DIVISOR CLASS GROUPS

PETE L. CLARK

1. Krull domains and class groups

In algebraic geometry one meets both the Picard group and the divisor class group. In favorable situations these two groups coincide – especially, they coincide for nonsingular algebraic varieties – but in general they are distinct and the distinction is of some importance in analyzing singularities.

We have already defined the Picard group of an arbitrary domain R. The definition we gave was invertible fractional ideals modulo principal fractional ideals. We also noted in passing that a nonzero R-submodule M of K is a fractional ideal exactly when it is locally free of rank 1, and if I and J are fractional ideals, $IJ \cong_R I \otimes_R J$. In other words, the Picard group can equally well be interpreted as the group of isomorphism classes of rank 1 locally free modules.[1]

In fact this definition extends naturally to the context of **locally ringed spaces**: one has a notion of rank 1 locally free $\mathcal{O}_X$-modules, which again form a group under tensor product. Note that in geometric discussions the preferred synonym for "rank one locally free $\mathcal{O}_X$-module" is **line bundle**. As an example, if X is a nonsingular, geometrically integral projective variety over a field k, then $\operatorname{Pic}(X)$ lies in a short exact sequence

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \to NS(X) \to 0,$$

where $\operatorname{Pic}^0(X)$ is the subgroup of line bundles which are "algebraically equivalent to zero," a geometric condition which we will not define here, and the quotient $NS(X)$, the **Néron-Severi group**, is a finitely generated abelian group. In fact $\operatorname{Pic}^0(X)$ can be endowed with the structure of a projective variety compatibly with the group structure: i.e., it is an **abelian variety**, called the **Picard variety** of X.

In case X has dimension one – i.e., is an algebraic curve – $\operatorname{Pic}^0(X)$ is called the **Jacobian** of X. In this case it is decidedly more elementary to say what "algebraically equivalent" to zero means. In fact it means "degree zero." This in turn is best understood by thinking in terms of **Weil divisors**, i.e., as a motivation for understanding the equality $\operatorname{Pic}(X) = \operatorname{Cl}(X)$ of the Picard group with the **divisor class group** that holds in this case.

We use this geometry as motivation (only) for the need to work with Pic and Cl together. In the rest of these notes, we concentrate only on the case of integral affine schemes, that is to say on integral domains. I say that a domain is **normal**

[1] If P is any property for modules over a commutative ring R, we say that a module is "locally P" iff $M \otimes R_{\mathfrak{p}}$ has property P for all prime ideals $\mathfrak{p}$ of R. Similarly for a property P of commutative rings.

if it is integrally closed in its quotient field.

In fact I know two different ways to define the class group of an integral domain R. The first definition requires R to be Noetherian (but not necessarily normal), whereas the second definition requires R to be a **Krull domain**. Exactly what a Krull domain is we shall see shortly, but I will tell you now the following:

Proposition 1. *(Facts about Krull domains)*
a) A Krull domain is normal.
b) A normal Noetherian domain is a Krull domain.
c) A UFD is a Krull domain.
d) If A is a Krull domain, and $R = A[\{t_i\}_{i \in I}]$ is a polynomial ring in any (possibly infinite) number of indeterminates, then R is a Krull domain.

Thus one can think of the notion of a Krull domain as a sort of well-calculated relaxation of the Noetherian hypothesis on a normal Noetherian domain. Being able to entertain non-Noetherian Krull domains is useful if you are, say, trying to exhibit a Krull domain with any given ideal class group.

On the class of normal Noetherian domains these two constructions coincide. In particular, the (natural globalization) of this construction associates an abelian group $\mathrm{Cl}(X)$ to each **normal** algebraic variety and this is exactly the construction of [Har, §II.6]. I must confess that I have not myself seen much literature on the case of the class group of a non-integrally closed Noetherian domain, but I feel confident that it is out there somewhere. In particular, some of the constructions involving ideal theory of an order $\mathcal{O}$ in a number field would seem to have more natural geometric explanations in terms of comparing $\mathrm{Pic}(\mathcal{O})$ to $\mathrm{Cl}(O)$.

In order to be able to refer to one or the other of these two constructions I will call the first construction – valid for any Noetherian domain R – the **Chow class group** of R and the second construction – valid for any Krull domain – the **Krull class group** of R. This is not standard terminology. In fact, the reader should be warned that in the literature either $\mathrm{Cl}(R)$ or $\mathrm{Pic}(R)$ is often called just "the class group of R," even in situations when both $\mathrm{Cl}(R)$ and $\mathrm{Pic}(R)$ are defined and unequal (as is the case for a nonmaximal order)! For instance, in Cox's book he speaks of "the class group of the quadratic order $\mathcal{O}(D)$", but really it is $\mathrm{Pic}(\mathcal{O}(D))$.

