

**Sequences and Series:**

**An Introduction to**

**Mathematical Analysis**

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minor(?) changes by Pete L. Clark

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# Chapter 1

## Sequences

ADDED BY PLC. We define the *integers*

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\},$$

the *natural numbers*

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

and the *positive integers*

$$\mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

All else is the work of man.

### 1.1 The general concept of a sequence

We begin by discussing the concept of a sequence. Intuitively, a sequence is an ordered list of objects or events. For instance, the sequence of events at a crime scene is important for understanding the nature of the crime. In this course we will be interested in sequences of a more mathematical nature; mostly we will be interested in sequences of numbers, but occasionally we will find it interesting to consider sequences of points in a plane or in space, or even sequences of sets.

Let's look at some examples of sequences.

**Example 1.1.1.** *Emily flips a quarter five times, the sequence of coin tosses is HTTHT where H stands for "heads" and T stands for "tails".*

As a side remark, we might notice that there are  $2^5 = 32$  different possible sequences of five coin tosses. Of these, 10 have two heads and three tails. Thus the probability that in a sequence of five coin tosses, two of them are heads and three are tails is  $10/32$ , or  $5/16$ . Many probabilistic questions involve studying sets of sequences such as these.

**Example 1.1.2.** John picks colored marbles from a bag, first he picks a red marble, then a blue one, another blue one, a yellow one, a red one and finally a blue one. The sequence of marbles he has chosen could be represented by the symbols *RBBYRB*.

**Example 1.1.3.** Harry the Hare set out to walk to the neighborhood grocery. In the first ten minutes he walked half way to the grocery. In the next ten minutes he walked half of the remaining distance, so now he was  $3/4$  of the way to the grocery. In the following ten minutes he walked half of the remaining distance again, so now he has managed to get  $7/8$  of the way to the grocery. This sequence of events continues for some time, so that his progress follows the pattern  $1/2, 3/4, 7/8, 15/16, 31/32$ , and so on. After an hour he is  $63/64$  of the way to the grocery. After two hours he is  $4095/4096$  of the way to the grocery. If he was originally one mile from the grocery, he is now about 13 inches away from the grocery. If he keeps on at this rate will he ever get there? This brings up some pretty silly questions; For instance, if Harry is 1 inch from the grocery has he reached it yet? Of course if anybody manages to get within one inch of their goal we would usually say that they have reached it. On the other hand, in a race, if Harry is 1 inch behind Terry the Tortoise he has lost the race. In fact, at Harry's rate of deceleration, it seems that it will take him forever to cross the finish line.

**Example 1.1.4.** Harry's friend Terry the Tortoise is more consistent than Harry. He starts out at a slower pace than Harry and covers the first half of the mile in twenty minutes. But he covers the next quarter of a mile in 10 minutes and the next eighth of a mile in 5 minutes. By the time he reaches  $63/64$  of the mile it has taken less than 40 minutes while it took Harry one hour. Will the tortoise beat the hare to the finish line? Will either of them ever reach the finish line? Where is Terry one hour after the race begins?

**Example 1.1.5.** Build a sequence of numbers in the following fashion. Let the first two numbers of the sequence be 1 and let the third number be

$1 + 1 = 2$ . The fourth number in the sequence will be  $1 + 2 = 3$  and the fifth number is  $2 + 3 = 5$ . To continue the sequence, we look for the previous two terms and add them together. So the first ten terms of the sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$$

This sequence continues forever. It is called the Fibonacci sequence. This sequence is said to appear ubiquitously in nature. The volume of the chambers of the nautilus shell, the number of seeds in consecutive rows of a sunflower, and many natural ratios in art and architecture are purported to progress by this natural sequence. In many cases the natural or biological reasons for this progression are not at all straightforward.

The sequence of positive integers,

$$1, 2, 3, 4, 5, \dots$$

and the sequence of odd positive integers,

$$1, 3, 5, 7, 9, \dots$$

are other simple examples of sequences that continue forever. The symbol ... (called an “ellipsis”) represents this infinite continuation. Such a sequence is called an *infinite sequence*. In this book most of our sequences will be infinite and so from now on when we speak of sequences we will mean infinite sequences. If we want to discuss some particular *finite sequence* we will specify that it is finite.

Since we will want to discuss general sequences in this course it is necessary to develop some notation to represent sequences without writing down each term explicitly. The fairly concrete notation for representing a general infinite sequence is the following:

$$a_1, a_2, a_3, \dots$$

where  $a_1$  represents the first number in the sequence,  $a_2$  the second number, and  $a_3$  the third number, etc. If we wish to discuss an entry in this sequence without specifying exactly which entry, we write  $a_i$  or  $a_j$  or some similar term.

To represent a finite sequence that ends at, say, the 29<sup>th</sup> entry we would write

$$a_1, a_2, \dots, a_{29}.$$

Here the ellipsis indicates that there are several intermediate entries in the sequence which we don't care to write out explicitly. We may also at times need to represent a series that is finite but of some undetermined length; in this case we will write

$$a_1, a_2, \dots, a_N$$

where  $N$  represents the fixed, but not explicitly specified length.

A slightly more sophisticated way of representing the abstract sequence  $a_1, a_2, \dots$  is with the notation:

$$\{a_i\}_{i=1}^{\infty}.$$

The finite sequence  $a_1, a_2, \dots, a_N$  is similarly represented by:

$$\{a_i\}_{i=1}^N.$$

Since in this text we study mostly infinite sequences, we will often abbreviate  $\{a_i\}_{i=1}^{\infty}$  with simply  $\{a_i\}$ . Although this looks like set notation you should be careful not to confuse a sequence with the set whose elements are the entries of the sequence. A set has no particular ordering of its elements but a sequence certainly does. For instance, the sequence  $1, 1, 1, 1, \dots$  has infinitely many terms, yet the set made of these terms has only one element.

When specifying any particular sequence, it is necessary to give some description of each of its terms. This can be done in various ways. For a (short) finite sequence, one can simply list the terms in order. For example, the sequence  $3, 1, 4, 1, 5, 9$  has six terms which are easily listed. On the other hand, these are the first six terms of the decimal expansion of  $\pi$ , so this sequence can be extended to an infinite sequence,  $3, 1, 4, 1, 5, 9, \dots$ , where it is understood from the context that we continue this sequence by computing further terms in the decimal expansion of  $\pi$ . Here are a few other examples of infinite sequences which can be inferred by listing the first few terms:

$$1, 2, 3, 4, \dots$$

$$2, 4, 6, 8, \dots$$

$$5, 10, 15, 20, \dots$$

$$1, 1, 2, 3, 5, 8, 13, \dots$$

Well maybe it is not so obvious how to extend this last sequence unless you are familiar with the Fibonacci sequence discussed in Example 1.1.5. This last example demonstrates the drawback of determining a sequence by



inference, it leaves it to the reader to discover what method you used to determine the next term.

A better method of describing a sequence is to state how to determine the  $n^{\text{th}}$  term with an *explicit formula*. For example, the sequence 1, 2, 3, 4, ... is easily specified by saying  $a_n = n$ . Formulas for the second and third sequence above can be specified with the formulas  $a_n = 2n$  and  $a_n = 5n$  respectively. An explicit formula for the  $n^{\text{th}}$  term of the Fibonacci sequence, or the  $n^{\text{th}}$  term in the decimal expansion of  $\pi$  is not so easy to find. In exercise 1.2.20 we will find an explicit formula for the Fibonacci sequence, but there is no such explicit formula for the  $n^{\text{th}}$  term in the decimal expansion of  $\pi$ .

**Example 1.1.6.** *The  $n^{\text{th}}$  term in a sequence is given by  $a_n = (n^2 + n)/2$ . The first five terms are 1, 3, 6, 10, 15.*

**Example 1.1.7.** *The  $n^{\text{th}}$  term in the sequence  $\{b_n\}$  is given by  $b_n = 1 - \frac{1}{n^2}$ . The first six terms of this sequence are*

$$0, 3/4, 8/9, 15/16, 24/25, 35/36.$$

A third way of describing a sequence is through a *recursive formula*. A recursive formula describes the  $n^{\text{th}}$  term of the sequence in terms of previous terms in the sequence. The easiest form of a recursive formula is a description of  $a_n$  in terms of  $a_{n-1}$ . Many of our earlier examples of numerical sequences were described in this way.

**Example 1.1.8.** *Let's return to Example 1.1.3 above. Each 10 minutes, Harry walks half of the remaining distance to the neighborhood. Let's denote the fraction of the total distance that Harry has travelled after  $n$  chunks of ten minutes by  $a_n$ . So  $a_1 = 1/2, a_2 = 3/4, a_3 = 7/8$ , etc. Then the fraction of the total distance that remains to be travelled after  $n$  chunks of ten minutes is  $1 - a_n$ . Since the distance travelled in the next ten minutes is half of this remaining distance, we see that*

$$a_{n+1} = a_n + \frac{1}{2}(1 - a_n) = \frac{1}{2}(1 + a_n).$$

*Notice that this formula is not enough by itself to determine the sequence  $\{a_n\}$ . We must also say how to start the sequence by supplying the information that*

$$a_1 = 1/2.$$

Now, with this additional information, we can use the formula to determine further terms in the sequence:

$$\begin{aligned} a_2 &= \frac{1}{2}(1 + a_1) = \frac{1}{2}(1 + 1/2) = 3/4 \\ a_3 &= \frac{1}{2}(1 + a_2) = \frac{1}{2}(1 + 3/4) = 7/8 \\ a_4 &= \frac{1}{2}(1 + a_3) = \frac{1}{2}(1 + 7/8) = 15/16, \end{aligned}$$

etc.

**Example 1.1.9.** Let's have another look at the Fibonacci sequence from Example 1.1.5 above. Here the  $n^{\text{th}}$  term is determined by two previous terms, indeed

$$a_{n+1} = a_n + a_{n-1}.$$

Now we can't get started unless we know the first two steps in the sequence, namely  $a_1$  and  $a_2$ . Since we are told that  $a_1 = 1$  and  $a_2 = 1$  also, we can use the recursion formula to determine

$$a_3 = a_2 + a_1 = 1 + 1 = 2.$$

And now since we have both  $a_2$  and  $a_3$  we can determine

$$a_4 = a_3 + a_2 = 2 + 1 = 3,$$

and similarly

$$\begin{aligned} a_5 &= a_4 + a_3 = 3 + 2 = 5, \\ a_6 &= a_5 + a_4 = 5 + 3 = 8, \\ a_7 &= a_6 + a_5 = 8 + 5 = 13, \end{aligned}$$

and so on.

To conclude this section we mention two more families of examples of sequences which often arise in mathematics, the *arithmetic* (the accent is on the third syllable!) sequences and the *geometric* sequences.

An *arithmetic* sequence has the form  $a, a + b, a + 2b, a + 3b, \dots$  where  $a$  and  $b$  are some fixed numbers. An explicit formula for this arithmetic sequence is given by  $a_n = a + (n - 1)b$ ,  $n \in \mathbb{Z}^+$ , a recursive formula is given by  $a_1 = a$

and  $a_n = a_{n-1} + b$  for  $n > 1$ . Here are some examples of arithmetic sequences, see if you can determine  $a$  and  $b$  in each case:

$$\begin{aligned} &1, 2, 3, 4, 5, \dots \\ &2, 4, 6, 8, 10, \dots \\ &1, 4, 7, 10, 13, \dots \end{aligned}$$

The distinguishing feature of an arithmetic sequence is that each term is the *arithmetic mean* of its neighbors, i.e.,  $a_n = (a_{n-1} + a_{n+1})/2$ , (see exercise 13).

A *geometric* sequence has the form  $a, ar, ar^2, ar^3, \dots$  for some fixed numbers  $a$  and  $r$ . An explicit formula for this geometric sequence is given by  $a_n = ar^{n-1}$ ,  $n \in \mathbb{N}$ . A recursive formula is given by  $a_1 = a$  and  $a_n = ra_{n-1}$  for  $n > 1$ . Here are some examples of geometric sequences, see if you can determine  $a$  and  $r$  in each case:

$$\begin{aligned} &2, 2, 2, 2, 2, \dots \\ &2, 4, 8, 16, 32, \dots \\ &3, 3/2, 3/4, 3/8, 3/16, \dots \\ &3, 1, 1/3, 1/9, 1/27, \dots \end{aligned}$$

Geometric sequences (with positive terms) are distinguished by the fact that the  $n^{\text{th}}$  term is the *geometric mean* of its neighbors, i.e.,  $a_n = \sqrt{a_{n+1}a_{n-1}}$ , (see exercise 14).

