HANDOUT FIVE: VECTOR FIELDS

PETE L. CLARK

1. Introduction to Vector Fields in the plane and in space

We have already studied several kinds of functions of several variables: vector valued functions of a scalar variable – i.e., parameterized curves in the plane and in space; scalar-valued functions of two or more variables (the case of two variables $z = f(x, y)$ giving the graph of a surface in space; and functions from the plane to space, parameterized surfaces. Here we consider functions V from the plane to itself and from space to itself, which are called **vector fields**.

A vector field in the plane is given by a pair of functions of two variables, $V(x, y) = (P(x, y), Q(x, y)$. We picture it as follows: at each point (x_0, y_0) in the plane, we get a vector $(P(x_0, y_0), Q(x_0, y_0))$ at that point. So in all every point in the plane (or every point in a certain region of the plane) has a vector attached to it: overall we get a "field" of vectors.

A similar story holds in space: a vector field gives a vector at every point of space, so to describe it we need three functions of x, y and z: $V(x, y, z) = (P(x, y, z), Q(x, y, z,), R(x, y, z))$.

Example:

$$F(x, y, z) = \frac{-(x, y, z)}{||(x, y, z)||^3} = \left(\frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \right).$$

This vector field gives, up to a multiplicative constant, the force felt by a particle at any point due to gravitational attraction to a mass centered at the origin, i.e., it is a reformulation of Newton's **inverse square law**. That is, at any point, the vector at that point points in the direction of the origin, and its magnitude is inversely proportional to the square of the distance from that point to the origin[1]. Note also that the vector field is **not** defined at the origin, nor could it be extended to the origin in a continuous manner, for two reasons: one the one hand the magnitude of the vector field approaches infinity, and on the other hand therer are vectors in the field arbitrarily close to the origin pointing in every direction, so the direction of the vector field cannot be continuously extended to the origin either. In such a situation – namely, when there is a point $P_0 = (x_0, y_0, z_0)$ at which a vector field is not defined and could not be defined in a continuous way – P_0 is called a **singularity** of the vector field.

We saw on p. 13 of handout four that if $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$, then

[1]Notice that the norm of the numerator is $|| - (x, y, z)|| = d$, the distance to the origin, so the norm of the entire expression is $\frac{d}{d^3} = \frac{1}{d^2}$.

$V = \nabla(f)$. We will see in the next unit that the existence of this function f leads to a law of **conservation of energy** for a particle travelling through a gravitational field: at any point, the sum of the kinetic energy $1/2mv^2$ and the potential energy $-f$ is a constant.

Other examples of vector fields modelling fields are $V_2 = (0, 0, -mg)$, the constant downward vector field modelling the constant force due to gravity of a particle close to the surface of the earth, and $V_3 = \frac{(x,y,z)}{||(x,y,z)||^3}$ – the same as Newton's law except without a minus sign, which models the situation in which we have a **positive charge** at the origin and want the force felt by a positively charged particle at a given point of space: this is a repulsive force, and that it also is inversely proportional to the square of the distance is known as **Coulomb's Law**.

On the other hand, when trying to picture a vector field in the plane or in space, it is convenient to have a different physical interpretation in mind: we think of V as the **velocity field** of some sort of fluid: that is, at any point P_0, the field $V(P_0)$ gives the velocity vector for the flow: i.e., it tells the fluid which way to go and how fast.

Viewing a vector field $V(x, y)$ or $V(x, y, z)$ as a velocity field sets up a fundamental geometric problem: given a velocity field, place a particle at a certain **initial point** P_0 and release it: what is its trajectory? That is, we want to find a curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ such that $\mathbf{r}(0) = P_0$ and with the property that for all t

$$\mathbf{r}'(t) = \mathbf{v}(t) = V(\mathbf{r}(t)). \tag{1}$$

Such a curve is called an **integral curve** and Equation (1) is called the **flow equation**: it is a system of ordinary differential equations in the variables x, y and z: see Section 11.1 of your text.

We now give some examples of vector fields in the plane.