2. Preliminaries on localization at height one primes

In this section we collect some results on localization at height one primes, with and without the Noetherian hypothesis. The reader may prefer to read on and look back at this section only when needed.

We begin by recalling some basic facts about localization. First, one can recover any integral domain by intersecting, in its fraction field, the localizations at all maximal ideals:

Proposition 2. *Let R be an integral domain with fraction field K. Then*

$$R = \bigcap_{\mathfrak{m}} R_{\mathfrak{m}},$$

the intersection taking place over all maximal ideals $\mathfrak{m}$ of R.

Proof: [Rei, §8.7].

Second, Noetherianity is a "global to local" property:

Theorem 3. *If a commutative ring R is Noetherian, and $\mathfrak{p}$ is any prime ideal of R, then the local ring $R_{\mathfrak{p}}$ is Noetherian.*

Proof: This should be familiar: the point is that every ideal I of $R_{\mathfrak{p}}$ may be identified, under pullback $I \mapsto I \cap R$, with an ideal of R: the image is precisely the set of ideals of R contained in $\mathfrak{p}$. If the partially ordered set of ideals of R satisfies (ACC), then so of course does every subset.

Remark: A ring R (even a domain) such that $R_{\mathfrak{p}}$ is Noetherian for each prime $\mathfrak{p}$ need not itself be Noetherian. In other words, it is not possible to check the Noetherian condition locally. Not only is this not a problem – because the Hilbert basis theorem and the fact that completions of Noetherian rings are Noetherian, one is rarely sweating to show that a ring R is Noetherian – it makes sense, since Noetherianity is in spirit a global property, sort of roughly analogous to compactness (or better, paracompactness).

On the other hand, normality is truly a local property:

Theorem 4. *For an integral domain R, TFAE:*
(i) R is normal.
(ii) For every prime ideal $\mathfrak{p}$ of R, the local ring $R_{\mathfrak{p}}$ is normal.
(iii) For every maximal ideal $\mathfrak{m}$ of R, the local ring $R_{\mathfrak{m}}$ is normal.

Proof: [Rei, §8.7].

Theorem 5. *For a Noetherian one-dimensional local domain R, TFAE:*
(i) R is normal.
(ii) The maximal ideal of R is principal.
(iii) R is a PID.
(iv) R is a DVR.

Proof: [Rei, §8.4].

The situation in which all of these results apply is when R is normal, Noetherian and one-dimensional, the latter meaning that each nonzero prime ideal is maximal. Then $R = \bigcap_{0 \neq \mathfrak{p}} R_{\mathfrak{p}}$, and each $R_{\mathfrak{p}}$ is a DVR. This is of course the class of **Dedekind domains**.

In a Dedekind domain we know that the fractional ideals form a group under multiplication, and moreover this group is the free abelian group $\bigoplus_{\mathfrak{p}} \mathbb{Z}[\mathfrak{p}]$ generated by the nonzero prime ideals. Given any nonzero element $f \in K^{\times}$, we can factor the principal fractional ideal fR into primes, getting an element $\sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(f)[\mathfrak{p}]$. Modding out the group of all fractional ideals by the subgroup of principal fractional ideals we get the class group of R.

This construction is at the same time prototypical for the Picard group construction and the class group construction. In a general domain R – in fact for any

domain which is not a Dedekind domain – not every fractional ideal will be invertible, so we modify the definition by taking specifically the group of invertible fractional ideals and modding out by the subgroup of principal (hence necessarily invertible) fractional ideals. By definition this gives $\operatorname{Pic}(R)$. If R is not a Dedekind domain, then there will exist ideals – even invertible ideals – which do not factor into primes. The analogue of factorization of an ideal into primes in a more general ring is **primary decomposition** (which holds in any Noetherian ring); however it is not clear to me how to use this analogy to define a class group.

The first key idea is that for a general domain R, we are interested not in localizations of arbitrary primes or at maximal primes, but at **height one** primes. Recall A prime ideal $\mathfrak{p}$ of an integral domain R is said to have height one if it is nonzero and there does not exist any prime ideal $\mathfrak{q}$ such that

$$0 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}.$$

Let us write $\Sigma(R)$ for the set of height one prime ideals of R.[2]

Theorem 6. *Let R be a normal Noetherian domain. Then*

$$R = \bigcap_{\mathfrak{p} \in \Sigma(R)} R_{\mathfrak{p}}.$$

In particular, every integrally closed Noetherian domain can be written as an intersection of DVR's with common fraction field.

Proof: See [Rei, §8.10].

