HANDOUT EIGHT: GREEN'S THEOREM

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1. The two forms of Green's Theorem

Green's Theorem is another higher dimensional analogue of the fundamental theorem of calculus: it relates the line integral of a vector field around a plane curve to a double integral of "the derivative" of the vector field in the interior of the curve. It admits two different but completely equivalent formulations, a "flux" version for normal line integrals and a "circulation" version for tangent line integrals. We have already predicted the former version as an "integral form" of the fact that divergence is equal to flux density. Here it is again:

Theorem 1. *(Green's Theorem: Flux Form) Let R be a region in the plane with boundary curve C and $F = (P, Q)$ a vector field defined on R. Then*

$$\int\int_R \mathrm{Div}(F) dxdy = \int_C F \cdot \mathbf{n}. \tag{1}$$

We recall that $\int_C F \cdot \mathbf{n}$ means the normal line integral around the closed curve C. That is, if $\mathbf{r}(t) = (x(t), y(t))$ is a parameterization and the velocity vector is $\mathbf{v}(t) = (x'(t), y'(t))$, then $\mathbf{n}(t) = R(\mathbf{v}(t)) = (y'(t), -x'(t))$ is the **rightward normal**, the velocity vector turned 90 degrees to the right. Note that there are two possible orientations of the curve C, such that changing the orientation will change the line integral by a minus sign. The orientation which makes the theorem true is the so-called **positive** orientation: that is, we walk in such a way so as to keep the interior of the region on our left. Recall that the physical interpretation of this result is that by integrating the net flux density (flow out minus flow in) at every point of the region we get the total flux, i.e., the total net flow of fluid out of the boundary of the surface.

There is another formulation of Green's theorem in terms of circulation, or curl. To get it from Theorem 1, apply the Theorem to the vector field $R(F)$ obtained by turning every vector of F 90 degrees to the right. Then the right hand side is

$$\int_C R(F) \cdot R(d\mathbf{r}) = \int_C F \cdot d\mathbf{r},$$

by the dot product formula, since if we rotate both F and $d\mathbf{r}$ 90 degrees to the right, we change neither their lengths nor the angle between them. Thus the right hand side becomes a usual (tangent) line integral, and we get

$$\int\int_R \mathrm{Div}(R(F)) dxdy = \int_C F \cdot d\mathbf{r}.$$

But as we observed in Handout 5,

$$\mathrm{Div}(R(F)) = \mathrm{Div}(Q, -P) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \mathrm{curl}(F),$$

i.e., the divergence of the rotated vector field is the (scalar part) of the curl of the original vector field. Substituting this in, we get the second form of Green's Theorem.

Theorem 2. *(Green's Theorem: Circulation Form) Let R be a region in the plane with boundary curve C and $F = (P, Q)$ a vector field defined on R. Then*

$$\int\int_R \mathrm{curl}(F) dxdy = \int\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy = \int_C F \cdot d\mathbf{r}. \tag{2}$$

In a similar way, the flux form of Green's Theorem follows from the circulation form: we substitute $L(F)$ in place of F in Equation (2) and use the fact that

$$\mathrm{curl}(L(F)) = \mathrm{curl}(-Q, P) = \frac{\partial}{\partial x}(P) - \frac{\partial}{\partial y}(-Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \mathrm{Div}(F).$$

Recal that the curl of a planar vector field is technically a vector pointing in the $\hat{\mathbf{k}}$ direction, and $\hat{\mathbf{k}}$ is the normal vector $\mathbf{N}$ to the plane. Then a more precise version of Theorem 2 is

$$\int\int_R \mathrm{curl}(F) \cdot \mathbf{N} dA = \int_C F \cdot d\mathbf{r}. \tag{3}$$

Later we will consider **surface integrals**, which are to ordinary double integrals as line integrals are to ordinary single integrals: i.e., we can integrate a function along a surface in space. We will see that Equation (3) is still valid for a surface in space whose boundary is a smooth curve: this more general formulation is called **Stokes' Theorem**[1].

We do want to give the proof of Green's Theorem, but even the statement is complicated enough so that we begin with some examples.

Example: We verify Green's Theorem (in circulation form) for the vector field $F_a(x, y) = (x/r^a, y/r^a)$ $(r = \sqrt{x^2 + y^2})$ on the circle of radius r centered at the origin.

Solution: Let R_r be the disk of radius r, whose boundary C_r is the circle of radius r, both centered at the origin. We have devoted much attention to these vector fields earlier in the course: recall that since they are purely radial, the field is perpendicular to the tangent vector at every point, so $\int_{C_a} F_a d\mathbf{r} = 0$. To verify Green's Theorem we need to see that the double integral of the curl over R_a is zero. But we calculated earlier that this field is irrotational – indeed it is a gradient field – so the curl is zero and we are integrating the zero function over the disk: both sides are zero.

