HANDOUT THREE: GEOMETRY OF SPACE CURVES

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1. Unit speed parameterizations

In the last lecture we studied properties of a particle in motion in (two or) three-dimensional space. Recall the fundamental point that the same curve can have many different parameterizations. Some statements about particle motion depend on the particular choice of parameterization: for instance, Kepler's second law says that as a planet revolves around the sun, it sweeps out equal areas in equal times, and we noted in the last handout that for a particle moving in an elliptical orbit to have this property, the speed *cannot* be constant (but rather is maximized when the particle is closest to the sun and minimized when it is farthest away). More fundamentally, the **speed** of a particle depends strongly on the parameterization: e.g. the parametric curve

$$\mathbf{r}(t) = (\cos kt, \sin kt)$$

has speed

$$\sqrt{(-k\sin(kt))^2 + (k\cos(kt))^2} = |k|\sqrt{\cos^2(kt) + \sin^2(kt)} = |k|.$$

On the other hand, if we reparameterize a curve, then the tangent line will not change: suppose instead of $\mathbf{r}(t) = (x(t), y(t), z(t))$ we took some function $g(t)$ and considered instead $\mathbf{r}(g(t)) = (x(g(t)), y(g(t)), z(g(t))$. Then by the chain rule

$$d/dt(\mathbf{r}(g(t)) = (x'(g(t))g'(t), y'(g(t))g'(t), z'(g(t))g'(t)) = g'(t)(x'(g(t)), y'(g(t)), z'(g(t))$$

and for any time t_0, the new velocity vector comes out as a scalar multiple of the old velocity vector evaluated at time $g(t_0)$.

If we assume that $g(t)$ is an *increasing* function of t, then $g'(t)$ is always positive, so the **unit tangent vector**

$$T(t) := \hat{\mathbf{v}}(t) = \mathbf{v}(t)/||\mathbf{v}(t)||$$

is well-defined among all reparameterizations of the curve by increasing functions $g(t)$. (Don't get thrown off by the fancy langugage: the point is that if we allowed arbitrary reparameterizations, the particle could travel "back and forth" along the curve, whereas we insist that it always travel along the curve in the same direction.)

Imagine we have some curve in three-dimensional space. It makes sense to ask for a parameterization where the speed is always one – i.e., so that the velocity vector $\mathbf{v}(t)$ is already a unit vector so is equal to the unit tangent vector $T(t)$. Recalling that speed is the derivative of arc-length, what this means it that we want the particle to be travelling along the curve in such a way so that the arclength of the particle between time t_0 and t_1 is equal to

$$\int_{t_0}^{t_1} ||\mathbf{v}'(t)||dt = \int_{t_0}^{t_1} 1 \quad dt = t_1 - t_0.$$

That is, if a particle travels with unit speed, then arclength accumulated in an interval of time from t_0 to t_1 *is* just the difference in time $t_1 - t_0$. This should explain why another name for unit speed parameterizations is **parameterization by arclength**.

I hope you agree that it is rather clear intuitively that every curve has a unit speed parameterization: we just "drive" along the curve at unit speed! In more analytical terms, given a parameterized curve $\mathbf{r}(t) = (x(t), y(t), z(t))$, we can give an "explicit" reparameterization which has unit speed. To do this, let $s(t) = \int_0^t ||\check{\ }'(w)||dw$ be the function which gives the arclength from time 0 to time t. This function is continuous and increasing, so has an inverse function $s^{-1}(t)$. Recall that this means that $s(s^{-1}(t)) = s^{-1}(s(t)) = t$, i.e., applying s and then s^{-1} in either order, we get back to where we started. Then indeed $\mathbf{r}(s^{-1}(t))$ is a reparameterization by arclength. (It pains me, but) we omit the proof of this, because it is not very useful in practice, because the arclength function $s(t)$ – let alone its inverse! – can so rarely be evaluated in terms of "elementary functions".[1] Thus in theory every curve has an explicit unit speed reparameterization, but in practice we usually do not actually do the reparameterization.[2]

One might well ask why we have discussed unit speed parameterizations at all if we are not going to use them in practice (except in the simplest cases). The answer is that the properties of a curve that we are going to study are best understood by contemplating a curve which is being traversed at unit speed. Then, when it comes to deriving the formulas, we do some chain rule calculation including a term like $ds/dt = ||\mathbf{v}'(t)||$ which keeps track of the fact that the parameterization may in fact have variable speed.