Proposition 7. *Let R be a Noetherian domain and $0 \neq x \in R$. Then the set of height one primes containing x is finite.*

Proof: If $\mathfrak{p}$ is a height one prime containing the nonzero element x, then certainly $\mathfrak{p}$ is a minimal element of the set of prime ideals containing (x), in other words it is a minimal prime ideal of the Noetherian ring $A/(x)$. But it is well-known that a Noetherian ring has only finitely many minimal prime ideals. This is best seen geometrically: if A is Noetherian, $\operatorname{Spec} A$ is a Noetherian topological space (open subsets satisfy ACC) and such a space has only finitely many irreducible closed subsets, which correspond to the minimal primes.

Corollary 8. *Let R be a normal Noetherian domain. Then R is the intersection in its fraction field K of all the DVR's $R_{\mathfrak{p}}$ obtained by localizing at height one primes, and for each $0 \neq f \in K$, we have $v_{\mathfrak{p}}(f) = 0$ for all but finitely many primes.*

For a general normal domain, the inclusion $R \subset \bigcap_{\mathfrak{p} \in \Sigma(R)} R_{\mathfrak{p}}$ may be proper. Moreover, a nonzero element x may be contained in infinitely many primes. This provides motivation for the following definition, which isolates the class of domains for which neither of these "pathologies" occur.

Definition: An integral domain R is a **Krull domain** if it satisfies:

(KD1) For each height one prime $\mathfrak{p}$, $R_{\mathfrak{p}}$ is a DVR.

[2] A better notation for the general picture would be $\Sigma^{(1)}(R)$, denoting height one, with $\Sigma^{(k)}(R)$ denoting height k ideals for any $k \in \mathbb{N}$. But we are only interested in height one primes here, so we simplify the notation.

(KD2) $R = \bigcap_{\mathfrak{p} \in \Sigma(R)} R_{\mathfrak{p}}$; and
(KD3) For each $0 \neq x \in R$, the set of height one primes containing x is finite.

We immediately deduce:

Theorem 9. *a) A Krull domain is normal.*
b) A normal Noetherian domain is a Krull domain.

Proof: It is an easy exercise to show that if $\{R_i\}$ is a family of normal domains inside a common fraction field K, their intersection $R = \bigcap_i R_i$ is normal: this gives part a). Part b) follows immediately from Corollary 8.

Here is a useful equivalent definition of a Krull domain. We begin with a field K and a family $\{v_i\}_{i \in I}$ of discrete valuations on K satisfying the following "finite character" property:

(FC) For $x \in K^\times$ such that $v_i(x) \geq 0 \ \forall i \in I$, $\{i \in I \mid v_i(x) > 0\}$ is finite.

Then:

Theorem 10. *Let K be a field and $\{v_i\}_{i \in I}$ be a family of discrete valuations on K satisfying (FC). Then*

$$R = \{x \in K \mid v_i(x) \geq 0 \ \forall i \in I\}$$

is a Krull domain.

Proof: CITATION MISSING.

3. The Chow divisor class group of a Noetherian domain

Unsurprisingly, it is easiest to define the divisor class group for a domain R which is both Noetherian and Krull, i.e., an integrally closed Noetherian domain, so it seems best to begin with this case and then discuss what modifications must be made when we weaken the hypotheses in either of the two above ways.

We define the **divisor group** $\operatorname{Div}(R) = \bigoplus_{\mathfrak{p}} \mathbb{Z}[\mathfrak{p}]$, i.e., a free abelian group on the set of height one prime ideals. In other words, an element of this group is a formal sum $\sum_{\mathfrak{p}} n_{\mathfrak{p}}[\mathfrak{p}]$ with each $n_{\mathfrak{p}} \in \mathbb{Z}$ and all but finitely many equal to 0. Such a guy is called a divisor, or, for emphasis, a **Weil divisor**.

As in the case of the class group (Pic = Cl) of a Dedekind domain, the goal is to associate to each $f \in K^\times$ a certain divisor $\operatorname{div}(f)$. The subset $\operatorname{Prin}(R) = \{\operatorname{div}(f) \mid f \in K^\times\}$ will turn out to be a subgroup, the subgroup of **principal divisors**, and we shall define

$$\operatorname{Cl}(R) := \operatorname{Div}(R)/\operatorname{Prin}(R).$$

So the key is to define $\operatorname{div}(f)$: in other words, for each height one prime $\mathfrak{p}$ we wish to associate in some reasonable way an integer, say, $\operatorname{ord}_{\mathfrak{p}}(f)$, so that for fixed f all but finitely many are 0. Well, all we have to do is localize at $\mathfrak{p}$: we get a ring $R_{\mathfrak{p}}$ which remains Noetherian, remains integrally closed, and is a domain in which the height one prime $\mathfrak{p}$ becomes the unique maximal ideal. In other words it is a one-dimensional integrally closed Noetherian local ring, hence by Theorem 5 a DVR. Therefore we can define $\operatorname{ord}_{\mathfrak{p}}(f)$ to be the valuation of f in this discrete valuation