Example 1: $V(x, y) = (x, y)$. That is, at any point in the plane, the vector at that point *is* that point; otherwise put, it points in the same direction (i.e., away from the origin) and has the same magnitude. Suppose we interpret this field as a velocity field and want to find the trajectories. As we just said that at every point a particle is pushed radially outward, the trajectories should be straight lines emanating outward from the origin. In terms of equations we have

$$x'(t) = P(x, y) = x.$$

$$y'(t) = Q(x, y) = y.$$

Here (quite luckily) each of the two equations involves only one variable at a time, so we just need to solve the differential equation $df/dt = f$ both times: the solution to this is $f(t) = Ce^t$, so $x(t) = C_1e^t$, $y(t) = C_2e^t$. Indeed $x(0) = C_1e^0 = C_1$ and $y(0) = C_2e^0 = C_2$, so the integral curve passing through the point (x_0, y_0) is $\mathbf{r}(t) = (x_0e^t, y_0e^t)$. This is a straight line since $y/x = \frac{y_0e^t}{x_0e^t} = y_0/x_0$. Note the special point $(x_0, y_0) = (0, 0)$, at which the vector field is zero. If we start at a point and the velocity vector is zero, then we never leave! For this reason a point (x_0, y_0) where $F(x_0, y_0) = (0, 0)$ is often called a **stationary point**.

Example 2: $V(x,y) = (-x,-y)$. This is exactly the opposite vector field. Geometrically, at any nonzero point we are getting pushed back towards the origin, so we expect the integral curves to be straight lines converging to the origin. Indeed, the system is $(x',y') = (-x,-y)$ which boils down to $dx/dt = -x,\ dy/dt = -y$, with solutions $x(t) = C_1 e^{-t},\ y(t) = C_2 e^{-t}$, and again we have $C_1 = x_0,\ C_2 = y_0$ and $y(t)/x(t) = y_0/x_0$ a constant. But this time

$$\lim_{t\to\infty} \mathbf{r}(t) = \lim_{t\to\infty} (x_0 e^{-t}, y_0 e^{-t}) = (0,0),$$

so all the trajectories approach the origin (the one that starts at the origin doesn't go anywhere).

Example 3: $\mathrm{V}_L(x,y) = (-y,x)$. Notice that $(-y,x)\cdot(x,y) = -yx + xy = 0$, so the velocity vector at a point $P_0 = (x_0,y_0)$ is perpendicular to the line segment $\overline{OP_0}$ – indeed, testing the point $(1,0) \mapsto V(1,0) = (0,1)$, it is 90 degrees to the left of $\overline{OP_0}$. It thus seems plausible that the trajectories will be counterclockwise circles centered at the origin. This time the system of differential equations we get is

$$x'(t) = -y(t),\ y'(t) = x(t)$$

so that $x = y' = (-x')' = -x''$, and also $y'' = -y$. A solution to this is $x(t) = R\cos t,\ y(t) = R\sin t$. So the trajectories are counterclockwise circles of radius R.

Example 4: $\mathrm{V}_R(x,y) = (y,-x)$. Again $(y,-x)\cdot(x,y) = 0$ so the velocity vector at a point P_0 is perpendicular to the radial vector $\overline{OP_0}$, but this time $(1,0) \mapsto V(1,0) = (0,-1)$, which turns 90 degrees to the *right*. Going back through the previous argument, we see that indeed the trajectories are clockwise circles.

Example 5: Put $r = \sqrt{x^2+y^2}$, and consider the vector field $V_a(x,y) = (\frac{-y}{r^a}, \frac{x}{r^a})$, where a is some constant. Since $V_a(x,y) = V_L(x,y)/r^a$, we are just rescaling the vector field of Example 3 so that at each point the vector field points 90 degrees to the left of the radial vector $\overline{OP_0}$ but has norm $||(-y,x)/r^a|| = ||(-y,x)||/r^a = r/r^a = r^{1-a}$.

2. Divergence and Curl

Let $V(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field in space. We view a vector field $F(x,y)$ in the plane as a special case of this, with $R \equiv 0$. Here are two different ways to, roughly speaking, "take a derivative" of a vector field:

The divergence:

$$\mathrm{Div}(V) = \nabla \cdot V = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (P,Q,R) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The curl:

$$\mathrm{curl}(V) = \nabla \times V = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

Note well that the divergence of a vector field is a *scalar*-valued function of three variables, whereas the curl of a vector field is another vector field.