**Example 1.1.10.** *If a batch of homebrew beer is inoculated with yeast it can be observed that the yeast population grows for the first several hours at a rate which is proportional to the population at any given time. Thus, if we let  $p_n$  denote the yeast population measured after  $n$  hours have passed from the inoculation, we see that there is some number  $\alpha > 1$  so that*

$$p_{n+1} = \alpha p_n.$$

*That is,  $p_n$  forms a geometric sequence.*

*Actually, after a couple of days, the growth of the yeast population slows dramatically so that the population tends to a steady state. A better model for the dynamics of the population that reflects this behavior is*

$$p_{n+1} = \alpha p_n - \beta p_n^2,$$

where  $\alpha$  and  $\beta$  are constants determined by the characteristics of the yeast. This equation is known as the discrete logistic equation. Depending on the values of  $\alpha$  and  $\beta$  it can display surprisingly varied behavior for the population sequence,  $p_n$ . As an example, if we choose  $\alpha = 1.2$ ,  $\beta = .02$  and  $p_0 = 5$ , we get

$$\begin{aligned} p_1 &= 5.5 \\ p_2 &= 5.995 \\ p_3 &= 6.475199500 \\ p_4 &= 6.931675229 \\ p_5 &= 7.357047845 \\ p_6 &= 7.745934354 \\ p_7 &= 8.095131245 \\ p_8 &= 8.403534497 \\ p_9 &= 8.671853559 \\ p_{10} &= 8.902203387. \end{aligned}$$

Further down the road we get  $p_{20} = 9.865991756$ ,  $p_{30} = 9.985393020$ ,  $p_{40} = 9.998743171$ , and  $p_{100} = 9.999999993$ . Apparently the population is leveling out at about 10. It is interesting to study the behavior of the sequence of  $p_n$ 's for other values of  $\alpha$  and  $\beta$  (see exercise 15).

### EXERCISES 1.1

1. a) How many sequences of six coin tosses have three heads and three tails?  
 b) How many different sequences of six coin tosses are there altogether?  
 c) In a sequence of six coin tosses, what is the probability that the result will consist of three heads and three tails?
2. (For students with some knowledge of combinatorics.) In a sequence of  $2n$  coin tosses, what is the probability that the result will be exactly  $n$  heads and  $n$  tails?
3. Let  $\{a_n\}$  be the sequence given explicitly by  $a_n = 2n - 1$  for all  $n \in \mathbb{Z}^+$ . Write out  $a_1, a_2, a_3, a_4$ , and  $a_5$ . Describe this sequence in words.

4. a) Let the sequence  $\{a_n\}$  be given recursively by

$$a_1 = 1, a_{n+1} = \frac{3 + a_n}{2} \text{ for all } n \in \mathbb{Z}^+.$$

Write down the first five terms of the sequence.

- b) Let the sequence  $\{a_n\}$  be given recursively by

$$a_1 = 0, a_{n+1} = \frac{3 + a_n}{2} \text{ for all } n \in \mathbb{Z}^+.$$

Write down the first five terms of the sequence.

5. a) Let  $\{a_n\}$  be the sequence given explicitly by  $a_n = \frac{1}{2}(n^2 - n)$  for all  $n \in \mathbb{Z}^+$ . Find explicitly  $a_1, a_2, a_5$  and  $a_{10}$ .  
 b) Show that  $a_{n+1} = \frac{1}{2}(n^2 + n)$  for all  $n \in \mathbb{N}$ , and find a formula for  $a_{n+1} - a_n$ .  
 c) Conclude that

$$a_1 = 0, a_{n+1} = a_n + n \text{ for all } n \in \mathbb{Z}^+$$

gives a recursive formula for  $a_{n+1}$  in terms of  $a_n$ .

6. Let  $\{a_n\}$  be the sequence given explicitly by  $a_n = n^2$  for all  $n \in \mathbb{Z}^+$ . Use the method developed in exercise 5 to find a recursive formula for  $a_{n+1}$  in terms of  $a_n$ .  
 7. Let  $\{a_n\}$  be the sequence given explicitly by  $a_n = \frac{n(n+1)(2n+1)}{6}$  for all  $n \in \mathbb{Z}^+$ . Use the method developed in exercise 5 to find a recursive formula for  $a_{n+1}$  in terms of  $a_n$ .  
 8. a) Let  $\{a_n\}$  be the sequence of positive integers:  $a_n = n$  for all  $n \in \mathbb{Z}^+$ . Define a new sequence  $\{b_n\}$  by

$$b_n = a_{2n-1} \text{ for all } n \in \mathbb{Z}^+.$$

Write down explicitly  $b_1, b_2, b_3$ , and  $b_{10}$ . Give an explicit formula for  $b_n$ .

- b) Let  $\{a_n\}$  be the sequence of positive integers:  $a_n = n$  for all  $n \in \mathbb{Z}^+$ . Define a new sequence  $\{c_n\}$  by

$$c_n = a_{n^2} \quad \forall n \in \mathbb{Z}^+.$$

Write down explicitly  $c_1, c_2, c_3$ , and  $c_{10}$ . Give an explicit formula for  $c_n$ .

9. Let  $\{a_n\}$  be the sequence given explicitly by  $a_n = n^2$  for all  $n \in \mathbb{Z}^+$ . Define a new sequence  $\{b_n\}$  by

$$b_n = a_{2n} \text{ for all } n \in \mathbb{Z}^+.$$

Write down explicitly  $b_1, b_2, b_3$ , and  $b_{10}$ . Give an explicit formula for  $b_n$ .

10. Let  $\{a_n\}$  be the sequence given recursively by

$$a_1 = 1, \quad a_{n+1} = \frac{a_n^2 + 2}{2a_n} \text{ for all } n \geq 1.$$

Write out the first five terms of this sequence. First find them as fractions and then, using a calculator, give five place decimal approximations to them. Compare these numbers to the decimal expansion for  $\sqrt{2}$ .

11. (Thanks to Mo Hendon.) Joe is trying to sell his old car for \$1000 and Mo offers him \$500. Joe says, “Ok, let’s split the difference, I’ll sell it to you for \$750.” But Mo says, “That is still too much, but I’ll offer to split the difference now and pay you \$625.” Joe and Mo continue to dicker in this manner for a long time. Write down a recursive formula for the  $n^{\text{th}}$  offer in terms of the previous two offers. Do you think they can ever settle on a price?
12. Little Jhonen<sup>1</sup> is at home and decides he wants to go hang out at the local pool hall. So he sets off toward the pool hall, but when he is halfway there, he remembers that his mother told him not to go there anymore and turns back towards home. When he is halfway from the midpoint back to his home, he changes his mind once more and heads to the pool hall. Continuing in this way, Jhonen changes his mind whenever he has traversed half of the distance from his turning point to the new destination. If  $D$  is the distance from Jhonen’s house to the pool hall, what fractions of  $D$  are the distance from his house to the first four turning points? What happens to Jhonen eventually?
13. a) Let an arithmetic sequence be given by the recursive formula

$$a_1 = a, \quad a_n = a_{n-1} + b \text{ for all } n \geq 2.$$

---

<sup>1</sup>Changed from “Johnny” in the Pete L. Clark version. Cf. [https://en.wikipedia.org/wiki/Jhonen\\_Vasquez](https://en.wikipedia.org/wiki/Jhonen_Vasquez)

Show that  $a_n = (a_{n+1} + a_{n-1})/2$  for all  $n \geq 2$ .

b) Suppose that the arithmetic sequence is given by the explicit formula.  $a_n = a + (n - 1)b$  for all  $n \in \mathbb{Z}^+$ . Again, show that  $a_n = (a_{n+1} + a_{n-1})/2$  for all  $n \geq 2$ .

**Comment on Proving Equalities:**

When proving two quantities are equal, as in the above exercise, it is usually best to begin with the expression on one side of the equality and manipulate that expression with algebra until you arrive at the other side of the original equality. Logically, it doesn't matter which side you start on, as long as you progress directly to the other side. The choice is yours, but often you will find that one direction seems to be easier to follow than the other. In fact, usually when you are figuring out how to prove an equality you will start off playing with both sides until you see what is going on, but in the end, the proof should not be written up

in that fashion. For example, in part a of the above exercise, you may wish to start with the expression  $(a_{n+1} + a_{n-1})/2$  and substitute for the term  $a_{n+1}$  the fact that  $a_{n+1} = a_n + b$ . Then use the fact that  $b + a_{n-1} = a_n$  to see that the original expression is equal to  $(a_n + a_n)/2$ . To write this up formally, one might say:

Replacing  $n$  with  $n + 1$  in the expression  $a_n = a_{n-1} + b$  we see that  $a_{n+1} = a_n + b$ . Thus we see that  $(a_{n+1} + a_{n-1})/2 = (a_n + b + a_{n-1})/2$ . Furthermore, since  $a_{n-1} + b = a_n$ , we can conclude that  $(a_n + b + a_{n-1})/2 = (a_n + a_n)/2 = a_n$ . Combining the first and last equalities yields

$$(a_{n+1} + a_{n-1})/2 = (a_n + b + a_{n-1})/2 = (a_n + a_n)/2 = a_n.$$

14. a) Let  $a$  and  $r$  be positive real numbers and define a geometric sequence by the recursive formula

$$a_1 = a, \quad a_n = ra_{n-1} \text{ for all } n \geq 2.$$

Show that  $a_n = \sqrt{a_{n+1}a_{n-1}}$  for all  $n \geq 2$ .

b) Let  $a$  and  $r$  be positive real numbers define (again) a geometric sequence by the explicit formula  $a_n = ar^{n-1}$  for all  $n \in \mathbb{Z}^+$ . Show that (again)  $a_n = \sqrt{a_{n+1}a_{n-1}}$  for all  $n \geq 2$ .

15. (Calculator needed) Find the first 20 terms of the sequences given by  $p_{n+1} = \alpha p_n - \beta p_n^2$  where  $\alpha, \beta$ , and  $p_0$  are given below. In each case write a sentence or two describing what you think the long term behavior of the population will be.
- a)  $\alpha = 2, \beta = 0.1, p_0 = 5$ .
  - b)  $\alpha = 2.8, \beta = 0.18, p_0 = 5$ .
  - c)  $\alpha = 3.2, \beta = 0.22, p_0 = 5$ .
  - d)  $\alpha = 3.8, \beta = 0.28, p_0 = 5$ .

If you have MAPLE available you can explore this exercise by changing the values of a and b in the following program:

```
[> restart;
[> a:=1.2; b:= 0.02;
[> f:=x->a*x - b*x^2;
[> p[0]:=5;
[> for j from 1 to 20 do p[j]:=f(p[j-1]); end do;
```

## 1.2 The sequence of positive integers

ADDED BY PLC For my own take on mathematical induction, see <http://www.math.uga.edu/~pete/3200induction.pdf>.

A very familiar and fundamental sequence is that of the *positive integers*,

$$a_1 = 1, a_2 = 2, \dots, a_n = n \text{ for all } n \in \mathbb{Z}^+.$$

The nature of the existence of the natural numbers is a fairly tricky issue in the foundations of mathematics which we won't delve into here, but we do want to discuss a defining property of the natural numbers that is extremely useful in the study of sequences and series:

### The Principle of Mathematical Induction

Let  $S$  be a subset of  $\mathbb{Z}^+$  satisfying

(PMI1)  $1 \in S$ , and

(PMI2)  $\forall n \in \mathbb{Z}^+$ , if  $n \in S$ , then  $n + 1 \in S$ .

Then  $S = \mathbb{Z}^+$ .



**Example 1.2.1.** a) Let  $S = \{1, 2, 3, 4, 5\}$ . Then  $S$  satisfies (PMI1)

but not (PMI2):  $5 \in S$  but  $6 \notin S$ . b) Let  $S = \{2, 3, 4, 5, \dots\}$ . Then  $S$  satisfies (PM2) but not (PM1). c) Let  $S = \{1, 3, 5, 7, \dots\}$ , the set of odd natural numbers. Then  $S$  satisfies property (PMI1) but not (PMI2).

The Principle of Mathematical Induction (which we shall henceforth abbreviate by PMI) is not only an important defining property of the natural numbers, it also gives a beautiful method of proving an infinite sequence of statements. In the present context, we will see that we can use PMI to verify explicit formulae for sequences which are given recursively.