ring. In other words, if $\mathfrak{p}R_{\mathfrak{p}} = (\pi)$ is the maximal ideal, then the fractional ideal (f) is necessarily of the form $(\pi)^a$ for a unique integer a, and that's the integer we want. Notice that

$$\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g), \ \operatorname{div}(1/f) = -\operatorname{div}(f), \tag{1}$$

so that $\operatorname{Prin}(R)$ is at least a subgroup of the direct product $\prod_{\mathfrak{p}} \mathbb{Z}$. It remains to be shown that it is actually contained in the direct *sum*, i.e., that for each fixed f we have $\operatorname{ord}_{\mathfrak{p}}(f) = 0$ for all but at most finitely many height one primes $\mathfrak{p}$. Just writing $f = \frac{x}{y}$ with $x, y \in R$ and noting that $\operatorname{ord}_{\mathfrak{p}}(f) = \operatorname{ord}_{\mathfrak{p}}(x) - \operatorname{ord}_{\mathfrak{p}}(y)$, we are reduced to Corollary 8.

This completes the construction. We pause to remark that if R is a Dedekind domain, then $\operatorname{Div}(R)$ is precisely the group of fractional R-ideals and $\operatorname{Prin}(R)$ is the subgroup of principal fractional ideals, so $\operatorname{Pic}(R) = \operatorname{Cl}(R)$ in this case. In other words, it is indubitably correct and unambiguous to speak about "the class group" of a Dedekind domain.

3.1. **Chow class group of a Noetherian domain.**

Now suppose R is a Noetherian domain – but not necessarily integrally closed – with fraction field K. We define the group of Weil divisors, $\operatorname{Div}(R)$ exactly as before: it is the free abelian group $\bigoplus_{\mathfrak{p}} \mathbb{Z}$ on the set of height one prime ideals. Moreover, for $f \in K^\times$ we again wish to define a divisor $\operatorname{div}(f)$. We wish it to say the same formal properties as in the previous case, especially (1), so that the subset $\operatorname{Prin}(R) = \{\operatorname{div}(f) \mid f \in K^\times\}$ is a subgroup of $\operatorname{Div}(R)$, and again define

$$\operatorname{Cl}(R) = \operatorname{Div}(R)/\operatorname{Prin}(R).$$

Again, it comes down to some (reasonable!) assignment $\mathfrak{p} \mapsto \operatorname{ord}_{\mathfrak{p}}(f) \in \mathbb{Z}$ such that all but finitely many are zero. Again we wish this integer to be a "local" invariant of f in $R_{\mathfrak{p}}$. Here $R_{\mathfrak{p}}$ will again be a one-dimensional Noetherian local ring, but it need not be integrally closed, so not a DVR. In fact, as we have mentioned before, since integral closure is a local property, if R itself is not integrally closed, at least one $R_{\mathfrak{p}}$ is guaranteed not to be a DVR. So the plot must thicken.

I don't have any especially good way of motivating what one actually does, so here goes: writing $f = \frac{x}{y}$ with $x, y \in R$ as usual, we define $\operatorname{ord}_{\mathfrak{p}}(x)$ to be the **length** of the $R_{\mathfrak{p}}$-module $R_{\mathfrak{p}}/xR_{\mathfrak{p}}$; similarly for y, and we define

$$\operatorname{ord}_{\mathfrak{p}}(f) = \operatorname{ord}_{\mathfrak{p}}(x) - \operatorname{ord}_{\mathfrak{p}}(y).$$

Recall that the **length** of an R-module M is the length of a composition series for M, i.e., given a sequence of R-submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_\ell = M$$

such that each M_i/M_{i-1} is simple (has no nonzero, proper submodules), we define the length of M to be ℓ. Two concerns here are: (i) does every R-module admit at least one composition series, and (ii) do any two composition series have the same length? The answer to (i) is generally no, but there is a nice characterization: an R-module M has a composition series iff it is both Noetherian and Artinian, i.e., satisfies both (ACC) and (DCC) on submodules. It is not too hard to show that a quotient of a one-dimensional Noetherian domain by a nonzero principal ideal is

both Artinian and Noetherian (e.g. [Liu, Lemma 7.1.26]). However, you should certainly check that in the case that $R_{\mathfrak{p}}$ is a DVR, for $x \in R$, the length of $R_{\mathfrak{p}}/xR_{\mathfrak{p}}$ is exactly $\operatorname{ord}_{\mathfrak{p}}(x)$ in the previous sense: this is easy and enlightening. As for (ii), I hope you are not entirely surprised to hear as long as an R-module has finite length, not only the length itself but the set of composition factors M_i/M_{i-1} – unordered, but taken with multiplicity – is independent of the choice of composition series: this is module-theoretic analogue of the **Jordan-Hölder theorem** for finite groups, and the proof is the same.

