \subset X$ be nonempty and bounded below by a. Let S be the set of lower bounds for T. Then $(-\infty, a] \subset S$, so S satisfies (IS1).

Case 1: Suppose S does not satisfy (IS2): there is $x \in S$ with no $y \in X$ such that $[x, y] \subset S$. Since S is downward closed, x is the top element of S and $x = \inf(T)$.

Case 2: Suppose S does not satisfy (IS3): there is $x \in X$ such that $(-\infty, x) \in S$ but $x \notin S$, i.e., there exists $t \in T$ such that $t < x$. Then also $t \in S$, so t is the least element of T: in particular $t = \inf T$.

Case 3: If S satisfies (IS2) and (IS3), then $S = X$, so $T = \{1\}$ and $\inf T = 1$. $\square$

Remark 1: Since a linearly ordered set X is Dedekind complete iff its order dual is Dedekind complete, there is a corresponding "downward" version of Theorem 2.

Remark 2: Taking X to be well-ordered, we recover transfinite induction. Taking $X = [a, b] \subset \mathbb{R}$, we recover Proposition 1.

A **bounded open interval** in a linearly ordered set X is an interval of the form (a, b) or $[0, b)$ (if X has a bottom element) or $(a, 1]$ (if X has a top element). The bounded open intervals form a base for the **order topology** on X. A linearly ordered set X is **dense** if for all $a < b \in X$, there exists c with $a < c < b$.

Lemma 3. *A subset of a Dedekind complete linearly ordered set is Dedekind complete if it is closed in the order topology.*

Proof. Left to the reader. $\square$

Theorem 4. *For a linearly ordered set X, TFAE:*
(i) X is dense and Dedekind complete.
(ii) X is connected in the order topology.

Proof. (i) $\implies$ (ii): Step 1: We suppose $0 \in X$. Since X is dense, a subset $S \subset X$ which contains 0 and is both open and closed in the order topology is inductive. Since X is Dedekind complete, by Theorem 2, $S = X$. This shows X is connected! Step 2: We may assume $X \neq \varnothing$ and choose $a \in X$. By Lemma 3, Step 1 applies to show $[a, \infty)$ connected. A similar downward induction argument shows $(-\infty, a]$ is connected. Since $X = (-\infty, a] \cup [a, \infty)$ and $(-\infty, a] \cap [a, \infty) \neq \varnothing$, X is connected.

(ii) $\implies$ (i): If the order is not dense, there are $a < b$ in X with $[a, b] = \{a, b\}$, so $A = (-\infty, a]$, $B = [b, \infty)$ is a separation of X. Suppose we have $S \subset X$, nonempty, bounded below by a and with no infimum. Let L be the set of lower bounds for S, and put $U = \bigcup_{b \in L}(-\infty, b)$, so U is open and $U < S$. We have $a \neq \inf(S)$, so $a \in U$, and thus $U \neq \varnothing$. If $x \notin U$, then $x \geq L$ and, indeed, since L has no maximal element, $x > L$, so there exists $s \in S$ such that $s < x$. Since the order is dense there is y with $s < y < x$, and then the entire open set (y, ∞) lies in the complement of U. Thus U is also closed. Since X is connected, $U = X$, contradicting $U < S$. $\square$

Theorem 5. *For a nonempty linearly ordered set X, TFAE:*
(i) X is complete.
(ii) X is compact in the order topology.

Proof. (i) $\implies$ (ii): Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. Let S be the set of $x \in X$ such that the covering $\mathcal{U} \cap [0, x]$ of $[0, x]$ admits a finite subcovering. $0 \in S$, so S satisfies (IS1). Suppose $U_1 \cap [0, x], \ldots, U_n \cap [0, x]$ covers $[0, x]$. If there exists $y \in X$ such that $[x, y] = \{x, y\}$, then adding to the covering any element U_y containing y gives a finite covering of $[0, y]$. Otherwise some U_i contains x and hence also $[x, y]$ for some $y > x$. So S satisfies (IS2). Now suppose that $x \neq \mathbb{B}$ and $(-\infty, x) \subset S$. Let $i_x \in I$ be such that $x \in U_{i_x}$, and let $y < x$ be such that $(y, x] \in U_{i_x}$. Since $y \in S$, there is a finite $J \subset I$ with $\bigcup_{i \in J} U_i \supset [a, y]$, so $\{U_i\}_{i \in J} \cup U_{i_x} \supset [a, x]$. Thus $x \in S$ and S satisfies (IS3). Thus S is an inductive subset of the Dedekind complete ordered set X, so $S = X$. In particular $\mathbb{T} \in S$, hence the covering has a finite subcovering.

(ii) $\implies$ (i): For each $x \in X$ there is a bounded open interval I_x containing x. If X is compact, $\{I_x\}_{x \in X}$ has a finite subcovering, so X is bounded, i.e., has 0 and 1. Let $S \subset X$. Since $\inf \varnothing = 1$, we may assume $S \neq \varnothing$. Since S has an infimum iff $\overline{S}$ does, we may assume S is closed and thus compact. Let L be the set of lower bounds for S. For each $(b, s) \in L \times S$, consider the closed interval $C_{b,s} := [b, s]$. For any finite subset $\{(b_1, s_1), \ldots, (b_n, s_n)\}$ of $L \times S$, $\bigcap_{i=1}^{N} [b_i, s_i] \supset [\max b_i, \min s_i] \neq \varnothing$. Since S is compact there is $y \in \bigcap_{L \times S} [b, s]$ and then $y = \inf S$. $\square$

Corollary 6. *(Heine-Borel) For a linearly ordered set X, TFAE:*
(i) X is Dedekind complete.
(ii) A subset S of X is compact in the order topology iff it is closed and bounded.

Proof. (i) $\implies$ (ii): A compact subset of any ordered space is closed and bounded. Conversely, if X is Dedekind complete and $S \subset X$ is closed and bounded, then by Lemma 3, S is complete and then by Theorem 5, S is compact.

(ii) $\implies$ (i): If $S \subset X$ is nonempty and bounded above, let $a \in S$. Then $S' = S \cap [a, \infty)$ is bounded, so $\overline{S'}$ is compact and thus $\overline{S'}$ is complete by Theorem 5. The least upper bound of $\overline{S'}$ is also the least upper bound of S. $\square$

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