**Example 1.2.2.** Consider the sequence given recursively by

$$a_1 = 1, \quad a_{n+1} = a_n + (2n + 1) \text{ for all } n \in \mathbb{Z}^+.$$

The  $n^{\text{th}}$  term,  $a_n$ , is the sum of the first  $n$  odd natural numbers. Writing out the first 5 terms we see

$$\begin{aligned} a_1 &= 1, \\ a_2 &= a_1 + 3 = 4, \\ a_3 &= a_2 + 5 = 9, \\ a_4 &= a_3 + 7 = 16, \\ a_5 &= a_4 + 9 = 25. \end{aligned}$$

Noticing a pattern here we might conjecture that, in general,  $a_n = n^2$ . Here is how we can use PMI to prove this conjecture:

Let  $S$  be the subset of natural numbers for which it is true that  $a_n = n^2$ , i.e.,

$$S = \{n \in \mathbb{Z}^+ \mid a_n = n^2\}.$$

We know that  $a_1 = 1$ , which is equal to  $1^2$ , so  $1 \in S$ . Thus  $S$  satisfies the first requirement in PMI. Now, let  $k$  be some arbitrary element of  $S$ . Then, by the description of  $S$  we know that  $k$  is some particular natural number such that  $a_k = k^2$ . According to the definition of the sequence,

$$a_{k+1} = a_k + (2k + 1),$$

so, since  $a_k = k^2$ , we can conclude that

$$a_{k+1} = k^2 + (2k + 1).$$

However, since  $k^2 + (2k + 1) = (k + 1)^2$ , we conclude that

$$a_{k+1} = (k + 1)^2,$$

i.e.,  $k + 1$  is an element of  $S$  as well. We have just shown that if  $k \in S$  then  $k + 1 \in S$ , i.e.,  $S$  satisfies the second requirement of PMI. Therefore we can conclude that  $S = \mathbb{Z}^+$ , i.e., the explicit formula  $a_n = n^2$  is true for every natural number  $n$ .

**Remark:** It is common to get confused at the last step and say  $a_{k+1} \in S$  instead of  $k + 1 \in S$ . Remember:  $S$  is the set of *subscripts* for which the statement is true.

#### Comments on the Subscript $k$ :

In general, letters like  $i, j, k, l, m$ , and  $n$  are used to denote arbitrary or unspecified natural numbers. In different contexts these letters can represent either an *arbitrary* natural number or a *fixed but unspecified* natural number. These two concepts may seem almost the same, but their difference is an important issue when studying PMI. To illustrate this here we are using two different letters to represent the different concepts. Where the subscript  $n$  is used we are talking about an arbitrary natural number, i.e., we are claiming that for this sequence,  $a_n = n^2$  for any natural number you choose to select. On the other hand, the subscript  $k$  is used above to denote a fixed but unspecified natural number, so we assume  $a_k = k^2$  for that particular  $k$  and use what we know about the sequence to prove that the general formula holds for the *next* natural number, i.e.,  $a_{k+1} = (k + 1)^2$ .

**Example 1.2.3.** Let  $\{a_n\}$  be the sequence defined recursively by  $a_{n+1} = a_n + (n + 1)$ , and  $a_1 = 1$ . Thus  $a_n$  is the sum of the first  $n$  natural numbers. Computing the first few terms, we see  $a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 10$ , and  $a_5 = 15$ . Let's now use PMI to prove that in general,  $a_n = \frac{n(n+1)}{2}$ . (You might want to check that this formula works in the above five cases.)

Let  $S$  be the set of natural numbers  $n$  so that it is true that  $a_n = \frac{n(n+1)}{2}$ , i.e.,

$$S = \{n \in \mathbb{Z}^+ \mid a_n = \frac{n(n+1)}{2}\}.$$

We wish to show that  $S = \mathbb{Z}^+$  by means of PMI. First notice that  $1 \in S$  since  $a_1 = 1$  and  $\frac{1(1+1)}{2} = 1$  also. That is, the formula is true when  $n = 1$ . Now assume that we know that some particular natural number  $k$  is an element of

*S*. Then, by the definition of *S*, we know that  $a_k = \frac{k(k+1)}{2}$ . From the recursive definition of the sequence, we know that  $a_{k+1} = a_k + (k + 1)$ . Substituting in  $a_k = \frac{k(k+1)}{2}$ , we get

$$\begin{aligned} a_{k+1} &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2k+2}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}. \end{aligned}$$

But this is the explicit formula we are trying to verify in the case that  $n = k + 1$ . Thus we have proven that whenever  $k$  is a member of *S* then so is  $k + 1$  a member of *S*. Therefore *S* satisfies the two requirements in PMI and we conclude that  $S = \mathbb{Z}^+$ .

#### Comments on Set Notation:

Although the issue of defining the notion of a *set* is a fairly tricky subject, in this text we will be concerned mostly with describing subsets of some given set (such as the natural numbers  $\mathbb{Z}^+$ , or the real numbers  $\mathbb{R}$ ) which we will accept as being given. Generally such subsets are described by a condition, or a collection of conditions. For example, the even natural numbers, let's call them *E*, are described as the subset of natural numbers which are divisible by 2. The notation used to describe this subset is as follows:

$$E = \{n \in \mathbb{Z}^+ \mid 2 \text{ divides } n\}.$$

There are four separate components to this notation. The brackets  $\{, \}$  contain the set, the first entry (in this example  $n \in \mathbb{Z}^+$ ) describes the set from which we are taking a subset, the vertical line  $\mid$  separates the first entry from the conditions (and is often read as “such that”), and the final entry gives the conditions that describe the subset (if there is more than one condition then they are simply all listed, with commas separating them). Of course sometimes a set can be described by two different sets of conditions, for instance an even natural number can also be described as twice another natural number. Hence we have

$$E = \{n \in \mathbb{Z}^+ \mid n = 2k \text{ for some } k \in \mathbb{Z}^+\}.$$

There are times when one has to be a bit tricky in re-labeling indices to apply PMI exactly as stated. Here is an example of this.

**Example 1.2.4.** *Here we use induction to prove<sup>2</sup> that  $n^2 + 5 < n^3$  for all  $n \geq 3$ . If we start off as usual by letting*

$$S = \{n \in \mathbb{Z}^+ \mid n^2 + 5 < n^3\}$$

*we will be in big trouble since it is easy enough to see that 1 is not an element of  $S$ . So it can't possibly be true that  $S = \mathbb{Z}^+$ . A formal way of getting around this problem is to shift the index in the inequality so that the statement to prove begins at 1 instead of 3. To do this, let  $n = m + 2$ , so that  $m = 1$  corresponds to  $n = 3$  and then substitute this into the expression we want to prove. That is, if we want to prove that  $n^2 + 5 < n^3$  for all  $n \geq 3$ , it is equivalent to prove that  $(m + 2)^2 + 5 < (m + 2)^3$  for all  $m \in \mathbb{Z}^+$ . Thus we can now proceed with PMI by setting*

$$\tilde{S} = \{m \in \mathbb{Z}^+ \mid (m + 2)^2 + 5 < (m + 2)^3\},$$

*and showing that properties 1.) and 2.) hold for  $\tilde{S}$ . Actually, this formality is generally a pain in the neck. Instead of being so pedantic, we usually proceed as follows with the original set  $S$ :*

*1'.) Show that  $3 \in S$ .*

*2'.) Show that if  $k \in S$  then so is  $k + 1 \in S$ .*

*Then we can conclude that  $S = \{n \in \mathbb{Z}^+ \mid n \geq 3\}$ . Of course, to do this we are actually applying some variation of PMI but we won't bother to state all such variations formally.*

*To end this example, let's see that 1'.) and 2'.) are actually true in this case. First, it is easy to see that  $3 \in S$  since  $3^2 + 5 = 14$  while  $3^3 = 27$ . Next, let us assume that some particular  $k$  is in  $S$ . Then we know that  $k^2 + 5 < k^3$ . But*

$$\begin{aligned} (k + 1)^2 + 5 &= (k^2 + 2k + 1) + 5 \\ &= (k^2 + 5) + (2k + 1) \\ &< k^3 + (2k + 1) \\ &< k^3 + (2k + 1) + (3k^2 + k) \\ &= k^3 + 3k^2 + 3k + 1 \\ &= (k + 1)^3, \end{aligned}$$

---

<sup>2</sup>See the end of section 1.3 for a review of the properties of inequalities.

where the first inequality follows from the assumption that  $k^2 + 5 < k^3$ , and the second inequality follows from the fact that  $3k^2 + k > 0$  for  $k \in \mathbb{N}$ .

For the next example we will need a more substantial variation on the Principle of Mathematical Induction called the Principle of Complete Mathematical Induction (which we will abbreviate PCMI).

**The Principle of Complete Mathematical Induction**

Let  $S$  be a subset of  $\mathbb{Z}^+$ , satisfying

(PCI1)  $1 \in S$ , and

(PCI2) if  $1, 2, 3, \dots, n$  are all elements of  $S$ , then  $n + 1$  is an element of  $S$ .

Then  $S = \mathbb{Z}^+$ .

**Example 1.2.5.** Let  $\{a_n\}$  be the sequence given by the two-term recursion formula  $a_{n+1} = 2a_n - a_{n-1} + 2$  for  $n > 1$  and  $a_1 = 3$  and  $a_2 = 6$ . Listing the first seven terms of this sequence, we get 3, 6, 11, 18, 27, 38, 51, ... Perhaps a pattern is becoming evident at this point. It seems that the  $n^{\text{th}}$  term in the sequence is given by the explicit formula  $a_n = n^2 + 2$ . Let's use PCMI to prove that this is the case. Let  $S$  be the subset of natural numbers  $n$  such that it is true that  $a_n = n^2 + 2$ , i.e.,

$$S = \{n \in \mathbb{Z}^+ \mid a_n = n^2 + 2\}.$$

We know that  $1 \in S$  since we are given  $a_1 = 3$  and it is easy to check that  $3 = 1^2 + 2$ . Now assume that we know  $1, 2, 3, \dots, k$  are all elements of  $S$ . Now, as long as  $k > 1$  we know that  $a_{k+1} = 2a_k - a_{k-1} + 2$ . Since we are assuming that  $k$  and  $k - 1$  are in  $S$ , we can write  $a_k = k^2 + 2$  and  $a_{k-1} = (k - 1)^2 + 2$ . Substituting these into the expression for  $a_{k+1}$  we get

$$\begin{aligned} a_{k+1} &= 2(k^2 + 2) - [(k - 1)^2 + 2] + 2 \\ &= 2k^2 + 4 - (k^2 - 2k + 3) + 2 \\ &= k^2 + 2k + 3 \\ &= (k + 1)^2 + 2. \end{aligned} \tag{1.1}$$

Thus  $k + 1 \in S$  as well. We aren't quite done yet! The last argument only works when  $k > 1$ . What about the case that  $k = 1$ ? If we know that  $1 \in S$  can we conclude that  $2 \in S$ ? Well, not directly, but we can check that  $2 \in S$  anyway. After all, we are given that  $a_2 = 6$  and it is easy to check that  $6 = 2^2 + 2$ . Notice that if we had given a different value for  $a_2$ , then the

entire sequence changes from there on out so the explicit formula would no longer be correct. However, if you aren't careful about the subtlety at  $k = 1$ , you might think that you could prove the formula by induction no matter what is the value of  $a_2$ !

**Remark:**

Although hypotheses 2.) of PMI and PCMI are quite different, it turns out that these two principles are logically equivalent: Assuming that the natural numbers satisfy PMI one can prove that they also must satisfy PCMI, and vice versa, if we assume PCMI we can prove PMI. We'll outline a proof of this equivalence in exercise 23.