We leave the verification of (1) as an (easy) exercise; finally, the argument that $\operatorname{div}(f) \subset \bigoplus_{\mathfrak{p}} \mathbb{Z}$ is much the same as above: there are only finitely many primes containing a nonzero element $x \in R$, and for any other prime $\mathfrak{p}$, $xR_{\mathfrak{p}} = R_{\mathfrak{p}}$, so $R_{\mathfrak{p}}/xR_{\mathfrak{p}} = 0$, so has length 0.

Thus we have defined $\operatorname{Cl}(R)$ for an arbitrary Noetherian domain in a way which agrees with $\operatorname{Cl}(R)$ in the special case in which R is integrally closed.

3.2. $\operatorname{Pic}(R)$ **versus** $\operatorname{Cl}(R)$**.** Comparison with $\operatorname{Pic}(R)$: I claim there is a canonical homomorphism

$$\operatorname{Pic}(R) \to \operatorname{Cl}(R),$$

which is generally not an isomorphism. We will first define a homomorphism from the group $I(R)$ of invertible fractional ideals of R to the group $\operatorname{Div}(R)$ and then check that this factors through the quotient $I(R)/\operatorname{Prin}(R)$.

Recall that a fractional ideal I is invertible iff it is locally principal, so in particular for each height one prime $\mathfrak{p}$ we have $I_{\mathfrak{p}} = x_{\mathfrak{p}} R_{\mathfrak{p}}$ is a principal fractional ideal in the local ring. Clearly then the map we want is

$$I \mapsto (\operatorname{ord}_{\mathfrak{p}} x_{\mathfrak{p}}) \in \operatorname{Div}(R).$$

I leave it to you to check that this induces a homomorphism $\operatorname{Pic}(R) \to \operatorname{Div}(R)$.

We say that a ring R is **locally factorial** if for every maximal ideal $\mathfrak{m}$ of R, the local ring $R_{\mathfrak{m}}$ is a UFD (aka a "factorial ring"). Now:

Theorem 11. *(Comparison Theorem)*
a) The map $I(R) \to \operatorname{Div}(R)$ is an isomorphism if R is locally factorial.
b) A regular local ring is a UFD.
c) Therefore if R is regular – e.g., the coordinate ring of a nonsingular affine variety – then $\operatorname{Pic}(R) \xrightarrow{\sim} \operatorname{Cl}(R)$.

Proof: See [Eis, p. 261].

In general, the map $\operatorname{Pic}(R) \to \operatorname{Cl}(R)$ need not be either injective or surjective.

3.3. **Brief remarks on the geometric case.**

In a similar way, on can define the class group can be generalized to any Noetherian integral scheme. In fact, if you know what all these words mean, the construction should be clear. (Height one prime is synonymous with irreducible closed subscheme of codimension one, but on the other hand, the height of a point on a Noetherian scheme makes perfect sense: it is the height of the maximal ideal in the the local ring at that point.) In particular, one can define $\operatorname{Cl}(X)$ for any

algebraic variety.

Example: Suppose C is a nonsingular algebraic curve over a field k. Then the height one prime ideals are what the ancients called "points" on the curve (now we call them "closed points").[3] Therefore $\operatorname{Div}(C)$ is the free abelian group on the points, just as in your algebraic geometry class. In full generality there is an obvious homomorphism from any Weil divisor group to $\mathbb{Z}$: we just add up all the coefficients. This is called the **degree map** and the kernel is sometimes called Div^0 (of R or X or whatever). In the affine case, however, this is not very useful because it does not interact well with the notion of a principal divisor: there will be principal divisors of all possible degrees.

However, a nonzero element f of the function field $k(C)$ of the curve can be viewed as a finite map to the projective line so – as a finite map! – has a well-defined degree $d \in \mathbb{Z}^+$ which is the inverse image of a given point, counted with suitable multiplicities. (More precisely it's the length of a certain module, but let's skip it...) By definition, $\operatorname{div}(f) = f^{-1}(0) - f^{-1}(\infty)$; but both the first term (the divisor of zeros) and the second term (the polar divisor) have degree d, so their difference has degree 0. In other words, in the projective case we have $\operatorname{Prin}(C) \subset \operatorname{Div}^0(C)$, so we get a short exact sequence

$$0 \to \operatorname{Div}^0(C)/\operatorname{Prin}(C) \to \operatorname{Div}(C)/\operatorname{Prin}(C) \to \mathbb{Z} \to 0.$$

The first term is precisely the Jacobian of C: it can be given the structure of an abelian variety whose dimension is equal to the genus g of C, and here $\mathbb{Z} = NS(C)$, so that a divisor on a curve is algebraically equivalent to 0 iff it has degree 0. As in the affine case (i.e., Dedekind domains) it can be checked that $\operatorname{Cl}(C) = \operatorname{Pic}(C)$.