We may well ask: what is the use of Green's Theorem? It relates two quantities that we already in theory know how to compute. The point is that Green's

[1] Or occasionally, by those whose mathematical ability far exceeds their knowledge of punctutation, "Stoke's Theorem."

Theorem gives us a **choice**: would we rather compute the line integral or the double integral? In many cases, one of the two is much easier to evaluate than the other, and Green's Theorem makes some calculations routine that we would otherwise despair to complete.

Example: Evaluate the line integral $\int_C (x^5 + 3y)dx + (2x - e^{y^3})dy$, where C is the circle centered at $(1, 5)$ of radius 2.

Solution: In theory we can do this line integral: the circle is parameterized by $\mathbf{r}(t) = (2((\cos t)+1), 2((\sin t)+5))$, whose velocity vector is $\mathbf{v}(t) = (-2\sin t, 2\cos t)$. So the integral we want is

$$\int_0^{2\pi} ((2\cos t+1)^5 + 3(2\sin t+5)(-2\sin t dt) + (2(2\cos t+1) - e^{(2\sin t+5)^3})(2\cos t dt).$$

This line integral can be done – the subtitution $u = (2\sin t + 1)$ will take care of the exponential part – but we are not going to enjoy the calculation. On the other hand the curl of the vector field is $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - 3 = -1$, so by Green's Theorem all we need to do is integrate the constant function -1 over the disk of radius 2: we'll get -1 times the area of the disk, or -4π. What a relief!

2. The Proof of Green's Theorem

We will prove Green's Theorem in circulation form, i.e., Equation 2. We begin by proving the theorem in the case where the region R is of a special type: i.e., it is simultaneously **x-convex** and **y-convex** in the terminology we introduced in our study of double integrals[2]: that is, it is at the same time expressible as the region between two curves $y_2 = G(x)$ and $y_1 = g(x)$ bounding it above and below and as the region between two curves $x_2 = H(y)$ and $x_1 = h(y)$ bounding it on the right and on the left. Let C_T (T is for "top") be the curve bounding the region on the top, so it has a parameterization $\mathbf{r}_T(t) = (t, G(t))$, valid for $a \le t \le b$; similarly let C_B (B is for "bottom") be the curve bounding the region on the bottom, with a parameterization $\mathbf{r}_B(t) = (t, g(t))$ also for $a \le t \le b$. Together C_T and C_B make up the boundary C of the region R, but we must be careful with the orientation on the boundary: with the positive orientation the boundary is $C = C_B - C_T$. We have a completely analogous discussion with respect to the y-variable: there is a curve C_R (R is for "right") bounding the region on the right, with parameterization $\mathbf{r}_R(t) = (H(t), t)$ and a curve C_L (L is for "left") bounding the region on the left, with parameterization $\mathbf{r}_L(t) = (h(t), t)$, both for $c \le t \le d$. This time the positive orientation on the boundary means that $C = C_R - C_L$.

We start with the left hand side of Equation (2), namely with

$$\int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \int\int_R \frac{\partial Q}{\partial x} dA - \int\int_R \frac{\partial P}{\partial y} dA.$$

We evaluate the first integral by integrating with respect to x first, getting

$$\int\int_R \frac{\partial Q}{\partial x} dA = \int_{y=c}^{y=d} \int_{x=h(y)}^{x=H(y)} \frac{\partial Q}{\partial x} dx dy =$$

[2] In the unimaginative but unfortunately rather standard terminology used in the text, this region is simultaneously Type I and Type II.

$$\int_c^d Q(H(y)) - Q(h(y))dy.$$

And we evaluate the second integral by integrating with respect to y first, getting

$$-\int_{x=a}^{x=b} \int_{y=g(x)}^{y=G(x)} \frac{\partial P}{\partial y} dydx = -\int_a^b P(G(x)) + P(g(x))dx.$$

So the entire left hand side (the double integral of the curl) is

$$\int_c^d Q(H(y))dy - \int_c^d Q(h(y))dy - \int_a^b P(G(x))dx + \int_a^b P(g(x))dx. \tag{4}$$