**Example 1.2.6.** Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence, and define a sequence  $\{S_n\}_{n=1}^{\infty}$  recursively by

$$S_1 = a_1, \quad S_{n+1} = S_n + a_{n+1} \text{ for all } n \in \mathbb{Z}^+.$$

Then we have

$$S_n = a_1 + \dots + a_n \text{ for all } n \in \mathbb{Z}^+.$$

This is an easy induction proof: indeed  $S_1 = a_1$ . For  $n \in \mathbb{Z}^+$ , suppose that  $S_n = a_1 + \dots + a_n$ . Then

$$S_{n+1} = S_n + a_{n+1} = (a_1 + \dots + a_n) + a_{n+1} = a_1 + \dots + a_{n+1}.$$

Now we push our luck a bit: we can view the process of passing from a sequence  $\{a_n\}$  to the sequence  $\{S_n = a_1 + \dots + a_n\}$  as an operation on real sequences, the summation operator  $\Sigma$ . Now consider the (backward) difference operator  $\Delta$ : for a sequence  $\{a_n\}$ , we associate the sequence

$$\Delta\{a_n\} = (a_1, a_2 - a_1, a_3 - a_2, \dots, a_n - a_{n-1}, \dots).$$

The point is that  $\Sigma$  and  $\Delta$  are inverses of each other. Namely, for any sequence  $\{a_n\}$ , we have

$$\begin{aligned} \Sigma\Delta\{a_n\} &= \Sigma(a_1, a_2 - a_1, a_3 - a_2, \dots) \\ &= (a_1, a_1 + (a_2 - a_1), a_1 + (a_2 - a_1) + (a_3 - a_2), \dots, a_1 + (a_2 - a_1) + (a_3 - a_2) + \dots + (a_n - a_{n-1}), \dots) \\ &= (a_1, a_2, a_3, \dots) = \{a_n\}, \end{aligned}$$

and

$$\Delta\Sigma\{a_n\} = \Delta(a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots) =$$

$$(a_1, (a_1 + a_2) - a_1, (a_1 + a_2 + a_3) - (a_1 + a_2), \dots) = (a_1, a_2, a_3, \dots) = \{a_n\}.$$

Okay, but what's the point? It's this: often we use induction to show formulas of the type

$$a_1 + \dots + a_n = b_n \text{ for all } n \in \mathbb{Z}^+.$$

In our notation, we want to show

$$\Sigma\{a_n\} = \{b_n\}. \quad (1.2)$$

Because of what we just said, the summation identity (1.2) is equivalent to the differencing identity

$$\{a_n\} = \Delta\{b_n\}. \quad (1.3)$$

Indeed, applying  $\Delta$  to both sides of (1.2) gives

$$\{a_n\} = \Delta\Sigma\{a_n\} = \Delta\{b_n\},$$

and applying  $\Sigma$  to both sides of (1.3) gives

$$\Sigma\{a_n\} = \Sigma\Delta\{b_n\} = \{b_n\}.$$

In plainer terms, in order to prove that  $a_1 + \dots + a_n = b_n$  for all  $n \in \mathbb{Z}^+$ , it suffices to check that

$$b_1 = a_1 \text{ and } \forall n \geq 2, b_n - b_{n-1} = a_n.$$

This is the actual algebraic calculation that underlies an induction proof. For instance, to show

$$1 + 3 + \dots + (2n - 1) = n^2,$$

what we really need to check is that

$$1^2 = 1 \text{ and } \forall n \geq 2, n^2 - (n - 1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1.$$

Thus in the problem of finding a closed form expression for  $\Sigma\{a_n\}$ , the hard part is actually finding the sequence  $\{b_n\}$ . Just checking that a claimed  $\{b_n\}$  works is a completely mechanical "differencing" calculation. This is the discrete analogue of a situation familiar from calculus: finding an antiderivative of a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is usually hard, but checking that a claimed  $F(x)$  works is a completely mechanical differentiation calculation.

**EXERCISES 1.2**

1. Let  $\{a_n\}$  be the sequence given recursively by

$$a_1 = 1, a_{n+1} = a_n + (n+1)^2 \text{ for all } n \geq 1.$$

(So by Example 1.2.6, we have  $a_n = 1^2 + \dots + n^2$ .) Use PMI to show

$$\forall n \in \mathbb{Z}^+, a_n = \frac{n(n+1)(2n+1)}{6}.$$

2. a) Let  $\{a_n\}$  be the sequence given recursively by

$$a_1 = 1, a_{n+1} = a_n + (n+1)^3 \text{ for all } n \geq 1.$$

(So by Example 1.2.6, we have  $a_n = 1^3 + \dots + n^3$ .) Use PMI to show

$$\forall n \in \mathbb{Z}^+, a_n = \left(\frac{n(n+1)}{2}\right)^2.$$

b) Combine the result from part a) with the result in Example 1.2.3 to prove the remarkable fact that

$$\forall n \in \mathbb{Z}^+, (1 + 2 + 3 + \dots + n)^2 = 1^3 + 2^3 + 3^3 + \dots + n^3.$$

3. Let  $\{a_n\}$  be the sequence given recursively by

$$a_1 = 2, a_{n+1} = a_n + 2n + 2 \text{ for all } n \geq 1.$$

Find a formula for  $a_n$  and use induction to prove it.

4. Let  $\{a_n\}$  be the sequence given recursively by

$$a_1 = 5, a_{n+1} = a_n + 2n + 3 \text{ for all } n \geq 1.$$

Find a formula for  $a_n$  and use induction to prove it.

5. Let  $\{a_n\}$  be the sequence given recursively by

$$a_1 = 1, a_{n+1} = a_n + 2^n \text{ for all } n \geq 1.$$

Use induction to show that  $a_n = 2^n - 1$  for all  $n \in \mathbb{Z}^+$ .

In mathematics the word “show” is used synonymously with “prove”.



6. Let  $\{a_n\}$  be the sequence given recursively by

$$a_1 = 1, \quad a_{n+1} = a_n + 3^n \text{ for all } n \geq 1.$$

Use induction to show that  $a_n = \frac{3^n - 1}{2}$  for all  $n \in \mathbb{Z}^+$ .

7. a) Let  $r \in \mathbb{R} \setminus \{1\}$ , and let  $\{a_n\}$  be the sequence given recursively by

$$a_1 = 1, \quad a_{n+1} = a_n + r^n \text{ for all } n \geq 1.$$

Use induction to show that  $a_n = \frac{r^n - 1}{r - 1}$  for all  $n \in \mathbb{Z}^+$ .

b) What happens when  $r = 1$ ?

8. Let  $c, r, a_0 \in \mathbb{R}$ . Define the sequence  $\{a_n\}$  recursively by

$$a_{n+1} = c + ra_n \text{ for all } n \in \mathbb{N}.$$

Use induction to show that

$$a_n = \left( \frac{r^n - 1}{r - 1} \right) c + r^n a_0 \text{ for all } n \in \mathbb{N}.$$

9. Let  $r$  be a real number with  $0 < r < 1$ . Prove by induction that  $0 < r^n < 1$  for every  $n \in \mathbb{Z}^+$ .

(You may use: if  $0 < a < 1$  and  $0 < b < 1$ , then  $0 < ab < 1$ . See §1.3.)

10. Let  $\{a_n\}$  be the sequence given recursively by

$$a_1 = \frac{1}{2}, \quad a_{n+1} = a_n + \frac{1}{(n+1)(n+2)} \text{ for all } n \geq 1.$$

Show by induction that

$$a_n = \frac{n}{n+1} \text{ for all } n \geq 1.$$

11. Prove by induction that  $2n + 1 \leq 3n^2$  for all  $n \in \mathbb{Z}^+$ .

12. Prove by induction that  $2n^2 - 1 \leq n^3$  for all  $n \in \mathbb{Z}^+$ .

(Hint: You may need the result in problem 11.)

13. a) Prove by induction that  $2^n < n!$  for all  $n \geq 4$ .  
 b) Use induction to prove that  $\frac{2^n}{n!} < \frac{1}{n}$  for  $n \geq 5$ .
14. Let  $\{a_n\}$  be the sequence defined recursively by

$$a_1 = 2, \quad a_{n+1} = \frac{2a_n + 1}{a_n + 2} \text{ for all } n \geq 1.$$

Prove by induction that  $a_n \geq 1$  for all  $n \in \mathbb{Z}^+$ . (*Hint:* Look at  $a_{n+1} - 1$ .)

15. Let  $\{a_n\}$  be the sequence defined recursively by  $a_1 = 3$ , and  $a_{n+1} = \frac{3a_n + 4}{a_n + 3}$ , for  $n \geq 1$ . Prove by induction that  $a_n \geq 2$  for all  $n \in \mathbb{Z}^+$ .
16. Prove by induction: for all  $n \in \mathbb{N}$ , a set with  $n$  elements has  $2^n$  subsets.
17. Let  $\{a_n\}$  be the sequence given recursively by  $a_1 = 1, a_2 = 8$ , and  $a_{n+1} = 2a_n - a_{n-1} + 6n$  for  $n \geq 2$ . Using complete induction show that  $a_n = n^3$  for all  $n \geq 1$ .
18. Let  $\{a_n\}$  be the sequence given recursively by  $a_1 = 1, a_2 = 4, a_3 = 9$ , and  $a_{n+1} = 3a_n - 3a_{n-1} + a_{n-2}$  for all  $n \geq 3$ . Using complete induction show that  $a_n = n^2$  for all  $n \geq 1$ .
19. What is wrong with the following argument?

Old MacDonald claims that all cows have the same color. First of all, if you have just one cow, it certainly has the same color as itself. Now, using PMI, assume that all the cows in any collection of  $k$  cows have the same color and look at a collection of  $k + 1$  cows. Removing one cow from that collection leaves a collection of  $k$  cows, which must therefore all have the same color. Putting back the removed cow, and removing a different cow leaves another collection of  $k$  cows which all have the same color. Certainly then, the original collection of  $k + 1$  cows must all have the same color. By PMI, all of the cows in any finite collection of cows have the same color.

20. Let  $\{a_n\}$  be the Fibonacci sequence given recursively by  $a_1 = 1, a_2 = 1$ , and  $a_{n+1} = a_n + a_{n-1}$  for all  $n \geq 2$ . (See Example 1.1.5.) Using complete

induction show that

$$a_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) \text{ for all } n \in \mathbb{Z}^+.$$

21. (On arithmetic sequences.)

a) Suppose that a sequence is given recursively by  $a_1 = a$  and  $a_n = a_{n-1} + b$  for  $n > 1$ , where  $a$  and  $b$  are fixed real numbers. Prove that  $a_n = a + (n - 1)b$  for all  $n \geq 1$ .

b) Suppose that the sequence  $\{a_n\}$  satisfies  $a_n = (a_{n+1} + a_{n-1})/2$  for all  $n \geq 2$ . Prove that there is a  $b \in \mathbb{R}$  so that  $a_n = a_{n-1} + b$  for all  $n \geq 2$ . (Hint: Use the expression  $a_n = (a_{n+1} + a_{n-1})/2$  to show that  $a_{n+1} - a_n = a_n - a_{n-1}$  for  $n > 1$  then use induction to show that  $a_n - a_{n-1} = a_2 - a_1$  for all  $n \geq 2$ .)

22. (On geometric sequences.)

a) Suppose that a sequence is given recursively by  $a_1 = a$  and  $a_n = ra_{n-1}$  for all  $n \geq 2$ . Prove that  $a_n = ar^{n-1}$  for all  $n \in \mathbb{Z}^+$ .

b) Suppose that the sequence  $\{a_n\}$  satisfies  $a_n = \sqrt{a_{n+1}a_{n-1}}$  for all  $n \geq 2$ . Prove that there is an  $r \in \mathbb{R}$  so that  $a_n = ra_{n-1}$  for all  $n \geq 2$ . (Hint: Start with the case that  $a_n \neq 0$  for all  $n$ , and use the expression  $a_n = \sqrt{a_{n+1}a_{n-1}}$  to show that  $a_{n+1}/a_n = a_n/a_{n-1}$  for all  $n \geq 2$  then use induction to show that  $a_n/a_{n-1} = a_2/a_1$  for all  $n \geq 2$ . Don't forget to separately deal with the case that some  $a_n$  is zero.)

23. In this exercise we outline the proof of the equivalence of PMI and PCMI.

a) First we assume that the natural numbers satisfy PCMI and we take a subset  $T \subset \mathbb{Z}^+$  which satisfies hypotheses 1.) and 2.) of PMI, i.e., we know:

1.)  $1 \in T$

and that

2.) if  $n \in T$  then  $n + 1 \in T$ .

Show directly that  $T$  also satisfies the hypotheses of PCMI, hence by our assumption we conclude that  $T = \mathbb{Z}^+$ .

b) This direction is a little trickier. Assume that the natural numbers satisfy PMI and take  $T \subset \mathbb{Z}^+$  which satisfies hypotheses 1.) and 2.) of PCMI. Now define  $S$  to be the subset of  $\mathbb{Z}^+$  given by  $S = \{n \in \mathbb{Z}^+ \mid 1, 2, \dots, n \in T\}$ . Prove by PMI that  $S = \mathbb{Z}^+$ .

### 1.3 Sequences as functions

Let  $X$  be a set. Earlier we discussed the notion of an infinite sequence  $\{a_n\}$  informally as an infinite ordered list  $a_1, a_2, \dots$  of elements of  $X$ . We can now give a formal definition: a sequence is precisely a function

$$a_{(\cdot)} : \mathbb{Z}^+ \rightarrow X.$$

Namely, we map the positive integer  $n$  to the element  $a_n$  of  $X$ .

**Remark:** If a sequence is given by an explicit formula, then that formula gives an explicit formula for the corresponding function. But just as sequences may be defined in non-explicit ways, so too can functions. For instance, the consecutive digits in the decimal expansion of  $\pi$  define a sequence and therefore a corresponding function,  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ , yet we know of no simple explicit formula for  $f(n)$ .