The Comparison Theorem (Theorem 11) also generalizes to Noetherian integral schemes in a straightforward way. In particular, for any nonsingular Noetherian integral scheme X, one has a canonical isomorphism $\operatorname{Pic}(X) = \operatorname{Cl}(X)$. This is often summarized as an equality between Weil divisors and Cartier divisors, although I will not go so far into geometry as to give the (sheaf-theoretic) definition of a Cartier divisor here.

4. The class group of a Krull domain

If R is a Krull domain with quotient field K, it should be clear by now how to define $\operatorname{Cl}(R)$: we define $\operatorname{Div}(R)$ to be the free abelian group on $\Sigma(R)$, the set of height one primes. For $0 \neq f \in K$, we define

$$\operatorname{div}(f) = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(f)\ [\mathfrak{p}],$$

$\operatorname{Prin}(R) = \{\operatorname{div}(f) \mid f \in K^\times\}$ and $\operatorname{Cl}(R) = \operatorname{Div}(R)/\operatorname{Prin}(R)$. There is really nothing further to say, since every nice property that we had to show in the case of a Noetherian normal domain is built into the definition of a Krull domain!

We would however like to mention the following alternate construction of the class group, which is more closely analogous to the construction of the Picard group in that it is a group of equivalence classes of certain ideals modulo principal ideals.

[3] Actually, this is not quite true if k is not algebraically closed, but I don't want to get into it.

5. Complete normalization

Definition: Let $R \subset S$ be a ring extension. An element $x \in R'$ is **almost integral** over R if there exists a finitely generated R-submodule M of S such that for all $n \in \mathbb{Z}^+$, $x^n \in M$.

Since x is integral over $R \iff R[x]$ is finite over R, indeed if x is integral over R it is almost integral over R. Conversely, suppose x is almost integral over R and R is Noetherian: then, since $R[x] \subset M$ and M is finitely generated, so is $R[x]$, so x is integral over R.

Thus integral implies almost integral, and the two coincide for Noetherian rings.

For $R \subset S$, the **complete normalization** of R in S is the set of all elements of S which are almost integral over R. We say a domain R is **completely normal** if every element of the fraction field of R which is almost integral over R is already an element of R.

Proposition 12. *For an extension of domains $R \subset S$, the complete normalization of R in S is a subring of S which is normal.*

Proof: [LM, Prop. 4.18].

Warning: Unfortunately it need **not** hold that the complete normalization of a domain in its fraction field is completely normal.

Theorem 13. *The complete normalization of a domain R in its fraction field K is the set of elements $x \in K$ such that there exists $0 \neq r \in R$ such that $rx^n \in R$ for all $n \in \mathbb{Z}^+$.*

Proof: [LM, Thm. 4.20].

6. Valuation rings

Theorem 14. *Let R be an integral domain with fraction field K. The normalization of R in K is the intersection of all (not necessarily discrete!) valuation rings of K containing R.*

Proof: [LM, Cor. 5.8].

A subgroup H of an ordered abelian group G is **isolated** if for each $0 \leq x \in H$ and $0 \leq y \in G$, $0 \leq y \leq x$ implies $y \in H$.

If an ordered abelian group G has only a finite number of isolated subgroups, the number of proper isolated subgroups is called the **rank** of G.

Proposition 15. *A nonzero ordered abelian group has rank one iff there is an order embedding from G into the additive group of the real numbers.*

Proof: [LM, Prop. 5.15].

Theorem 16. *Let v be a valuation on a field K with value group G and valuation ring R. Then there is a bijective, order-reversing correspondence between isolated subgroups of G and proper prime ideals of R.*

Proof: [LM, Thm. 5.17]. An immediate corollary is that the Krull dimension of a valuation ring is equal to its rank.

Theorem 17. *A valuation ring is completely normal iff it has rank at most one.*

Proof: [LM, Thm. 5.19].

7. Prüfer Domains

Theorem 18. *For an integral domain R, TFAE:*
*(i) Every nonzero finitely generated ideal is invertible (R is **Prüfer**).*
(ii) Every nonzero ideal generated by two elements is invertible.
(iii) If for a finitely generated nonzero ideal A and arbitrary ideals B and C of R we have $AB = AC$, then $B = C$.
(iv) For each prime ideal P of R, R_P is a valuation ring.
(iv′) For each maximal ideal $\mathfrak{m}$ of R, $R_{\mathfrak{m}}$ is a valuation ring. (v) For all ideals A, B, C of R, $A(B \cap C) = AB \cap AC$.
(vi) For all ideals A, B of R, $(A + B)(A \cap B) = AB$.
(viii) If $A \subset C$ are ideals of R, with C finitely generated, there exists an ideal B of R such that $A = BC$.
(ix) For all ideals A, B, C of R with A, B finitely generated,

$$C : (A \cap B) = C : A + C : B.$$

(x) For all ideals A, B, C of R, $A \cap (B + C) = A \cap B + A \cap C$.