Now we will work the right hand side to get it in this form. We split the line integral into two pieces, using for one piece the decomposition of the boundary $C = C_B - C_T$ and on the other piece the decomposition $C = C_R - C_L$:

$$\int_C F \cdot d\mathbf{r} =$$

$$= \int_C Pdx + Qdy = \int_C Pdx + \int_C Qdy = \int_{C_B - C_T} Pdx + \int_{C_R - C_L} Qdy =$$

$$\int_{C_B} Pdx - \int_{C_T} Pdx + \int_{C_R} Qdy - \int_{C_L} Qdy. \tag{5}$$

But now we win: using the four parameterizations $\mathbf{r}_T, \mathbf{r}_B, \mathbf{r}_L, \mathbf{r}_R$ from above, we see that each of the four terms of (4) is equal to a corresponding term in (5), indeed

$$\int_{C_B} Pdx = \int_a^b P(g(x))dx, \ \int_{C_T} Qdx = \int_a^b P(G(x))dx,$$

$$\int_{C_L} Qdy = \int_c^d Q(h(y))dy, \ \int_{C_R} Qdy = \int_c^d Q(H(y))dy,$$

so equations (4) and (5) are the same. This proves Green's Theorem for "xy-convex" regions.

The proof for a general region follows from this by a dissection argument: indeed, by adding straight line boundaries we can cut up any region into a finite collection of regions each of which is "xy-convex," say $R = R_1 \cup \ldots \cup R_N$. On each little region R_i we just showed that

$$\int\int_{R_i} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \int_{C_i} Pdx + Qdy.$$

So by the additivity of integrals we have

$$\int\int_{R=R_1\cup\ldots\cup R_n} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \sum_{i=1}^n \int\int_{R_i} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sum_{i=1}^n \int_{C_i} Pdx + Qdy.$$

We want to say that this last expression is the total line integral along the boundary of R, but what about all these extra lines that we added? It turns out okay: the way we have chosen orientations, we go around all these extra lines exactly twice, once in each direction, so their contributions to the line integral cancel out exactly, and we end up with the line integral around the original boundary $\int_C Pdx + Qdy$. This completes the proof of Green's Theorem.

Actually, the proof proves *more* than we probably had in mind originally: Green's theorem holds for any region in the plane which can be sewn out of finitely many xy-convex regions. In particular nothing stops us from "sewing in holes": Green's Theorem applies equally well to regions whose boundary is more than one simple closed curve, e.g. the region between two circles. In Section 5 we will see that some of the most useful applications come from applying Green's theorem to such regions.

We wanted to present the proof in its entirety because it is an argument which is both important and beautiful: it uses many of the techniques we have learned so far: that we can interchange the order of integration in a multiple integral; that the same curve can have different parameterizations; and that we must pay attention to issues of orientation. This same idea of proof (start with a convex region, evaluate the same mutiple integral as an iterated integral in several different orders) works also for the proofs of Stokes' Theorem and Gauss' Theorem (coming soon) although the details are (even more) complicated, and we will not repeat these proofs.

3. Applications to flux and divergence of planar vector fields

Still commenting on the proof of Green's Theorem, we remark that actually the *second* part (on dividing the region) is just as important as the first: interpreting Green's Theorem in divergence form, it has the important physical interpretation that flux is *additive*: if we glue two regions together along their common boundary, the total flux through the new region is the same as the sum of the fluxes of the original two regions. Keeping this principle in mind, we could have proved Green's Theorem in a *completely* geometric way by filling up the region R wih more and more rectangles (whose area approaches zero) and using the fact that we already showed, that the integral of the divergence equals the flux *in the limit* as the area of the rectangle goes to zero. To use the language of physicists (and engineers), what we earlier showed was the **differential form** of Green's Theorem (i.e., for "infinitesimally small" boxes) and what we just showed is the **integral form** of Green's Theorem (i.e., overall, or adding everything up). So the integral form follows from the differential form by a limiting process involving boxes whose area goes to zero – i.e., exactly the sort of limiting process that is implicit in the definition of a double integral.

Conversely, the differential statement follows immediately from Green's Theorem, reconfirming and generalizing our geometric interpretation of divergence: Let P be a point and R_ϵ be a small region centered at that point with area A_ϵ and boundary curve C_ϵ. Then we get

$$\lim_{\to 0} \frac{\int_{C_\epsilon} F \cdot n}{A_\epsilon} = \lim_{\epsilon \to 0} \frac{\int\int_{A_\epsilon} \mathrm{Div}(F) dA}{A_\epsilon} = \mathrm{Div}(F)(P),$$

since as the diameter of R_ϵ goes to zero, the minimum value and the maximum value of $\mathrm{Div}(F)$ both converge to $\mathrm{Div}(F)(P)$.