Thinking of sequences in terms of their corresponding functions gives us an important method of visualizing sequences, namely by their *graphs*. Since the natural numbers  $\mathbb{Z}^+$  are a subset of the real numbers  $\mathbb{R}$  we can graph a function  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$  in the plane  $\mathbb{R}^2$  but the graph will consist of only isolated points whose  $x$ -coordinates are natural numbers.

The interpretation of sequences as functions leads to some of the following nomenclature.

**Definition 1.3.1.** *The real sequence  $\{a_n\}$  is said to be:*

- i.) *increasing if  $a_{n+1} \geq a_n$  for all  $n \in \mathbb{Z}^+$*
- ii.) *strictly increasing if  $a_{n+1} > a_n$  for all  $n \in \mathbb{Z}^+$ ,*
- iii.) *decreasing if  $a_{n+1} \leq a_n$  for all  $n \in \mathbb{Z}^+$*
- iv.) *strictly decreasing if  $a_{n+1} < a_n$  for all  $n \in \mathbb{Z}^+$ .*

**Proposition 1.3.2.** *Let  $\{a_n\}$  be a real sequence.*

a) *If there is  $C \in \mathbb{R}$  such that  $a_n = C$  for all  $n \in \mathbb{Z}^+$ , then  $\{a_n\}$  is both increasing and decreasing.*

b) *If a real sequence  $\{a_n\}$  is both increasing and decreasing, then we have  $a_n = a_1$  for all  $n \in \mathbb{Z}^+$ .*

*Proof.* a) Let  $n \in \mathbb{Z}^+$ . Then  $a_{n+1} = C = a_n$ . Thus  $a_n \leq a_{n+1}$  and also  $a_n \geq a_{n+1}$ .

b) We proceed by contradiction using the Well-Ordering Principle: if it is not true that  $a_n = a_1$  for all  $n \in \mathbb{Z}^+$ , then the set  $\{n \geq 2 \mid a_n \neq a_1\}$  has a least element, say  $N$ . Thus  $a_1 = a_2 = \dots = a_{N-1}$  and  $a_{N-1} \neq a_N$ . So either  $a_{N-1} < a_N$  – in which case the sequence  $\{a_n\}$  fails to be decreasing – or  $a_{N-1} > a_N$  – in which case the sequence  $\{a_n\}$  fails to be increasing.  $\square$

**Remark:**

1.) It is important to notice that in these definitions the condition must hold for *all*  $n \in \mathbb{Z}^+$ . So if a sequence sometimes strictly increases and sometimes strictly decreases, then it is neither increasing nor decreasing.

2.) For the rest of this text we will be making extensive use of inequalities and absolute values. For this reason we have included the basic properties of these at the end of this section. Now might be a good time to review those properties.

**Example 1.3.3.** a) *The sequence given by  $a_n = n^2$  is strictly increasing since  $(n+1)^2 = n^2 + 2n + 1 > n^2$  for all  $n > 0$ .*

b) *The sequence given by  $a_n = 0$  for all  $n$  is both increasing and decreasing but it is not strictly increasing nor strictly decreasing.*

c) *A strictly increasing sequence is, of course, increasing, but it is not decreasing.*

d) *The sequence given by  $a_n = 3 + (-1)^n/n$  is neither increasing nor decreasing.*

**Example 1.3.4.** *Consider the sequence given recursively by*

$$a_{n+1} = \frac{4a_n + 3}{a_n + 2}$$

*with  $a_1 = 4$ . We claim that this sequence is decreasing. To check this we will*

show that  $a_{n+1} - a_n$  is negative. Now

$$\begin{aligned} a_{n+1} - a_n &= \frac{4a_n + 3}{a_n + 2} - a_n \\ &= \frac{4a_n + 3}{a_n + 2} - \frac{a_n^2 + 2a_n}{a_n + 2} \\ &= \frac{2a_n + 3 - a_n^2}{a_n + 2} \\ &= \frac{-(a_n - 3)(a_n + 1)}{a_n + 2} \end{aligned}$$

so  $a_{n+1} - a_n$  is negative as long as  $a_n$  is greater than 3. But an easy induction argument shows that  $a_n > 3$  for all  $n$  since

$$\begin{aligned} a_{k+1} - 3 &= \frac{4a_k + 3}{a_k + 2} - 3 \\ &= \frac{4a_k + 3}{a_k + 2} - \frac{3a_k + 6}{a_k + 2} \\ &= \frac{a_k - 3}{a_k + 2} \end{aligned}$$

which is positive as long as  $a_k > 3$ .

### Comment on Inequalities

In the above proof we have twice used the fact that when trying to prove that  $a > b$  it is often easier to prove that  $a - b > 0$ .

**Definition 1.3.5.** A real sequence  $\{a_n\}$  is monotone if it is either increasing or decreasing.

The following equivalent condition for an increasing sequence is often useful when studying properties of such sequences. Of course there are similar equivalent conditions for decreasing sequences and strictly increasing or decreasing sequences.

**Proposition 1.3.6.** A real sequence,  $\{a_n\}$ , is increasing if and only if  $a_m \leq a_n$  for all natural numbers  $m$  and  $n$  with  $m \leq n$ .

**Proof:** There are two implications to prove here. First we prove that if the sequence  $\{a_n\}$  satisfies the condition that  $a_m \geq a_n$  whenever  $m \geq n$  then

that sequence must be increasing. But this is easy, since if  $a_m \leq a_n$  whenever  $m \leq n$ , then in particular we know that  $a_{n+1} \leq a_n$  for all  $n \in \mathbb{Z}^+$ , i.e., the sequence is increasing.

Next we must prove that if the sequence  $\{a_n\}$  is known to be increasing then it must satisfy the condition  $a_m \geq a_n$  whenever  $m \geq n$ . Certainly if  $m = n$  then  $a_m = a_n$ , so we may assume  $m < n$ . One idea which works is to view the passage from  $m$  to  $n$  as a “step of length  $n - m$ ” and break it down into “ $n - m$  steps of length 1”. Doing so, we get

$$a_n - a_m = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \dots + (a_{m+1} - a_m).$$

Since  $\{a_n\}$  is increasing, each of the  $n - m$  terms on the right hand side is non-negative, so their sum is non-negative:  $a_n - a_m \geq 0$ , and thus  $a_m \leq a_n$ .  $\square$

The next property of sequences we wish to discuss is *boundedness*. In fact boundedness is an attribute of a subset, not a sequence, so we begin with a slight digression about this. For a function  $f : X \rightarrow Y$ , we define the *image*

$$f(X) = \{f(x) \mid x \in X\}.$$

This is precisely the subset of  $Y$  consisting of points which are actually mapped to by some element of  $X$ . (Alternate terminology: the *range*.)

Since a real sequence  $\{a_n\}$  is really a function  $a_{(\ )} : \mathbb{Z}^+ \rightarrow \mathbb{R}$ , it too has an *image*, namely  $\{a_n \mid n \in \mathbb{Z}^+\}$ . We must admit that the terminology here is highly nonstandardized: for what we call the image of a sequence, some call the *range*, the *trace*, or the *term set*. (And many people do not call it anything.) It is important to understand that some information is lost in the passage from a sequence to its image. In the case where the function  $n \mapsto a_n$  is injective – i.e., we have  $a_m \neq a_n$  for all  $m \neq n$  – then what is missing is precisely the ordering of the terms: e.g. the sequences

$$1, 2, 3, 4, 5, 6, 7, \dots$$

and

$$4, 3, 2, 1, 5, 6, 7, \dots$$

are not the same, but each has  $\mathbb{Z}^+$  as its image. For sequences with repeated terms, the difference between the sequence and the image can be more drastic: e.g. the image of the sequence

$$1, -1, 1, -1, 1, \dots$$

is the finite set  $\{1, -1\}$ .

Some properties of a sequence depend only on its image. Here is one:

A subset  $X \subset \mathbb{R}$  is *bounded above* if there is  $M \in \mathbb{R}$  such that  $x \leq M$  for all  $x \in X$ . We say that  $M$  is an *upper bound* for  $X$ . Similarly,  $X \subset \mathbb{R}$  is *bounded below* if there is an  $m \in \mathbb{R}$  such that  $m \leq x$  for all  $x \in X$ . We say that  $m$  is a lower bound for  $X$ . We say that  $X \subset \mathbb{R}$  is *bounded* if it is bounded above and below. Equivalently, there are real numbers  $m \leq M$  such that  $X$  is contained in the interval  $[m, M]$ .

**Proposition 1.3.7.** *Let  $X \subset \mathbb{R}$  be a finite subset. Then  $X$  is bounded.*

*Proof.* We begin by addressing the question “What is a finite set?” We endorse the Kroneckerian approach that the integers  $\mathbb{Z}$  are given to us, and it is our job to define everything else in terms of them. So we define

$$[0] = \emptyset$$

and for  $n \in \mathbb{Z}^+$ ,

$$[n] = \{1, \dots, n\}.$$

Then a finite set is a set  $X$  which can be placed in bijection with  $[n]$  for some  $n \in \mathbb{N}$ . When  $n = 0$ , this is a fancy way of saying  $X = \emptyset$ . Otherwise, we mean that there is a function  $\iota : [n] \rightarrow X$  such that every  $x \in X$  is of the form  $\iota(i)$  for exactly one  $i \in [n]$ . In perhaps plainer terms, we mean that we can list all the elements of  $X$  as  $x_1, \dots, x_n$ , without repetitions.

In light of all this, we see that what we want to show can be restated as: for all  $n \in \mathbb{N}$ , a subset  $X \subset \mathbb{R}$  is bounded. The empty set is bounded above and below by, say, 0, so no worries there. If  $X = \{x_1, \dots, x_n\}$ , then the evident bound is  $M = \max_{1 \leq i \leq n} |x_i|$ .

However, if you reflect on the above for a while, you may feel that we have just kicked the can a little down the road: how do we know that any finite, nonempty set of real numbers has a maximum? We can prove this by induction on  $n$ , starting at  $n = 1$ . The base case is clear: the largest element of  $\{x\}$  is  $x$ . Now we do the induction step: let  $n \in \mathbb{Z}^+$ , suppose that every  $n$  element subset of  $\mathbb{R}$  has a maximum, and let  $X = \{x_1, \dots, x_n, x_{n+1}\}$  be an  $n + 1$ -element subset. By induction, the set  $\{x_1, \dots, x_n\}$  has a maximum, call it  $M_n$ . This means precisely that  $M_n \geq x_i$  for all  $i$  and that  $M_n = x_i$  for some  $i$ . If  $M_n \geq x_{n+1}$  then  $M_n$  is the maximum for  $X$ . If  $M_n < x_{n+1}$ , then  $x_i < x_{n+1}$  for all  $1 \leq i \leq n$  and thus the maximum for  $X$  is  $x_{n+1}$ .  $\square$



**Comments on maxima:** An element  $M$  of a subset  $S \subset \mathbb{R}$  is called a *maximal element* of  $S$  (or a *maximum* of  $S$ ) if  $s \leq M$  for all  $s \in S$ . Thus, in the above proof, with  $S = \{|a_1|, |a_2|, \dots, |a_{N_1}|, |L| + 1\}$ , we can conclude that each of the elements,  $|a_1|, |a_2|, \dots, |a_{N_1}|, |L| + 1$  is less than or equal to the maximum,  $M$ . The subtlety is that *not every set has a maximal element*. Indeed, any set that is not bounded above (for example  $\mathbb{Z}^+$ ) will have no maximal element. But even sets which are bounded above may not have a maximal element. For example, the set  $\{1 - 1/n \mid n \in \mathbb{Z}^+\}$  is bounded above by 1, but has no maximal element.

Let  $X$  be any set, and let  $f : X \rightarrow \mathbb{R}$  be a function. We say that  $f$  is bounded above / bounded below / bounded if its image  $f(X)$  is bounded above / bounded below / bounded. Explicitly,  $f$  is bounded if there are real numbers  $m \leq M$  such that  $m \leq f(x) \leq M$  for all  $x \in X$ . Since a real sequence is a function  $a_{(\cdot)} : \mathbb{Z}^+ \rightarrow \mathbb{R}$ , this definition applies in particular to sequences. Since it is so important, we again spell it out: a real sequence  $\{a_n\}$  is bounded above if there exists  $M \in \mathbb{R}$  such that  $a_n \leq M$  for all  $n \in \mathbb{Z}^+$ ; a real sequence is bounded below if there exists  $m \in \mathbb{R}$  such that  $m \leq a_n$  for all  $n \in \mathbb{Z}^+$ ; and a real sequence is bounded if it is bounded above and below: equivalently, there are real numbers  $m \leq M$  such that  $a_n \in [m, M]$  for all  $n \in \mathbb{Z}^+$ .