Clearly it is no harder to compute the divergence or the curl of a vector field than the gradient of a function: either way we're just taking some partial derivatives. On the other hand, the gradient has a useful geometric interpretation as the path of steepest ascent. It would be nice to have some similar geometric and/or physical intuition for the divergence and the curl. Let's try to see what they are by looking at some of our examples from the last section.

Before we do this, however, we note the special case of the curl of a planar vector field $V(x,y) = (P(x,y), Q(x,y), 0)$: the only terms in the definition of the curl in which we are not either differentiating the z-component R – which is zero, or with respect to z – which is 0 – is the last, so we get

$$\operatorname{curl}(P(x,y), Q(x,y), 0) = (0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\hat{\mathbf{k}}.$$

Thus, although the curl of a planar vector field is technically a vector-valued function, it always points in the same direction, so it is useful to think of the "scalar curl" $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ and remember that its direction is always perpendicular to the plane.

Example 1: $V(x,y) = (x,y)$. Then $\operatorname{Div}(V) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2$. Also $\operatorname{curl}(V) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0$. Thus, for this "radially outward" vector field, the divergence is positive at every point and the curl is zero.

Example 2: $V(x,y) = (-x,-y)$. Then $\operatorname{Div}(V) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2$, whereas $\operatorname{curl}(V) = \frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(-x) = 0$. This "radially inward" vector field has constant negative divergence and zero curl.

Example 3: $V_L(x,y) = (-y,x)$. Now $\operatorname{Div}(V) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$, whereas $\operatorname{curl}(V_L) = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2$. This "purely rotational" vector field has no divergence but positive curl (and note that we are rotating in a positive direction).

Example 4: $V_R(x,y) = (y,-x) = -V_L(x,y)$. Indeed $\operatorname{Div}(-V) = -\operatorname{Div}(V)$ and $\operatorname{curl}(-V) = -\operatorname{curl}(V)$ for any vector field V, as you are invited to check, so this clockwise rotational vector field has zero divergence and negative curl.

From these examples it seems that divergence is positive or negative according to whether there is an outward or inward flow at a point, and the (scalar part of the) curl is positive or negative according to whether there is a counterclockwise or clockwise rotation about the point. This is the right idea, but we need to be careful that we understand why the picture works for *all points* of these vector fields rather than just at the origin. Indeed, we are saying that a point has positive divergence if it is overall a "source" for a fluid: it has more fluid flowing out than flowing in. This is clear at the origin for Example 1; why is it true at some other point (x_0, y_0)? The answer is that the flow is purely radial, and the *magnitude* of the flow is increasing with the distance from the origin. That is, consider the ray joining the origin to (x_0, y_0). In between the origin and (x_0, y_0) the speed is *smaller* than it is at (x_0, y_0), and beyond (x_0, y_0) the speed is *larger* than it is at that point. This does mean that fluid is flowing out faster than it is flowing in, so the divergence is positive.

Divergence as flux density: We can argue more generally that divergence at a point represents a "net flow" as follows: suppose we have a fluid flowing in three-dimensional space and we draw an imaginary box around the fluid at a certain point $P_0 = (x_0, y_0, z_0)$. Then we have the notion of the **flux** through the surface of the box, which is the total amount of fluid leaving the box minus the total amount of fluid entering the box. Now imagine we do this with a variable box of volume V: as $V \to 0$ we are zooming in on the net flow about the point P_0. Indeed, it makes sense to define the **flux density** as the limit as $V \to 0$ of the flux through a region of surface area A divided by the surface area V.

We claim that the flux density at P_0 is exactly the divergence at P_0. Indeed our box has six faces, so we need to compute the flux through each face, which is approximately $(F \cdot \mathbf{n})\Delta S$, where $\mathbf{n}$ is the outward normal vector for the face and ΔS is the surface area. Suppose our point (x_0, y_0, z_0) is at the bottom-left corner of the box, and the sides of the box have length Δx, Δy and Δz. Then the surface area of the top and bottom faces is $\Delta x \Delta y$, the outward normal for the top face is $\hat{\mathbf{k}}$ and the outward normal for the bottom face is $-\hat{\mathbf{k}}$. So the flux along the top face is approximately $F \cdot \hat{\mathbf{k}} = \mathrm{Q}(\mathrm{x}_0, y_0, z_0 + \Delta z)\Delta x \Delta y$.[2] The flux through the bottom face is $F \cdot (-\hat{\mathbf{k}}) = -Q(x_0, y_0, z_0)\Delta y \Delta z$, so the net flux through the top and bottom faces is the difference of these, or:

$$Q(x_0, y_0, z_0 + \Delta z)\Delta x \Delta y - Q(x_0, y_0, z_0)\Delta x \Delta y =$$

$$\Delta x \Delta y \left(Q(x_0, y_0, z_0 + \Delta z) - Q(x_0, y_0, z_0)\right) = (\Delta x \Delta y \Delta z)\left(\frac{Q(x_0, y_0, z_0 + \Delta z) - Q(x_0, y_0, z_0)}{\Delta z}\right).$$

But $(\Delta x \Delta y \Delta z)$ is the volume of the box, and as Δz goes to zero the other factor approaches $\frac{\partial Q}{\partial z}$, so we get $\lim_{\Delta x \Delta y \Delta z \to 0}$ (flux through top and bottom faces)/(volume of the box) is $/dz$. Now we have also to compute the same limit using the left and right faces, getting $\frac{\partial P}{\partial x}$ and through the front and back faces, getting $\frac{\partial Q}{\partial y}$. Now we have accounted for all six faces in the surface, so the total flux density is $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \mathrm{Div}(V)$, and we've shown that the flux density is equal to the divergence!

Now that we have justified our phyiscal intuition about the divergence, we introduce the terminology that a vector field whose divergence is identically zero is **incompressible**.

Here is the corresponding physical intuition for the curl of a vector field: a curl measures the rotation of a vector field about a point in the following sense. First consider the case of a planar vector field: if we stuck a paddlewheel in the fluid at the point P_0, then it will turn in the direction of the curl: i.e., counterclockwise if the curl is positive and clockwise if the curl is negative. It is understood that the axis of rotation is the z-axis. If we now have a three-dimensional vector field, the

[2] It's only *approximately* this because the x and y-coordinates are varying along this face, and we are assuming they are constant. The exact value of the flux is given by a **surface integral**, which we will see later in the course, and this argument will reappear in the form of the **Divergence Theorem**.

curl captures the net rotation, which will have *some axis*: in other words, the axis is such that if we orient the paddlewheel in the direction of curl(F) it will turn with maximum speed; if we orient it perpendicular to curl(F) it will not turn at all, and in general it will turn with a speed of $||\operatorname{curl}(F)|| \sin\theta$, where θ is the angle between the axis of the paddlewheel and curl(F) (this comes from the cross product formula $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||||\mathbf{w}|| \sin\theta$).

In line with this interpretation, we say that a vector field F with curl$(F) \equiv 0$ is **irrotational**.

It is worth asking how one could prove this statement: how does one give a rigorous argument about a "paddlewheel"? (Indeed, when I took multivariable calculus, more than ten years ago now, I regarded this statement about the curl measuring rotation with great suspicion.) But it will turn out that at the very end of the course we will be in a position to understand rigorously this geometric interpretation of the curl, as well as to revisit the interpretation of the divergence as flux density. Indeed, these statements are the geometry behind the two most important results in the course, Stokes' Theorem (for the curl) and the Divergence Theorem (for the divergence, of course!).

We want to give another example to show that a vector field being irrotational is actually quite subtle: we really cannot tell just by looking at a rough sketch of it. For instance, on p. 484 of your text, it is pointed out that just because the trajectories of a vector field are circles, it does not necessarily mean that there is nonzero curl. This is because the curl measures the tendency to rotate *locally* about that point, not the tendency for all trajectories to rotate about some central axis. (The curl is defined in terms of partial derivatives at a point, so is a statement about very small neighborhoods about that point; it could not "see" the long-term behavior of rotation.) However from the arrows drawn in for the vector field in Figure 9.42b one cannot conclude whether the vector field is irrotational or not. We see explore this by considering the family of vector fields from the last section.