Proof: [LM, Thm. 6.6, Cor. 6.7].

Let R be an integral domain with fraction field K. By an **overring** of R, we mean a domain S intermediate between R and K: $R \subset S \subset K$.

Proposition 19. *Let S be an overring of the domain R. TFAE: (i) T is a flat R-algebra.*
(ii) For all maximal ideals $\mathfrak{m}$ of S, $S_{\mathfrak{m}} = R_{\mathfrak{m} \cap R}$.
(iii) $S = \bigcap_{\mathfrak{m}} R_{\mathfrak{p} \cap R}$, the intersection running over all maximal ideals of S.

Proof: [LM, Prop. 4.14].

Theorem 20. *An integral domain is Prüfer iff every overring of R is flat.*

Proof: [LM, Thm. 6.10].

Corollary 21. *Every overring of a Prüfer domain is a Prüfer domain.*

Proof: This follows immediately from the theorem.

Corollary 22. *Let R be a Prüfer domain and S an overring. Let Δ be the set of all prime ideals $\mathfrak{p}$ of R with $\mathfrak{p}S \neq S$. Then*

$$S = \bigcap_{\mathfrak{p} \in \Delta} R_{\mathfrak{p}}.$$

Proof: [LM, Cor. 6.12].

Theorem 23. *For an integral domain R, TFAE:*
(i) R is Prüfer.
(ii) Every overring of R is normal.

Proof: [LM, Thm. 6.13].

8. Divisorial ideals

For a fractional ideal I of a domain R, put

$$I^* = (R : I) = \{a \in K \mid aI \subset R\}.$$

We always have $I^*I \subset R$, and equality holds iff I is invertible. In particular, for any invertible ideal, I^* *is* the inverse of I. I^* is called[4] the **quasi-inverse** of I.

Two fractional ideals I and J are **quasi-equal** if $I^* = J^*$. Notice that two invertible ideals are quasi-equal iff they are equal. In general, quasi-equality is an equivalence relation on the set of fractional ideals: we write $I \sim J$.

A **divisor** of R is an equivalence class of quasi-equal fractional ideals. for a fractional ideal I, we write $[I]$ for its divisor. Let $D(R)$ denote the set of all divisors of R.

For any $f \in K^\times$, we define $\operatorname{div}(f)$ to be the class in $D(R)$ of fR. Because principal fractional ideals are invertible, $\operatorname{div}(f) = \operatorname{div}(g) \iff fR = gR$.

Proposition 24. *Two fractional ideals I and J are quasi-equal iff they are contained in the same principal fractional ideals.*

Proof: If $f \in K^\times$, then $I \subset fR$ iff $\frac{1}{f} \in (R : I)$.

For a fractional ideal I, put $\overline{I} = \bigcap_{I \subset fR} fR$. Notice $\overline{I} \supset I$.

Proposition 25. *$\overline{I}$ is a fractional ideal which is quasi-equal to I.*

Proof: Since $\overline{I}$ contains I, it is evidently a nonzero R-submodule of K. If $0 \neq x \in R$ is such that $xI \subset R$, then $I \subset \frac{1}{x}R$ so $\overline{R} \subset \frac{1}{x}R$ and thus $x\overline{I} \subset R$, so $\overline{I}$ is a fractional ideal. That it is quasi-equal to I is built into the definition.

Corollary 26. *For a fractional ideal I, $\overline{I}$ is characterized as the unique fractional ideal quasi-equal to I and containing all fractional ideals quasi-equal to I.*

Proposition 27. *For any fractional ideal I, $\overline{I} = (I^*)^* = (R : (R : I))$.*

Now an important definition: We say a fractional ideal I is **divisorial** if $I = \overline{I}$.

It follows from the above that any divisor has a unique divisorial representative, so that we may identify $D(R)$ with the set of divisorial fractional ideals.