Similarly, the equation

$$\int\int_R \mathrm{curl}(F) dA = \int_C F d\mathbf{r}$$

tells us that the curl of a vector field in the plane is what we said it was: suppose that curl(F) is positive at some point (x_0, y_0). Then, since the curl varies continuously from point to point, there is some small disk R about that point on which the curl is positive, so that the left hand side of Green's Theorem is positive. That means that the line integral of the vector field around the boundary is positive: travelling counterclockwise in a circle of sufficiently small radius about a point of positive curl, positive work is done. If you think about it for a second, you will agree that we have found a rigorous mathematical way of expressing that the paddlewheel fixed at (x_0, y_0) will turn counterclockwise!

This argument is for planar vector fields only. That the curl does what we say it does in three space – including the bit about the axis of rotation – will come as a consequence of our later generalized form of Green's Theorem, namely Stokes' Theorem.

4. Using Green's Theorem to Compute Areas

Recall that if R is any plane region, then the double integral $\int\int_R 1 dA$ computes the area of R. Since we now possess an almost magical ability to convert double integrals into line integrals, we might try to exploit our powers to compute areas of regions via line integrals around the boundary. In order to do this, we need only find a vector field $F = (P, Q)$ with the property that $\mathrm{curl}(F) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \equiv 1$. There are many such vector fields – indeed we can choose the function $P(x, y)$ arbitrarily and solve for Q, getting $Q = \int (1 + \frac{\partial P}{\partial y}) dx$.

The simplest choice is $P = 0$, $Q = x$, gving the formula

$$\mathrm{area}(R) = \oint_{t_{\min}}^{t_{\max}} (0, x(t)) \cdot (x'(t), y'(t)) dt = \oint_{t_{\min}}^{t_{\max}} x(t)y'(t)dt. \tag{6}$$

A more traditional choice is $F = \frac{1}{2}(-y, x)$, giving

$$\mathrm{area}(R) = \frac{1}{2} \oint_{t_{\min}}^{t_{\max}} x(t)y'(t) - x'(t)y(t)dt. \tag{7}$$

Note that as a consequence of the right hand sides of equations (6) and (7) both being equal to the area of R, they must also be equal to each other. This amounts to the identity

$$\oint x(t)y'(t) + x'(t)y(t)dt = 0.$$

This can be seen using integration by parts:

$$\int_{t_{\max}}^{t_{\min}} xdy + ydx = xy|_{t_{\min}}^{t_{\max}} = x(t_{\max})y(t_{\max}) - x(t_{\min})y(t_{\min}),$$

which is zero in our case: since the curve is closed (as the circles on the integral signs are there to remind us), the initial and terminal points coincide: $(x(t_{\max}), y(t_{\max})) = (x(t_{\min}), y(t_{\min}))$.

Example: We use both formulas to compute the area of the (interior of the!) ellipse with semiaxes a and b – i.e., so that with the most convenient choice of coordinates it is given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Either way, we use the parameterization

$\mathring{(t)} = (a\cos t, b\sin t)$ with $0 \le t \le 2\pi$.

Using (6) we get the integral

$$\int_0^{2\pi} (a\cos t)(b\cos t)dt = ab\int_0^{2\pi} \cos^2 t dt = \pi ab,$$

where we have used that $\int_0^{2\pi} \cos^2 t dt = \pi$. One can do this integral using the trig identity $\cos^2 t = \frac{1}{2}(1+\cos 2t)$, or in many other ways. Here's an elegant one: since $\sin^2(t+\pi/2) = \cos^2(t)$ and both $\sin^2(t)$ and $\cos^2(t)$ have period π, it must be that $I_1 = \int_0^{2\pi} \sin^2(t)dt = \int_0^{2\pi} \cos^2(t)dt = I_2$ (think about it). But then $2I_1 = I_1 + I_2 = \int_0^{2\pi} \cos^2(t) + \sin^2(t)dt = \int_0^{2\pi} 1dt = 2\pi$, so $I_1 = I_2 = \pi$.

Using the formula (7), we get the integral

$$\frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t) - (-a\sin t)(b\sin t) = \frac{ab}{2}\int_0^{2\pi} cos^2(t) + \sin^2(t)dt = \pi ab.$$

Remark: There *is* a surprisingly practical reason to prefer formula (7) to formula (6) (or to any of the other infinitely many formulas we get using vector fields of unit curl): there is a mechanical device, called a **planimeter**, consisting of two metal arms joined by a flexible elbow. One end is fixed in place outside of the plane region, and the other end has attached to it a little wheel – aha! – which spins as it is dragged around the boundary of the curve.

For an explanation of the mathematics behind the planimeter – as well as instructions on how to build one! – consult the webpage

`http://www.math.harvard.edu/~ knill/math21a2000/planimeter`.