Example: For any number a, we put $F_a(x,y) = (\frac{-y}{r^a}, \frac{x}{r^a})$, where $r = \sqrt{x^2+y^2}$. Note that all of these vector fields differ from the $a = 0$ case just by rescaling: in particular all the trajectories are circles. We computed in the last section that when $a = 0$ the curl was identically equal to 2. Let's see what happens in general: the scalar part of the curl is $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. Let us compute $\frac{dr}{dx}$ and $\frac{dr}{dy}$ in advance. We have

$$\frac{dr}{dx} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r},$$

and similarly

$$\frac{dr}{dy} = \frac{y}{r}.$$

So

$$\frac{\partial Q}{\partial x} = d(x/r^a)/dx = 1/r^a + x(d(r^{-a}/dx) = r^{-a} + x(-a)r^{-a-1}dr/dx) =$$

$$r^{-a} - a(x)r^{-a-1}(x/r) = r^{-a} - ax^2r^{-a-2} = \frac{r^2 - ax^2}{r^{a+2}}.$$

Similarly

$$-\frac{\partial P}{\partial y} = -d(-y/r^a)/dy = d(y/r^a)/dy = \frac{r^2 - 2ax^2}{r^{a-1}}.$$

So the scalar part of the curl is

$$\frac{2(r^2 - a(x^2 + y^2))}{r^{a+2}} = \frac{(2-a)(x^2+y^2)}{r^{a+2}}.$$

That is, when $a < 2$ the curl is positive – in particular this covers the case $a = 0$ – when $a > 2$ the curl is negative: in other words, in this case, despite the fact that any given particle travels around in a counterclockwise circle, a paddlewheel nailed to any point will spin *clockwise*. Most of all, at the special value $a = 2$ we get an irrotational vector field,

$$F_\star(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

This very special vector field will come up again several times in the course.

Example: If $f = f(x,y,z)$ is a function of three variables and $F = \nabla(f)$, then $\mathrm{curl}(F) = 0$. That is, gradient vector fields are irrotational. We leave this calculation as an exercise (Exercise 29 in Section 9.7).

3. The turning operators L and R

In this section we expose a little secret about vector fields: for planar vector fields, the curl and the divergence can be understood in terms of each other just by **turning** the vector field. For this, we introduce the following two simple operators:

$$L(x,y) = (-y,x), \ \ R(x,y) = (y,-x).$$

As we saw above, for any vector in the plane $\mathbf{v} = (x,y)$, $L\mathbf{v}$ just gives us the vector which is rotated 90 degrees to the **left**, whereas $R\mathbf{v}$ is the vector rotated 90 degrees to the right.

L and R can also be applied to vector fields:

$$L : (P(x,y), Q(x,y) \mapsto (-Q(x,y), P(x,y)).$$

$$R : (P(x,y), Q(x,y) \mapsto (Q(x,y), -P(x,y)).$$

Geometrically, this just means that $L(F)$ is obtained from F just by spinning each vector 90 degrees to the left, and similarly $R(F)$ is obtained by F just by spinning each vector 90 degrees to the right. Note that this is *not* the same as spinning the entire plane 90 degrees: for instance, the vector fields $F_1(x,y)$ and $F_2(y,x) = (-y,x)$ are both symmetric about all rotations through the origin, but $L(F_1) = F_2$, and $R(F_2) = F_1$.

Recall now that $F_1(x,y) = (x,y)$ has constant divergence 2 and constant curl 0, whereas $F_2(x,y) = (-y,x)$ has constant divergence 0 and constant curl 2. But $L(F_1) = F_2$. This is an instance of the following simple but useful fact: For any planar vector field $F = (P(x,y), Q(x,y))$,

$$\mathrm{curl}(L(F)) = \mathrm{Div}(F)$$

$$\mathrm{Div}(R(F)) = \mathrm{curl}(F).$$

This works in general for the same reason as the above example: $F = (P, Q)$ implies

$$L(F) = (-Q, P)$$

, so $\mathrm{curl}(L(F)) = \frac{\partial}{\partial x}(P) - \frac{\partial}{\partial y}(-Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \mathrm{Div}(F)$.Similarly, $R(F) = (Q, -P)$, so

$$\mathrm{Div}(R(F)) = \frac{\partial}{\partial x}(Q) + \frac{\partial}{\partial y}(-P) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \mathrm{curl}(F).$$

Nothing like this works in space, since there are infinitely many different possible axes for a rotation.

This observation is certainly not very deep: there is no more content than the fact that $(x, y) \cdot (-y, x) = 0$. Nevertheless it will prove useful late in the day: it will help us to understand the relationship between Green's Theorem, Stokes' Theorem and the Divergence Theorem.