2. Curvature

We come to our first example: we want a definition of the **curvature** of a curve in the plane or in $\mathbb{R}^3$. This is a good example of a struggle to give a precise definition of a concept which we certainly have an intuitive idea. Namely, curvature should be some kind of measure of how sharply the path is turning, and at least it should have the properties that the curvature of a circle is the same at every point and the curvature of a straight line is zero.

The "sharpness of turning" sounds like the rate of change of the tangent line, i.e., it should have something to do with a second derivative. But this is not exactly right: for instance, consider the parameterization $\mathbf{r}(t) = (t, t^2)$ of the parabola $y = x^2$. The problem is that $\mathbf{r}'(t) = (1, 2t)$ and $\mathbf{r}''(t) = (0, 2)$, i.e., the second derivative is

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[1] Recall that in Friday's class we computed the arclength integral for the standard parameterization of an ellipse, getting $s(t) = \int_0^t \sqrt{a^2 + (b^2 - a^2)\sin^2 w} dw$. You might try giving this integral to a software program like Mathematica and Maple and seeing what it tells you.

[2] Josh Laison told me the following joke: "Q: What is the difference between theory and practice? A: In theory there is no difference."

constant. But intuitively the curvature of a parabola is not constant: the parabola turns most sharply at its vertex – here $(0, 0)$ and looks more and more like a straight line the farther we get away from the vertex. On the other hand, at least a function has $\mathbf{r}''(t) = 0$ if and only if it is a straight line, which should have curvature zero. This leads us to believe that the correct curvature formula should be $\mathbf{r}''(t)$ times something else.

In fact the problem with trying to get the curvature from $\mathbf{v}'(t)$ is that the parameterization need not have constant speed: if the particle is speeding up and slowing down along the path (as indeed happens in our example, the speed at t being $||(1, 2t)|| = \sqrt{1 + 4t^2}$). In fact if our path had unit speed, then the definition of curvature as $\mathbf{v}'(t) = T'(t)$ would be correct. In general, we want to keep track of the rate of change of the unit tangent vector not with respect to time, but rather the rate of change per unit length of arc:

$$\kappa(t) = ||dT/ds||.$$

Now, since $T'(t) = dT/dt = dT/dsds/dt$, we get

$$\kappa(t) = ||dT/ds|| = \frac{||dT/dt||}{|ds/dt|} = \frac{||dT/dt||}{||v(t)||}.$$

Example: If $\mathbf{r}(t) = (x_0 + at, y_0 + bt, z_0 + ct)$ is a straight line, then $T(t) = (a, b, c)/||(a, b, c)||$ is a constant, so $||dT/dt|| = 0$ – a straight line has zero curvature, as it should.

Example: We compute the curvature of a circle of radius R, namely $\mathbf{r}(t) = (Rt, R\sin t)$. Then the velocity vector is $(-R\sin t, R\cos t)$, which has norm R, so the unit tangent vector is just $T(t) = (-\sin t, \cos t)$. So $||dT/dt|| = ||(,-)|| = 1$. Therefore the curvature at any point t is one over the speed, so is constant for all t and is equal to $1/R$. Because of this example, one sometimes defines the **radius of curvature** of a curve at a point t to be $\frac{1}{\kappa t)}$, the idea being that if the radius of curvature is R, then the sharpness of turning of the curve at that point is precisely the same as for a circle of radius R.

3. The principal normal and the binormal

Recall that we showed in class on Friday that if a particle has constant speed $||\mathbf{v}(t)||$, then its acceleration is $a(t) = \mathbf{r}''(t)$ is perpendicular to the tangent vector: $a(t) \cdot v(t) = a(t) \cdot T(t) = 0$. If our curve does *not* have unit speed parameterization, then the derivative of the unit tangent vector will have this property: namely $dT/dt \cdot T = 0$.