5. Irrotational versus Conservative Vector Fields Revisited

Our study of when a vector field is conservative in the last unit ended somewhat indefinitively: we showed that a vector field is conservative (which, recall, means that every line integral along a closed curve is zero) if and only if it is the gradient of some function, and that a *necessary* condition for this was that the vector field be irrotational, i.e., have zero curl at every point. It looked for a while like the converse should be true: given a vector field with $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, we gave a procedure for finding a function f such that $\nabla(f) = F$, but unfortunately this procedure worked only "locally": near any point we can find such a function f, but in order for the vector field to be conservative, we need a single function f defined and continuous on the entire region of definition of the vector field, and we saw that for the vector field

$$F_\star = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

we could not find an f defined for all nonzero (x, y).

Green's Theorem clarifies these matters considerably: suppose F is a vector field defined on a region R with the property that every simple closed curve C in R is the boundary of some subregion S: equivalently, suppose that for every simple closed curve, the vector field has no singularities inside the curve. This condition on a region R is called **simply-connected**: it can be expressed intuitively (but accurately) by saying that the region R has **no holes**. Suppose that F is an irrotational vector field on such a region. Then for any simple closed curve C in the region, C is the boundary of S, so

$$\int_C F \cdot d\mathbf{r} = \int\int_S \mathrm{curl}(F) dA = 0,$$

that is the line integral of F around every closed path is zero.[3] We summarize this argument by the following useful result:

Theorem 3. *Let R be a region in the plane which is simply connected (no holes!). Then every vector field F defined on R with* $\mathrm{curl}(F) = 0$ *is conservative, i.e., of the form $\nabla(f)$ for some f defined on all of R.*

Thus, if our region has no holes, in order to determine whether $F = (P, Q)$ is convervative, we need only check whether $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$: unless we need the potential function f, we do not have to go through the calculus of finding it.

We emphasize that this theorem is giving conditions for a system of partial differential equations – namely $\frac{\partial f}{\partial x} = P$, $\frac{\partial f}{\partial y} = Q$ such that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ – has a globally defined solution: this is simultaneously the type of result that we most want for applications and is the hardest to derive, so, truly, hurray for Green's Theorem.

But Green's theorem is still useful when there are holes in the region.

Example: Let C be any simple closed curve containing the origin, positively oriented; compute $\int_C F_\star d\mathbf{r}$, for our special vector field $F_\star$.

Solution: We have already computed that $\mathrm{curl}(F_\star) = 0$ – so our field is irrotational; moreover we know it's not conservative, because we computed that the line integral around *any* circle is 2π. So take a very small circle C_ϵ around the origin, where small means that C_ϵ lies entirely inside our simple closed curve C. Then we can consider the region R that lies *between* the two curves C and C_ϵ: this region has more than one boundary component – this is allowed! – and with correct orientations the boundary of R, which we denote ∂R, is equal to $C - C_\epsilon$. So Green's Theorem tells us that

$$\int_C F_\star \cdot d\mathbf{r} - \int_{C_\epsilon} F\star \cdot d\mathbf{r} = \int_{C - C_\epsilon} F_\star \cdot d\mathbf{r} =$$

$$\int_{\partial R} F_\star \cdot d\mathbf{r} = \int\int_R \mathrm{curl}(F_\star) dA = \int\int_R 0 dA = 0,$$

That is, for any simple closed curve circling clockwise around the origin,

$$\int_C F \cdot d\mathbf{r} = \int_{C_\epsilon} F \cdot d\mathbf{r} = 2\pi.$$

[3]Technically we showed this only for *simple* closed paths, but any path which intersects itself can be viewed as the sum of two paths which each have one less self-intersection point: e.g. the line integral around a figure-eight is the sum of the line integrals around the upper and lower circles which comprise it, so if the line integral around every simple closed path is zero, the vector field is conservative.

Similarly, the integral of $F_\star$ along any simple closed curve circling clockwise around the origin will have line integral -2π. Since any closed curve can be split up into finitely many simple closed curves which either wind clockwise once around the origin, wind counterclockwise once around the origin, or do not circle around the origin at all, it follows that the integral of $F_\star$ along *any* closed curve is $2\pi n$ for some integer (positive or negative whole number) n. This number n is called the **winding number** of C around the origin.

6. Extra: Still More Irrotational Vector Fields and *A Beautiful Mind*

So we have trained ourselves to be suspicious of irrotational vector fields over regions with "holes": the line integral might still depend on the path. But we don't want to be *too* suspicious: e.g. the irrotational radial vector field $F_a(x, y) = (x/r^a, y/r^a)$ is conservative for any value of a. It is natural to ask how many other examples we can produce of irrotational vector fields which are *not* conservative.