The idea here is that $D(R)$ is, in a rather general context, a substitute for the group of invertible fractional ideals. Namely, we can define a multiplication operation on divisors: for fractional ideals I and J, put

$$[I] \cdot [J] := [IJ].$$

We have to check that this is well-defined on equivalence classes. To see this, observe

$$(R : IJ) = ((R : I) : J) = ((R : \overline{I}) : J) = ((R : J) : \overline{I}) = ((R : \overline{J}) : \overline{I}) = (R : \overline{IJ}).$$

[4] By some, at least – I do not find the terminology very appealing

This gives $D(R)$ the structure of a commutative monoid, with $[R]$ as the identity element. We write $[I] \leq [J]$ if $\overline{J} \leq \overline{I}$ (note the order reversal!). This gives a partial ordering which is compatible with the monoid structure in the following sense:

Proposition 28. *For A, B, C fractional ideals of R, with $[A] \leq [B]$, $[AC] \leq [BC]$.*

Proof: It suffices to show that $\overline{BC} \subset \overline{AC}$. By hypothesis we have $\overline{J} \subset \overline{I}$: that is, $(R : A) \subset (R : B)$. Then

$$(R : AC) = ((R : A) : C) \subset ((R : B) : C) = (R : BC),$$

so $\overline{BC} \subset \overline{AC}$.

In fact $D(R)$ is a **lattice ordered monoid**: that means that it is a monoid, endowed with a compatible partial order, which is a lattice: any two fractional ideals have a unique greatest lower bound:

$$\max([I], [J]) = [I \cap J],$$

$$\min([I], [J]) = [I + J].$$

Theorem 29. *For a domain R, the monoid of divisors $D(R)$ is a group iff R is completely normal.*

Proof: [LM, Thm. 8.7].

Corollary 30. *If R is a Krull domain, $D(R)$ is a lattice ordered group.*

Proof: We know that a Krull domain is an intersection of DVR's and that every DVR is normal and Noetherian, hence completely normal. It is immediate to see that the intersection of completely normal domains inside a common fraction field is a completely normal domain, so Krull domains are completely normal.

Theorem 31. *Let R be an integral domain which is not a field. TFAE:*
(i) R is a Krull domain.
(ii) R is completely normal, and every nonempty set of divisorial integral ideals has a maximal element.

Proof: [LM, Thm. 8.12].

This is an elegant generalization of the fact that a Noetherian domain is Krull iff it is normal.

As one might expect, in a Krull domain the two divisor groups $D(R)$ and $\operatorname{Div}(R)$ are canonically isomorphic.

Namely, to each divisorial fractional ideal I and height one prime $\mathfrak{p}$, the localization $IR_{\mathfrak{p}}$ is necessarily of the form $(\pi)^{v_{\mathfrak{p}}}$ where $(\pi) = \mathfrak{p}R_{\mathfrak{p}}$ and $v_{\mathfrak{p}}$ is a unique integer. In fact a nonzero prime ideal of R is divisorial iff it has height one. See [LM, §VIII.2] for proofs.

Thus one can equally well define the divisor class group $\operatorname{Cl}(R)$ of a Krull domain as $D(R)/\operatorname{Prin}(R)$.

Theorem 32. *For an integral domain R, TFAE:*
(i) R is a Krull domain with $\mathrm{Cl}(R) = 0$.
(ii) R is a UFD.
(iii) R is a Krull domain and each divisorial ideal is principal.
(iv) R satisfies the ascending chain condition on ***principal*** *ideals (ACCP) and each irreducible element is prime.*
(v) R satisfies (ACCP) and the intersection of two principal ideals is principal.

Proof: [LM, §VIII.4].

Theorem 33. *A ring is Dedekind iff it is a one-dimensional Krull domain.*[5]

Proof: [ZSII, Thm. VI.27].

Theorem 34. *(Approximation theorem for Krull domains) Let R be a Krull domain with fraction field K. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be a finite set of height one primes. Let $n_1, \ldots, n_k \in \mathbb{Z}$. Then there exists an element $f \in K$ such that:*
(i) $v_{\mathfrak{p}_i}(f) = n_i$, $1 \leq i \leq k$.
(ii) $v_{\mathfrak{q}}(f) \geq 0$ for all $\mathfrak{q} \neq \mathfrak{p}_i$.

Theorem 35. *Let R be a Noetherian domain. Then its normalization is a Krull domain.*

Remark: If R has dimension at most one, then the Krull-Akizuki theorem gives a stronger result: the normalization is (of course normal and) Noetherian, i.e., a Dedekind domain. However, starting in dimension two the normalization of a Noetherian ring need not be Noetherian! This gives another hint that the class of Krull domains is for some purposes more natural than that of normal Noetherian domains.

References

[AM] Atiyah and MacDonald, *Introduction to commutative algebra.*
[Eis] Eisenbud, *Commutative algebra with a view towards algebraic geometry.*
[Har] Hartshorne, *Algebraic geometry.*
[LM] Larsen and McCarthy, *Multiplicative ideal theory.*
[Liu] Qing Liu, *Algebraic geometry and arithmetic curves.*
[Mat] Matsumura, *Commutative ring theory.*
[Rei] Miles Reid, *Undergraduate commutative algebra.*
[ZSII] Zariski and Samuel, *Commutative Algebra: Volume II.*

[5] Or a field, of course.