**Example 1.3.8.** *The set  $\mathbb{Z}^+$  is not bounded above. This is not at all surprising: it means precisely that there is no  $M \in \mathbb{R}$  such that  $n \leq M$  for all  $n \in \mathbb{Z}^+$ . Though our understanding of  $\mathbb{R}$  may be a bit vague and primitive, it is not so vague that this is not clear: indeed, if we represent a real number  $M$  by a decimal expansion*

$$M = a_0.a_1a_2\dots a_n\dots$$

*then  $M$  is no larger than the next integer  $a_0 + 1$  and thus  $M < a_0 + 2$ .*

*Here is something that you may find much more surprising: the assertion that  $\mathbb{Z}^+$  is not bounded above in  $\mathbb{R}$  is a very famous and important one, called the Archimedean<sup>3</sup> Property of the real numbers. The Archimedean Property is certainly useful: we will use it starting in §1.4. But why are we making such a big deal about it? We have to become a bit more interested in what the real numbers actually are in order to appreciate this, so: more later!*

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<sup>3</sup>After Archimedes of Syracuse, c. 287BC - c. 212BC, the greatest mathematician of the ancient world

**Example 1.3.9.** a) The sequence given by  $a_n = n^2$ ,  $n \in \mathbb{Z}^+$ , is bounded below by 0 but it is not bounded above. Thus  $\{a_n\}$  is not bounded.

b) The sequence given by  $a_n = -n$ ,  $n \in \mathbb{Z}^+$ , is bounded above by 0 but it is not bounded below. Thus  $\{a_n\}$  is not bounded.

c) The sequence given by  $a_n = \frac{1}{n}$ ,  $n \in \mathbb{Z}^+$ , is bounded below by 0 and bounded above by 1. Thus  $\{a_n\}$  is bounded.

d) The sequence given by  $a_n = \cos(n)$ ,  $n \in \mathbb{Z}^+$ , is bounded below by  $-1$  and bounded above by 1. Thus  $\{a_n\}$  is bounded.

**Proposition 1.3.10.** If the sequence  $\{a_n\}$  is bounded, then there is a real number  $B > 0$  such that  $-B \leq a_n \leq B$  for all  $n \in \mathbb{Z}^+$ .

*Proof.* Let  $M$  be an upper bound and  $m$  a lower bound for the bounded sequence  $\{a_n\}$ . Let  $B$  be the maximum of the two numbers  $|m|$  and  $|M|$ . Then since  $-|m| \leq m$  and  $M \leq |M|$ , for all  $n \in \mathbb{Z}^+$  we have

$$-B \leq -|m| \leq m \leq a_n \leq M \leq |M| \leq B. \quad \square$$

Given two real valued functions we can add or multiply them by adding or multiplying their values. In particular, if we have  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$  and  $g : \mathbb{Z}^+ \rightarrow \mathbb{R}$ , we define two new functions,  $(f + g)$  and  $fg$  by

$$(f + g)(n) = f(n) + g(n)$$

and

$$fg(n) = f(n)g(n).$$

If we use  $f$  and  $g$  to define sequences  $\{a_n\}$  and  $\{b_n\}$ , i.e.,  $a_n = f(n)$  and  $b_n = g(n)$ , then the sequence given by  $(f + g)$  has terms given by  $a_n + b_n$  and the sequence given by  $fg$  has terms given by  $a_n b_n$ . These two new sequences are called the *sum* and *product* of the original sequences.

In general, one can also compose two functions as long as the set of values of the first function is contained in the domain of the second function. Namely, if  $g : A \rightarrow B$  and  $f : B \rightarrow C$ , then we can define  $f \circ g : A \rightarrow C$  by  $f \circ g(a) = f(g(a))$  for each  $a \in A$ . Notice that  $g(a) \in B$  so  $f(g(a))$  makes sense. To apply the notion of composition to sequences we need to remark that the corresponding functions always have the domain given by  $\mathbb{Z}^+$ . Thus, if we wish to compose these functions then the first one must take its values in  $\mathbb{Z}^+$ , i.e., the first sequence must be a sequence of natural numbers. In the next few sections we will be particularly interested in the case that the first sequence is a strictly increasing sequence of natural numbers, in this case we call the composition a *subsequence* of the second sequence.

**Definition 1.3.11.** Let  $\{a_n\}$  be the sequence defined by the function  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$  and let  $g : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  be a strictly increasing function (with values in  $\mathbb{Z}^+$ ). Then the sequence  $\{b_n\}$  defined by the function  $f \circ g : \mathbb{Z}^+ \rightarrow \mathbb{R}$ , i.e.,  $b_n = f(g(n))$ , is called a subsequence of the sequence  $\{a_n\}$ .

**Example 1.3.12.** The sequence given by  $b_n = (2n + 1)^2$  is a subsequence of the sequence given by  $a_n = n^2$ . If we let  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$  be given by  $f(n) = n^2$  and  $g : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  be given by  $g(n) = 2n + 1$ , then the function corresponding to  $b_n$  is given by  $f \circ g$  since  $f \circ g(n) = (2n + 1)^2$ .

A good way to think about subsequences is the following. First list the elements of your original sequence in order:

$$a_1, a_2, a_3, a_4, a_5, \dots$$

The first element,  $b_1$ , of a subsequence can be any one of the above list, but the next element,  $b_2$ , must lie to the right of  $b_1$  in this list. Similarly,  $b_3$  must lie to the right of  $b_2$  in the list,  $b_4$  must lie to the right of  $b_3$ , and so on. Of course, each of the  $b_j$ 's must be taken from the original list of  $a_n$ 's. Thus we can line up the  $b_j$ 's under the corresponding  $a_n$ 's:

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \dots$$

$$b_1 \quad b_2 \quad b_3$$

A useful result which helps us review some of the above definitions is the following.

**Proposition 1.3.13.** Any subsequence of an increasing real sequence is increasing.

*Proof.* Let  $\{a_n\}_{n=1}^{\infty}$  be an increasing real sequence, let  $n_k : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  be an increasing function, and consider the subsequence

$$\{a_{n_k}\} : k \mapsto a_{n_k}.$$

Let  $k_1 \leq k_2$ . Since  $\{n_k\}$  is increasing we have  $n_{k_1} \leq n_{k_2}$ . Since  $\{a_n\}$  is increasing, we have  $a_{n_{k_1}} \leq a_{n_{k_2}}$ .  $\square$

**Remark 1.3.14.** There is a more general moral here, for those who are so inclined. If  $(X, \leq)$  and  $(Y, \leq)$  are sets equipped with a total ordering  $\leq$ , we can define a function  $f : X \rightarrow Y$  to be increasing if applying  $f$  preserves

the order relation: that is, for all  $x_1 \leq x_2$  in  $X$ , we have  $f(x_1) \leq f(x_2)$  in  $Y$ . Then it is absolutely immediate that if we have three totally ordered sets  $(X, \leq)$ ,  $(Y, \leq)$ ,  $(Z, \leq)$  and functions  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , then if  $f$  and  $g$  are increasing, so is the composite function  $g \circ f$ . Indeed, if  $x_1 \leq x_2 \in X$ , then since  $f$  is increasing, we have  $f(x_1) \leq f(x_2)$ , and since  $g$  is increasing, we have  $(g \circ f)(x_1) = g(f(x_1)) \leq g(f(x_2)) = (g \circ f)(x_2)$ . Proposition 1.3.13 is a special case of this.

**Lemma 1.3.15.** (*Rising Sun [NP88]*) *Each infinite sequence has a monotone subsequence.*

*Proof.* Let us say that  $m \in \mathbb{Z}^+$  is a *peak* of the sequence  $\{a_n\}$  if for all  $n > m$ , we have  $a_n < a_m$ . Suppose first that there are infinitely many peaks. Then the sequence of peaks forms a strictly decreasing subsequence, hence we have found a monotone subsequence. So suppose on the contrary that there are only finitely many peaks, and let  $N \in \mathbb{N}$  be such that there are no peaks  $n \geq N$ . Since  $n_1 = N$  is not a peak, there exists  $n_2 > n_1$  with  $a_{n_2} \geq a_{n_1}$ . Similarly, since  $n_2$  is not a peak, there exists  $n_3 > n_2$  with  $a_{n_3} \geq a_{n_2}$ . Continuing in this way we construct an infinite (not necessarily strictly) increasing subsequence  $a_{n_1}, a_{n_2}, \dots, a_{n_k}, \dots$ . Done!  $\square$

Before ending this section, we mention that there is another way of visualizing a sequence. Essentially this amounts to depicting the set of values of the corresponding function. However, if we merely show the set of values, then we have suppressed a great deal of the information in the sequence, namely, the ordering of the points. As a compromise, to retain this information one often indicates the ordering of the points by labeling a few of them. Of course, just as with a graph, any such picture will only show finitely many of the points from the sequence, but still it may give some understanding of the long term behavior of the sequence.

**Example 1.3.16.** *Consider the sequence given by  $a_n = 1/n$  for all  $n \in \mathbb{Z}^+$ . A graph of the first five terms of this sequence was given in figure 1.3.1. The first five values of this sequence are given by the projection of this graph to the  $y$ -axis as in figure 1.3.2.*

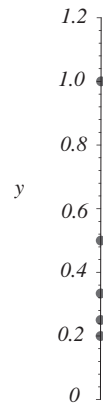


figure 1.3.2

Actually, it is customary to rotate this picture to the horizontal as in figure 1.3.3.

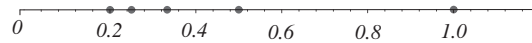


figure 1.3.3

To give an even better indication of the full sequence, we label these first five points and then indicate the location of a few more points as in figure 1.3.4.

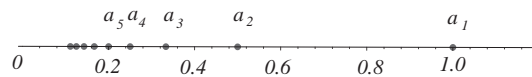


figure 1.3.4

## Review of the Properties of Inequalities and Absolute Values

In the next section we will be working extensively with inequalities and absolute values so here recall some of the basic properties of these.

### Properties of Inequalities

Let  $x, y$ , and  $z$  be real numbers. Then the following are true:

- i.) if  $x < y$  and  $y < z$ , then  $x < z$
- ii.) if  $x < y$ , then  $x + z < y + z$
- iii.) if  $x < y$  and  $z > 0$ , then  $xz < yz$
- iv.) if  $x < y$  and  $z < 0$ , then  $xz > yz$
- v.) if  $0 < x < y$  then  $0 < \frac{1}{y} < \frac{1}{x}$ .

It is also useful to recall that if  $x$  is a real number then exactly one of the following is true: either  $x > 0$ , or  $x < 0$ , or  $x = 0$ . This is called the property of *trichotomy*.

If  $x$  is a real number, we define the *absolute value* of  $x$ ,  $|x|$  by

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

### Properties of Absolute Values

Let  $x, y \in \mathbb{R}$ , then

- a)  $|xy| = |x||y|$
- b)  $-|x| \leq x \leq |x|$
- c)  $|x| \leq y$  if and only if  $-y \leq x \leq y$
- d)  $|x + y| \leq |x| + |y|$ .

Property d) is called the *triangle inequality*.

**EXERCISES 1.3**

1. From the properties of inequalities prove that<sup>4</sup> if  $a > b > 0$ , then  $a^2 > b^2$ .
2. From the properties of inequalities prove that if  $a > 0, b > 0$  and  $a^2 > b^2$ , then  $a > b$ .
3. For the following statements, either prove using the properties of inequalities, or provide a counterexample:
  - a) If  $a < b$  and  $c < d$ , then  $ac < bd$ .
  - b) If  $a < b$  and  $0 < c < d$ , then  $ac < bd$ .
  - c) If  $0 < a < b$  and  $0 < c < d$ , then  $ac < bd$ .
  - d) If  $0 < a < b$  and  $c < d < 0$ , then  $ac < bd$ .
4. For the following statements, either prove using the properties of inequalities, or provide a counterexample:
  - a) If  $a < b$  and  $c < d$ , then  $a/c < b/d$ .
  - b) If  $a < b$  and  $c > d$ , then  $a/c < b/d$ .
  - c) If  $a < b$  and  $c > d > 0$ , then  $a/c < b/d$ .
  - d) If  $0 < a < b$  and  $c > d > 0$ , then  $a/c < b/d$ .
5. From the properties of inequalities prove that if  $a < b < 0$ , then  $1/b < 1/a$ .
6.
  - a) Using the definition of the absolute value, show that if  $\epsilon > 0$  and  $|a| < \epsilon$ , then  $-\epsilon < a$  and  $a < \epsilon$ .  
(This is written more compactly as  $-\epsilon < a < \epsilon$ .)
  - b) Using the definition of the absolute value, show that if  $\epsilon > 0$  and  $|a - b| < \epsilon$ , then  $b - \epsilon < a$  and  $a < b + \epsilon$ . (Or more compactly,  $b - \epsilon < a < b + \epsilon$ .)
7. From the properties of absolute values prove that  $||a| - |b|| \leq |a - b|$  for all  $a, b \in \mathbb{R}$ .