Using $F_\star$ we can make some more: if K is any nonzero constant, then $KF_\star$ will still have zero curl and still not be equal to ∇f (since then $F_\star$ would be $\nabla f/K$). Also we can add any conservative vector field: if $G = KF_\star + \nabla f$, then (because $\operatorname{curl}(F_1 + F_2) = \operatorname{curl}(F_1) + \operatorname{curl}(F_2)$ and $\operatorname{curl}(KF_1) = K \operatorname{curl}(F_1)$) G is irrotational, and it can't be a gradient field, since $KF_\star + \nabla f = \nabla g$ implies $KF_\star = \nabla(g - f)$, which it isn't. Intuitively (and also in a sense that can be made precise using linear algebra) these examples G are *dependent* on $F_\star$. Are there any other irrotational nonconservative vector fields on the complement of the origin, i.e., any examples which are *independent* of $F_\star$?

Remarkably, the answer is no: if G is any irrotational vector field defined everywhere except at the origin, then there is a constant K such that $G - KF_\star = \nabla(f)$ is a gradient field.

Indeed we can figure out what K must be for G, as follows: by Green's theorem, we know that all integrals of G along simple closed counterclockwise curves have the same value, say $I = \int_C G \cdot d\mathbf{r}$. If we put $K := \frac{I}{2\pi}$, then the integral of $G - KF_\star$ along any closed curve C is

$$\int_C G - KFd\mathbf{r} = \int_C Gd\mathbf{r} - K\int_C F_\star = I - K(2\pi) = I - (\frac{I}{2\pi})(2\pi) = 0,$$

so $G - KF$ is conservative and hence is the gradient of some function. We have been saying all along that $F_\star$ is a "very special" vector field: now we know how special: it is essentially the only irrotational nonconservative vector field on the complement of the origin.

In general, if R is a region of the plane, we can ask for the number of **independent** irrotational nonconservative vector fields define don R: we say this number is N if there exist vector fields $F_1, \ldots, F_N$ defined on R which are irrotational with the following two properties:

- for any constants $C_1, \ldots, C_N$ which are not all zero the vector field

$$C_1F_1 + C_2F_2 + \ldots + C_NF_N \tag{8}$$

is not a gradient field;
• for any other irrotational vector field G defined on R, there are some constants $C_1, \ldots, C_N$ such that

$$G - (C_1 F_1 + C_2 F_2 + \ldots + C_N F_N) = \nabla(f) \tag{9}$$

for some function f.

This latter condition is saying that all irrotational vector fields "depend" on the F_i's up to a gradient vector field in the same way as we already saw for $N = 1$.

A yet more amazing result is that the number of independent nonconservative irrotational vector fields is always equal to the **number of holes** of the region. Indeed, if we consider the region R which is the plane minus a set of N points $P_1 = (x_1, y_1),\ P_2 = (x_2, y_2), \ldots,\ P_N = (x_N, y_N)$, then there is a translated version of our special vector field at each point:

$$F_{\star,P_i} = \frac{(y - y_i, x - x_i)}{(x - x_i)^2 + (y - y_i)^2}.$$

Since we have just translated everything to the point P_i, the line integral of $F_{\star,P_i})$ around a small counterclockwise circle enclosing P_i (and none of the other points) is 2π. Using this, it can be shown that any linear combination of the $F_{\star P_i}$'s as in (6) is still not conservative (this is not so hard, but we don't try to explain it here: we're just sketching some ideas), but given an arbitrary irrotational vector field G with singularities at these points, by keeping track of the values I_i of the line integrals around little circles around the points P_i, then $\frac{I_i}{2\pi}$ give the coefficients such that Equation (7) holds, i.e., by subtracting just the right combination of these special vector fields we turn G into a gradient field.

(The same result would hold if some or all of our "holes" were not just isolated points P_i but entire regions R_i with boundary a simple closed curve.)

To be sure, this is serious mathematics: the idea that the number of irrotational vector fields up to gradient fields on a region R counts the number of holes in the region R is due to the early 20th century mathematician **Gustave de Rham** and is called de Rham theory[4]. It is true not just for vector fields on subsets of the plane but for vector fields on surfaces and even higher-dimensional spaces.

Now one reason I bring this up is that in the movie *A Beautiful Mind* based on the biography of the mathematician John F. Nash Jr., the main character – played by Russell Crowe – learns that he is required to teach a multivariable calculus class five minutes before the start of the class. After flipping through the textbook in front of the students and throwing it into the trash, he writes the following problem in the blackboard in the hope that it will take the students "the rest of your natural lives" to solve it (thereby leaving him alone to do his own work):
Find a set X in $\mathbb{R}^3$ such that if:
V is the set of irrotational vector fields on $\mathbb{R}^3 \setminus X$ and
W is the set of gradient vector fields on $\mathbb{R}^3 \setminus X$,
then the dimension of V/W is 8.