4. Extra: Unit tangent vector fields on the sphere and the torus

If we have a surface S in space, then it makes sense to consider vector fields $F(x, y, z)$ defined on the surface. Indeed, the flux of a vector field through a closed surface is obtained by adding up the contribution of the component of the vector field which is **normal** to the surface at every point, a construction which is called a **surface integral** and will be studied later in the course.

But consider now exactly "the opposite" kind of vector field on a surface: namely a vector field which is **tangent** to the surface at every point. For instance $F(x, y) = (-y, x)$ is a tangent vector field to the unit circle at every point.

Suppose we look for tangent vector fields on a surface satisfying the additional condition that every vector is a **unit vector** at any point: call this a **unit tangent** vector field on the surface S. If we have a tangent vector field that is merely *nonzero* at every point, then we can just divide by the norm to get a unit tangent field: for instance, we can renormalize the above example to get $F_T = (\frac{-y}{r}, \frac{x}{r})$, where as usual $r = \sqrt{x^2 + y^2}$. So F_T is a unit tangent vector field on the unit circle.

But the circle is a curve, not a surface: what about a unit tangent vector field on the unit sphere? We can visualize what we are asking for as follows: suppose that the sphere is **hairy**: at each point we have a one inch hair emanating from that point. We want to **comb** the hair on the sphere, meaning we want the hair to lie flat against the surface of the sphere, and we want to do this in a continuous manner – no parting of the hair!

Here's a very simple vector field on the sphere: at each point, we go "north" with unit speed, where north means the direction of the shortest path from our starting point to the north pole[3] At first it seems like this will give a "combing," but there are two problem points: at the north pole, where it is no longer possible

[3]Such a path will be an arc of a **great circle**, which is obtained by slicing the sphere through the unique plane containing our starting point P, the north pole N and the center of the earth.

to go north, and also at the south pole when *all* directions are north according to the way we have defined it (because all the meridian lines of the sphere run through the north and the south pole): we are off by two points from combing the sphere.

If you think a bit, you can reduce the number of problem points to one, because it you remove just the north pole from the sphere, you can unfold what remains into a surface which looks like the plane: this is the so-called **stereographic projection**, which one way of passing from a map on the surface of the sphere (i.e., a globe) to a flat map suitable for a textbook: there is a lot of distortion and the angles change, but one can get from a nonvanishing tangent field on the plane to a nonvanishing tangent field on the sphere minus the north pole in this way. And there are plenty of nonvanishing vector fields in the plane, e.g. the constant vector field $F(x, y) = \hat{\mathbf{i}}$. But no matter which vector field you choose, you will find yourself in trouble when it comes to extending it to the north pole of the sphere.

In fact it is a theorem that you will never succeed: the **No Combing Theorem** says that there just does not exist a continuously varying unit tangent vector field on the unit sphere.[4] The no-combing theorem has real-life consequences: for instance, it implies the result that at any given time, there is at least one point on the surface of the earth where there is no wind blowing! Less prosaically, the fact that any unit vector field on the surface of the sphere must have a singularity will come up whenever one tries to study the "global"[5] behavior of differential equations on the surface of the earth.

The situation is much different on the **torus**, which, recall, is given parametrically as

$$\mathbf{R}(u, v) = ((R + a\cos u)\cos v, (R + a\cos u)\sin v, a\sin u),$$

with $0 < a < R$. Here if we take the tangent vectors in the u and v directions, we get

$$T_u = (R - a\sin u)\cos v, (R - a\sin u)\sin v, a\cos u)$$
$$T_v = (-(R + a\cos u)\sin v, (R + a)\cos v, 0)$$

In fact *both* T_u and T_v are nonvanishing for any value of u and v, so after dividing them by their norms, we find that we not only have one unit tangent vector field on the torus, we have *two* unit tangent vector fields which are moreover perpendicular at every point: $T_u \cdot T_v = 0$.

Thus the sphere and the torus are qualitatively different, as shown in the completely different behavior of tangent vector fields on them. The branch of mathematics that studies "qualitative differences" in surfaces (and other geometric objects) via the behavior of vector fields is called **differential topology**.

[4] In more innocent times this was called the "hairy ball theorem," but it is hard to say this with a straight face.

[5] We see in this example where the word "global" comes from!