---

<sup>4</sup>In this exercise set, unless otherwise specified  $a, b, c, d$  denote arbitrary real numbers.

8. Which of the following sequences are increasing, strictly increasing, decreasing, strictly decreasing, or none of the above? Justify your answers.
- $a_n = n^2 - n, \quad n \in \mathbb{Z}^+$
  - $c_n = \frac{1}{n+1}, \quad n \in \mathbb{Z}^+$
  - $b_n = \frac{(-1)^n}{n^2}, \quad n \in \mathbb{Z}^+$
  - $a_{n+1} = a_n + \frac{1}{n}, \quad \text{for } n > 1, \text{ and } a_1 = 1$
  - $b_n = 1, \text{ for all } n \in \mathbb{Z}^+.$
9. Which of the above sequences are bounded above, or bounded below; which are bounded? Give an upper bound and/or a lower bound when applicable.
10. For  $n \in \mathbb{Z}^+$ , let  $a_n = \frac{1}{n}$  and  $b_n = \frac{1}{n+1}$ . Give explicit formulae for the sequences  $c_n = a_n - b_n$  for all  $n \in \mathbb{Z}^+$  and  $d_n = (a_n)(b_n)$  for all  $n \in \mathbb{Z}^+$ . Write the first four terms in each sequence.
11. Let  $f, g : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  be functions given by

$$f : n \mapsto (n+1)(n+2), \quad g : n \mapsto 2n - 1.$$

Give an explicit formula for the sequence defined by  $f \circ g$ . Write out the first 6 terms of the sequence defined by  $f$  and that defined by  $f \circ g$ .

12. Let  $a_n = 2n$  for all  $n \in \mathbb{Z}^+$  and let  $b_n = 2^n$  for all  $n \in \mathbb{Z}^+$ . Show that  $b_n$  is a subsequence of  $a_n$  by writing it explicitly as a composition of functions.
13. a) Let  $Y \subset X \subset \mathbb{R}$ . Show: if  $X$  is bounded, then so is  $Y$ .  
b) Show: every subsequence of a bounded sequence is bounded.
14. Exhibit two unbounded real sequences  $\{a_n\}$  and  $\{b_n\}$  such that  $\{a_n + b_n\}$  is bounded.
15. Prove that the sequence  $\{|a_n|\}$  is bounded if and only if the sequence  $\{a_n\}$  is bounded.
16. Let  $g : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  be strictly increasing. Prove that  $g(n) \geq n$  for all  $n \in \mathbb{Z}^+$ .



17. a) Write a statement similar to Proposition 1.3.6 about decreasing sequences. (Be careful, only one inequality changes.)  
b) Prove your statement from part a)
18. a) Write a statement similar to Proposition 1.3.13 about decreasing sequences.  
b) Prove your statement from part a)
19. Let  $a_n$  be the sequence given recursively by  $a_{n+1} = \frac{3a_n+2}{a_n+2}$ , with  $a_1 = 3$ .  
a) Prove by induction that  $a_n > 2$  for all  $n \in \mathbb{Z}^+$ .  
b) Prove that the sequence is decreasing.  
c) What happens to the sequence if we start with  $a_1 = 1$ ?
20. Let  $a_n$  be the sequence given recursively by  $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$  with  $a_1 = 2$ .  
a) Prove by induction that  $a_n > 0$  for all  $n \in \mathbb{Z}^+$ .  
b) Prove that  $a_n^2 - 2 \geq 0$  for all  $n \in \mathbb{Z}^+$ .  
c) Prove that  $\{a_n\}$  is a decreasing sequence.
21. a) Show that the sequence given by  $a_n = n^2 - 7n + 11$  for all  $n \in \mathbb{Z}^+$  is eventually increasing.  
b) Show that the sequence given by  $a_n = n^3 - 5n^2 - 17n + 21$  is eventually increasing.  
c) Let  $P(x)$  be a polynomial, and define a sequence by  $a_n = P(n)$  for all  $n$ . Find necessary and sufficient conditions on the coefficients of  $P(x)$  for  $\{a_n\}$  to be eventually increasing.
22. Let  $P = \{P(n)\}$  be a sequence of statements: that is, for each  $n \in \mathbb{Z}^+$ ,  $P(n)$  is a statement. We say  $P$  holds eventually if there is  $N \in \mathbb{Z}^+$  such that  $P(n)$  holds for all  $n > N$ . Show: if  $P$  and  $Q$  are two sequences of statements such that  $P$  holds eventually and  $Q$  holds eventually then  $(P$  and  $Q)$  holds eventually.
23. a) Write out carefully what it means for a real sequence  $\{a_n\}$  to be eventually positive.  
b) If  $\{a_n\}$  and  $\{b_n\}$  are two eventually positive real sequences, show that  $\{a_n + b_n\}$  and  $\{a_n b_n\}$  are eventually positive.

- c) Find real sequences  $\{a_n\}$  and  $\{b_n\}$ , neither of which are eventually positive, such that  $\{a_n + b_n\}$  is eventually positive.
24. Show that an eventually bounded real sequence is bounded.
25. a) (Erdős-Szekeres [ES35]) Let  $r, s \in \mathbb{Z}^+$ . Let  $x_1, \dots, x_{(r-1)(s-1)+1}$  be a finite sequence of real numbers of length  $(r-1)(s-1)+1$ . Show that there is *either* an increasing subsequence of length  $r$  or a decreasing subsequence of length  $s$ .  
b) Show that there is a real sequence of length  $(r-1)(s-1)$  which admits *neither* an increasing subsequence of length  $r$  or a decreasing subsequence of length  $s$ .
26. For a real sequence  $\{a_n\}$ , show that the following are equivalent:  
(i) There is  $n \in \mathbb{Z}^+$  such that either  $a_n < a_{n+1} > a_{n+2}$  (a  **$\Lambda$ -configuration**) or  $a_n > a_{n+1} < a_{n+2}$  (a  **$V$ -configuration**).  
(ii) The sequence  $\{a_n\}$  is not monotone.
27. a) Show that every real sequence admits a subsequence which is *strictly* increasing, a subsequence which is *strictly* decreasing, or a subsequence which is constant. (Suggestion: consider separately the cases in which the sequence has finite image and infinite image.)  
b) We can divide real sequences into 8 classes altogether, according to whether they do or not do admit a strictly increasing subsequence, do or not admit a strictly decreasing subsequence, and do or do not admit a constant subsequence. Part a) asserts that one of these classes is empty, namely the sequences which do not admit any of these three things. Show that the seven other classes are nonempty.  
c) Show that a sequence which is *not* bounded above admits an increasing subsequence and that a sequence which is *not* bounded below admits a decreasing subsequence.

## 1.4 Sequences of Approximations: Convergence

In calculus we learn *Newton's method* for approximating zeros of a differentiable function,  $f : \mathbb{R} \rightarrow \mathbb{R}$ . The method recursively defines a sequence of numbers,  $\{x_n\}$ , which (hopefully) eventually give an arbitrarily good approximation to a point where the graph of the function touches the  $x$ -axis.

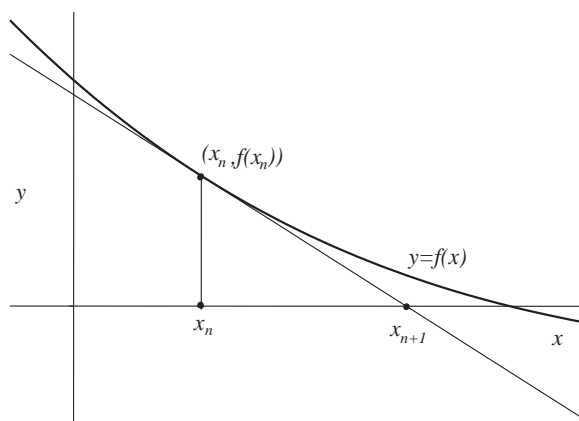


figure 1.4.1

Here is how the method works: First, since this will be a recursive formula, we have to pick an initial value  $x_1$ . Our choice here will often make use of some knowledge of the function and the region of its graph in which we expect it to touch the  $x$ -axis. Next, we derive the recursive formula for finding  $x_{n+1}$  in terms of  $x_n$ . We begin by writing out the equation for the tangent line to our function at the point  $(x_n, f(x_n))$  on its graph. Since the slope of the tangent line at this point is given by  $f'(x_n)$  and the line passes through the point  $(x_n, f(x_n))$ , we see that this equation is given by:

$$y = f(x_n) + f'(x_n)(x - x_n).$$

Finally, the next point in the sequence is determined by where this line crosses the  $x$ -axis (see figure 1.4.1). Thus,

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n).$$

Solving for  $x_{n+1}$  yields:

$$x_{n+1} = x_n - f(x_n)/f'(x_n).$$

(The sequence stops if at some point  $f'(x_n) = 0$ , in which case the tangent line is horizontal.)

**Example 1.4.1.** *Let's take  $f(x) = x^2 - 2$ . Since this function vanishes at  $\sqrt{2}$  and  $-\sqrt{2}$  we hope that by choosing the correct  $x_1$  we will be able to produce a sequence that gives better and better approximations to  $\sqrt{2}$ . Since  $f'(x) = 2x$ , the above recursion formula yields*

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = 1/2(x_n + \frac{2}{x_n}).$$

The choice  $x_1 = 2$  gives the sequence explored in exercise 1.3.20.

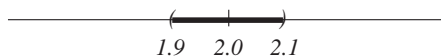
The rest of this section will be devoted to making precise the notion that this sequence of numbers “eventually gives an arbitrarily good approximation” to the zero of the function. The basic geometric idea is what we will call an “ $\epsilon$ -neighborhood” where  $\epsilon$  (the Greek letter *epsilon*) simply denotes a positive real number, but will be most interesting when it represents a *small* positive real number. With this in mind, the  $\epsilon$ -neighborhood around a real number  $L$  is just the interval of real numbers given by

$$\{x \in \mathbb{R} \mid L - \epsilon < x < L + \epsilon\} = (L - \epsilon, L + \epsilon).$$

Another important way of describing this same set is given by

$$\{x \in \mathbb{R} \mid |x - L| < \epsilon\}.$$

The  $\epsilon$ -neighborhood around  $L = 2$  with  $\epsilon = .1$  is shown in *figure 1.4.2*



*figure 1.4.2*

**Comments on interval notation:**

An *interval* of real numbers is the set of all real numbers between two specified real numbers, say  $a$  and  $b$ . The two specified numbers,  $a$  and  $b$ , are called the *endpoints* of the interval. One, both, or neither of the endpoints may be contained in the interval. Since intervals are commonly used in many areas of mathematics, it gets cumbersome to write out the full set notation each time we refer to an interval, so intervals have been given their own special shorthand notation. The following equations should be thought of as defining the notation on the left side of the equality:

$$\begin{aligned}(a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \\ [a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\ (a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\ [a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\}.\end{aligned}$$

Now, the size of the  $\epsilon$ -neighborhood around  $L$  is determined by  $\epsilon$ . The neighborhood gets smaller if the number  $\epsilon$  gets smaller. Our idea of some number  $a$  giving a *good approximation* to the number  $L$  is that  $a$  should be in some  $\epsilon$ -neighborhood around  $L$  for some small  $\epsilon$ . Our idea that a sequence “eventually gives an arbitrarily good approximation” to  $L$  should be that no matter how small we choose  $\epsilon$  we have that “eventually” all of the elements of the sequence are in that  $\epsilon$ -neighborhood. This is made mathematically precise in the following definition:

**Definition 1.4.2.** *The sequence  $\{a_n\}$  is said to converge to the real number  $L$  if the following property holds: For every  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  so that  $|a_n - L| < \epsilon$  for every  $n > N$ .*

*If the sequence  $\{a_n\}$  converges to  $L$ , we write*

$$\lim_{n \rightarrow \infty} a_n = L.$$

**Remark:** We will often simply write  $\lim a_n = L$  since it is understood that  $n \rightarrow \infty$ .

**Definition 1.4.3.** *A sequence  $\{a_n\}$  is said to be convergent if there is some real number  $L$ , so that  $\lim_{n \rightarrow \infty} a_n = L$ . If a sequence is not convergent it is called divergent.*

**Proposition 1.4.4.** a) The constant sequence  $\{a_n\}$ , given by  $a_n = c$  for all  $n \in \mathbb{Z}^+$  converges to  $c$ .

b) The sequence  $\{a_n\}$  given by  $a_n = \frac{1}{n}$  for all  $n \in \mathbb{Z}^+$  converges to 0.