[4] Or actually **de Rham cohomology**, but don't say this at a party: people will run.

In other words, find a region in $\mathbb{R}^3$ so that the number of irrotational vector fields up to gradient fields is 8. We just did this in the plane: take $\mathbb{R}^2$ minus any 8 different points. We will see soon enough as an application of Stokes' theorem that if we just take the cylinder on top of the whole picture – i.e., consider $\mathbb{R}^3$ minus the eight vertical lines projecting down to these points – then the result is the same. (One can show that this is in a certain sense the simplest solution.)

So now you know the answer to the most famous multivariable calculus problem in cinematic history. (You will not be surprised to learn that they hired an actual mathematician, David Bayer, as a consultant for the movie: the little snippets of math that you hear are remarkably true to life, unlike say, *Good Will Hunting*, where nothing looks or sounds quite right.) In a way, you now understand Russell Crowe's character more deeply than almost anyone who has seen the movie: he is the kind of guy who instead of stooping to teach a class, gives them a problem whose solution is part of the working vocabulary of most practicing mathematicians of the day but is so deep and intricate that an actual multivariable calculus student will almost certainly make no progress on it. I am tempted to speculate further on the implausible psychology of the beautiful young coed who is attracted to her instructor because he poses obnoxiously difficult math problems, but really decorum forbids me to say anything further on the matter.

7. Extra: Green's Theorem in Complex Variables

Closely related to the notion of a line integral of a planar vector field – and equally useful – is that of a **complex line integral** or **contour** integral of a function of a complex variable along a curve in the complex plane. This section, intended for students who have familiarity with such complex functions up to the Cauchy-Riemann equations, explains how the complex analogue of the fundamental theorem of calculus, namely **Cauchy's Integral Theorem**, follows from Green's Theorem.

We begin with $f(z)$ a function of a complex variable. That is, write $z = x + iy$, and $f(z) = P(z) + iQ(z)$. In fact, if we agree to identify $\mathbb{R}^2$ with the complex plane via $(x, y) \iff x + yi$, such functions correspond precisely to vector fields $F(x, y) = (P(x, y), Q(x, y))$.

Viewed in this way a complex line integral is however a pair of line integrals of vector fields. Let C be an oriented curve in the plane, as usual, except that since we are writing $z = x + iy$ its parameterization can be written more compactly as $z(t) = x(t) + iy(t)$. Then we define the complex line integral

$$\int_C f(z)dz = \int_{t_{\min}}^{t_{\max}} (P(z) + iQ(z))(x'(t) + iy'(t))dt.$$

When we write this out, we get two separate (ordinary) integrals,

$$\int_{t_{\min}}^{t_{\max}} P(z(t))x'(t) - Q(z(t))y'(t)dt + i\int_{t_{\min}}^{t_{\max}} P(z(t))y'(t) + Q(z(t))x'(t)dt.$$

However, we can still identify each integral as being the dot product of a certain vector field with the velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$, and we get line integrals

$$\int_C (P, -Q) \cdot d\mathbf{r} + i \int_C (Q, P) \cdot d\mathbf{r},$$

i.e., the corresponding vector fields are $F_1 = (P, -Q)$ and $F_2 = (Q, P)$.

Suppose C is the boundary of a region R in the plane such that $f(z)$ (and hence P and Q) is defined not just on the boundary C but on all of R. Then Green's Theorem applies: we compute $\mathrm{curl}(F_1) = -\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ and $\mathrm{curl}(F_2) = \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}$, and get that

$$\int_C f(z)dz = \int\int_R (-\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA + i \int\int_R (\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y})dA.$$

This expression is perhaps a bit surprising at first: neither expression is either the curl or the divergence of the original vector field $F = (P, Q)$, so it would not help us evaluate the line integral if F were irrotational or incompressible. But recall that the **Cauchy-Riemann equations** are

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \ \frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y},$$

and that a complex function $f(z) = P(x + iy) + iQ(x + iy)$ is **holomorphic** (or **analytic**) on a region R – i.e., the complex derivative $\lim_{h \rightarrow 0} \frac{f(z+h)-f(z)}{h}$ exists in a small neighorhood of each point in R – if and only if $f(z)$ satisfies the Cauchy-Riemann equations in R.[5] Therefore, for holomorphic functions both integrands are identically zero, so certainly the integrals are zero, and we have proved the following very important result.

Theorem 4. *(Cauchy's Integral Theorem) Let R be a closed region in the plane with smooth boundary ∂R (we allow the boundary to be any finite number of simple closed curves). Then if $f(z)$ is any complex function which is defined and holomorphic on all of R,*

$$\int_{\partial R} f(z)dz = 0.$$

Remark: We should therefore view the (two) Cauchy-Riemann equations as the complex analogue of the vector equation $\mathrm{curl}(F) = 0$. (If we have our heart set on it, we can write the Cauchy-Riemann equations as one equation, namely $\frac{\partial f}{\partial \overline{z}} = 0$, but we do not want to get into this.) Thus it is precisely the holomorphic functions which play the analogous role in the theory of complex line integrals that the irrotational vector fields play in the usual theory.

Some comments are due as for Green's Theorem: if the boundary of R is a simple closed curve C, then we are just getting $\int_C f(z)dz = 0$. In particular, if $f(z)$ is holomorphic on the entire complex plane (such functions are indeed called **entire**), then $\int_C f(z)dz = 0$ for all curves. In general, if the region has multiple boundary

[5] We recall that when we say that a function satisfies a partial differential equation at a point, we mean that the function is defined on at least a small open disk about that point and satisfies the equation on that entire disk.

components $C_1 \cup \ldots \cup C_r$, then we are getting a relation between the integrals over these curves, namely that

$$\int_{C_1} f(z)dz + \ldots + \int_{C_r} f(z)dz = 0.$$

Example: Let $f(z) = \frac{1}{z}$; note that this function has a singularity at 0, and nowhere else. If we were content to integrate around any simple closed curve not enclosing 0, then Cauchy's theorem guarantees that the integral is zero. But of course we are going to integrate around the unit circle! Take $z(t) = \cos t + i \sin t$, so

$$\int_C f(z)dz = \int_0^{2\pi} \frac{1}{\cos t + i \sin t}(-\sin t + i \cos t)dt.$$

By multiplying and dividing the fraction by $\cos t - i \sin t$,[6] the denominator becomes 1 and we get

$$\int_0^{2\pi} (\cos t - i \sin t)(-\sin t + i \cos t)dt = \int_0^{2\pi} idt = 2\pi i.$$

Now, as in the discussion of the previous section, Green's Theorem tells us that this is all we need to compute: if C is any curve which winds n times counterclockwise around the origin, then

$$\int_C \frac{dz}{z} = 2\pi in.$$

Finally, the considerations of the less section have analogues here (which are in fact older and better known in the complex setting): the function $f(z) = \frac{1}{z}$ is essentially the only holomorphic function on the complement of 0 for which line integrals around closed curves need not be zero, in the sense that every function $g(z)$ holomorphic except at zero can be written as $g(z) = \frac{R}{z} + h'(z)$ – here the analogue of "conservative vector fields are gradient fields" is the simpler "conservative holomorphic functions are derivatives of other holomorphic functions." (In fact even the reason we cannot integrate $\frac{1}{z}$ is the same as the reason we could not integrate our special vector field $F_\star$ – its antiderivative is defined locally, but involves the angular coordinate θ so cannot be continuously defined for an entire path around the origin.) And moreover, this number R has the property that for any closed curve C winding n times around the origin, $\int_C g(z)dz = 2\pi inR$. This number R is called the **residue** of $g(z)$ at 0, and one learns techniques (Laurent expansion) for calculating it explicitly in a complex variables course.

Moreover, there is the same generalization to functions $g(z)$ having singularities at finitely many points $z_1, \ldots, z_n$ in the complex plane: integrating $g(z)$ about a small counterclockwise circle centered at z_i gives a number of the form $2\pi i R_i$, and we call R_i the residue of $g(z)$ at z_i. Then we have the following result.

Theorem 5. *(Residue Theorem) Let $g(z)$ be a function holomorphic on a region R except for finitely many singularities at points $z_1, \ldots, z_n$. Let C be a curve in the region R, not passing through any of the singularities, which has winding numbers*

[6]This calculation would be less arduous if we allowed ourselves Euler's remarkable identity $e^{it} = \cos t + i \sin t$.

$n_1, \ldots, n_n$ with respect to each $z_1, \ldots, z_n$. Let $R_1, \ldots, R_n$ be the residues of $g(z)$ at these points. Then

$$\int_C g(z)dz = \sum_{i=1}^{n} 2\pi i R_i.$$

The residue theorem is *the* single biggest tool we have for evaluating (real-valued!) integrals and series, and I think it is good to know that it is the complex version of the weak path-independence property of irrotational vector fields.