*Proof.* a) Since for this sequence we have  $|a_n - c| = 0$  for all  $n$ , it follows that no matter what  $\epsilon$  is chosen, we can take  $N$  to be 1.

b) Let  $\epsilon > 0$ . By the *Archimedean Property* (see Example 1.3.8), there is a positive integer  $N > \frac{1}{\epsilon}$ . Then for every  $n > N$  we have  $n > 1/\epsilon$  also. But this is equivalent to  $1/n < \epsilon$ , i.e.,  $|a_n - 0| < \epsilon$ .  $\square$

Since the sequence  $\{a_n = 1/n\}$  converges to zero, we know that eventually all of the terms in the sequence are within some small neighborhood of 0. Since any other real number  $L \neq 0$  is a finite distance from 0, it seems impossible that the  $a_n$ 's could also eventually be close to  $L$ . We show this for  $L = .01$  in the next example.

**Example 1.4.5.** The sequence  $a_n = \frac{1}{n} \forall n \in \mathbb{Z}^+$  does not converge to 0.01.

**Proof:**

To prove that a sequence  $\{a_n\}$  does not converge to  $L$  we must show that the condition in the definition fails. Thus we must find just one  $\epsilon > 0$  so that for every  $N \in \mathbb{Z}^+$  there is an  $n > N$  with  $|a_n - L| \geq \epsilon$ .

In this case we have  $L = .01$ , let's take  $\epsilon = .001$  (actually any positive number less than .01 would work). Now, if  $n > 200$  we have  $1/n < .005$  and so  $|a_n - .01| > .01 - .005 = .005 > .001$ . Thus, no matter what  $N$  is chosen, we can find  $n$  which is larger than both  $N$  and 200 and get that  $|a_n - .01| \geq \epsilon$ . (Notice that there is lots of flexibility in the choice of  $\epsilon$  and in the choice of  $n$ .)  $\square$

**Comments on Negations:**

Given some mathematical (or logical) statement,  $P$ , the *negation of  $P$*  is the new statement that  $P$  fails. In the above example the statement  $P$  is “the sequence  $\{a_n\}$  converges to  $L$ ”, the negation of this statement is “the sequence  $\{a_n\}$  does not converge to  $L$ ”. Now, in this case,  $P$  is defined by a fairly complicated collection of conditions on  $\{a_n\}$  and  $L$ , and so to get a useful interpretation of the negation of  $P$ , we had to determine what it means for that collection of conditions to fail. Negating a definition can often be a tricky exercise in logic, but it also leads to a better understanding of the definition. After all, understanding how the conditions can fail gives a deeper understanding of when the conditions are satisfied. Thus, it is generally recommended that when reading mathematics, one should stop after each definition and consider the negation of that definition. There are some useful guidelines that can help when writing out a negation of a definition. Here are a two such basic rules:

1.) *If a definition requires that two conditions hold, then the negation of the definition will require that at least one of the two conditions fail.*

2.) *If a definition asks that either condition  $A$  or condition  $B$  should hold, then the negation of the definition will require that both of the two conditions must fail.*

Actually, these two rules can be made much more general:

1'.) *If a definition asks that some condition holds for every element of some set  $S$ , then the negation of the definition will require that the condition must fail for at least one element of  $S$ .*

2'.) If a definition asks that some condition holds for at least one element of some set  $S$ , then the negation of the definition will require that the condition must fail for every element of  $S$ . Now the definition that the sequence  $\{a_n\}$  converges to  $L$  says, “for every  $\epsilon > 0$ ” some condition, let’s call it A, holds, thus to negate this definition, we ask that condition A should fail for “at least one  $\epsilon > 0$ ”. Now condition A asks that “there exist some  $N \in \mathbb{Z}^+$ ” so that some other condition, let’s call it B, should hold. Thus for A to fail we need to show that B fails “for every  $N \in \mathbb{Z}^+$ ”. Finally condition B is the statement that “ $|a_n - L| < \epsilon$  for every  $n > N$ ”, so B fails if “there exists just one  $n > N$  so that  $|a_n - L| \geq \epsilon$ .” Putting this string of logic together, we come up with the statement,

*The sequence  $\{a_n\}$  does not converge to  $L$  if there exists an  $\epsilon > 0$  such that for every  $N \in \mathbb{Z}^+$  there exists an  $n > N$  so that  $|a_n - L| \geq \epsilon$ , just as stated at the beginning of the proof in example 1.4.5. One should note that, in practice, showing that this condition is satisfied is usually accomplished by finding a subsequence  $\{b_n\}$  of  $\{a_n\}$  so that there is an  $\epsilon > 0$  with  $|b_n - L| \geq \epsilon$  for all  $n \in \mathbb{Z}^+$ .*

Another important aspect of example 1.4.5 is that it raises the question of *uniqueness* of the limit, i.e., is it possible that a given sequence has two or more distinct limits?

**Proposition 1.4.6.** *Let  $\{a_n\}$  be a sequence and  $L$  and  $M$  real numbers with  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} a_n = M$ . Then  $L = M$ .*

*Proof.* Let us first explain the idea of the proof and then give a detailed implementation of it. Seeking a contradiction, we suppose that  $L \neq M$ . If so, we can choose an open interval  $I_L$  containing  $L$  and an open interval  $I_M$  containing  $M$  and such that  $I_L$  and  $I_M$  are disjoint (i.e., have no points in common). Then since  $a_n \rightarrow L$  and  $a_n \rightarrow M$ , eventually the terms of  $I_L$  and  $I_M$  lie in both  $I_L$  and  $I_M$  (cf. Exercise 22): contradiction.

Now to the implementation. Let  $\epsilon = \frac{|L-M|}{2}$ , i.e., half the distance between  $L$  and  $M$ . Then we have

$$(L - \epsilon, L + \epsilon) \cap (M - \epsilon, M + \epsilon) = \emptyset :$$

indeed, if  $x$  lies in both  $(L - \epsilon, L + \epsilon)$  and  $(M - \epsilon, M + \epsilon)$ , then

$$|x - L| < \epsilon = \frac{|L - M|}{2}, \quad |x - M| < \epsilon = \frac{|L - M|}{2},$$



so by the triangle inequality

$$|L - M| = |(x - L) - (x - M)| \leq |x - L| + |x - M| < \epsilon + \epsilon = |L - M|.$$

Thus  $|L - M| < |L - M|$ : no way.<sup>5</sup> Now, since  $a_n \rightarrow L$ , there is  $N_L \in \mathbb{N}$  such that for all  $n > N_L$  we have  $a_n \in (L - \epsilon, L + \epsilon)$ ; and since  $a_n \rightarrow M$ , there is  $N_M \in \mathbb{N}$  such that for all  $n > N_M$  we have  $a_n \in (M - \epsilon, M + \epsilon)$ . Take  $N = \max(N_L, N_M)$ . Then for all  $n > N$ , both conditions hold:

$$a_n \in (L - \epsilon, L + \epsilon) \cap (M - \epsilon, M + \epsilon) = \emptyset,$$

contradiction. Thus  $L = M$ . □

In the next section we will discuss some properties of limits that will enable us to evaluate some limits without having to revert to the definition. However, before going there, it will strengthen our understanding of definition 1.4.2 if we prove convergence of another example directly from the definition.

**Example 1.4.7.** *In this example we will examine a proof that*

$$\lim_{n \rightarrow \infty} \frac{2n + 1}{3n - 1} = 2/3.$$

*Before getting involved in the proof, using an arbitrary  $\epsilon > 0$ , let's practice with a particular  $\epsilon$ , say  $\epsilon = .01$ . So, letting  $a_n = (2n + 1)/(3n - 1)$ , we want to find (explicitly) a natural number  $N$  so that whenever  $n > N$  we have  $|a_n - 2/3| < .01$ . Now I might impress you with my acute foresight if I tell you right now that I am going to choose  $N = 55$  and then prove to you that this  $N$  works. Instead, I will be a bit more honest with you and show you how I come up with this value for  $N$ .*

*We want a condition on  $n$  that will guarantee that*

$$\left| \frac{(2n + 1)}{(3n - 1)} - \frac{2}{3} \right| < .01.$$

*Let's try to find an equivalent inequality from which it is easier to read off a condition on  $n$ . First, we simplify the expression inside the absolute values by noting*

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<sup>5</sup>There are many ways to justify this disjointness claim. In fact, if you think about the situation geometrically for a little while, it becomes self-evident. But I (PLC) wanted to show how one can work with inequalities using the triangle inequality, since that comes up all over the place in more advanced mathematics.

$$\begin{aligned} \left| \frac{2n+1}{3n-1} - \frac{2}{3} \right| &= \left| \frac{3(2n+1)}{3(3n-1)} - \frac{2(3n-1)}{3(3n-1)} \right| \\ &= \left| \frac{5}{3(3n-1)} \right|. \end{aligned}$$

Next we rewrite the inequality without the absolute values, namely we use the fact that the inequality

$$\left| \frac{5}{3(3n-1)} \right| < .01$$

is equivalent to the two inequalities

$$-.01 < \frac{5}{3(3n-1)} < .01. \quad (*)$$

Now we notice that since  $n$  is a natural number, it is always true that  $3n-1 > 0$ , hence  $\frac{5}{3(3n-1)} > 0$  as well. Thus, the first inequality is true for all natural numbers  $n$ . Multiplying the second inequality by the positive number  $3n-1$ , and then by 100, yields,  $\frac{500}{3} < 3n-1$ . Further simplification leads to the condition

$$n > \frac{503}{9} \approx 55.9.$$

This shows that if we take  $n$  to be any natural number greater than 55 we will get our desired inequality, thus we can take  $N = 55$  (or any natural number bigger than 55).

Now in general we can go through the above argument for any given particular value of  $\epsilon$ , or we can go through the algebra leaving the  $\epsilon$  in. Of course, the  $N$  that we get will depend on the choice of  $\epsilon$ . To see how this works in general, lets go back to equation (\*) and replace the .01's with  $\epsilon$ , thus we have

$$-\epsilon < \frac{5}{3(3n-1)} < \epsilon.$$

Again, the fraction in the middle is positive so the left inequality is trivial, and we begin simplifying the right inequality by multiplying by  $3(3n-1)$  to get  $5 < 3(3n-1)\epsilon$ . Now we divide by  $\epsilon$  (this corresponds to multiplying by 100 when we took  $\epsilon = .01$ ) to get  $5/\epsilon < 9n-3$ . Simplifying this, we see that we want

$$n > \frac{5}{9\epsilon} + \frac{1}{3}.$$

That is, we can take  $N$  to be any natural number greater than  $\frac{5}{9\epsilon} + \frac{1}{3}$ .

Finally, before leaving this example, let's write out the formal proof that  $\lim_{n \rightarrow \infty} \frac{2n+1}{3n-1} = \frac{2}{3}$ .

**Proof:** Let  $\epsilon > 0$  be given and choose  $N \in \mathbb{N}$  in such a way that  $N > \frac{5}{9\epsilon} + \frac{1}{3}$ . then if  $n > N$  we have

$$n > \frac{5}{9\epsilon} + \frac{1}{3},$$

which is equivalent to

$$3n - 1 > \frac{5}{3\epsilon}.$$

Since  $n$  is a natural number, we know that  $3n - 1 > 0$ , so the above inequality is equivalent to

$$\frac{5}{3(3n - 1)} < \epsilon.$$

Also, since the left hand side is positive, we have trivially that

$$-\epsilon < \frac{5}{3(3n - 1)} < \epsilon.$$

Now a short computation shows that

$$\frac{5}{3(3n - 1)} = \frac{3(2n + 1) - 2(3n - 1)}{3(3n - 1)} = \frac{2n + 1}{3n - 1} - \frac{2}{3},$$

so our inequalities give

$$-\epsilon < \frac{2n + 1}{3n - 1} - \frac{2}{3} < \epsilon,$$

i.e.,

$$\left| \frac{2n + 1}{3n - 1} - \frac{2}{3} \right| < \epsilon.$$

Thus we have shown that if  $\epsilon > 0$  is given, and we choose a natural number  $N > \frac{5}{6\epsilon}$ , then whenever  $n > N$  we have  $|a_n - 2/3| < \epsilon$ , i.e.,  $\lim a_n = 2/3$ .  $\square$

That wasn't so bad. Let's try a slightly more complicated example:

**Example 1.4.8.** *Let us prove that*

$$\lim_{n \rightarrow \infty} \frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} = \frac{2}{5}.$$

*To figure out what  $N$  should be, we start off like we did in the last example. First, let  $\epsilon > 0$  be given. We want to find  $N \in \mathbb{Z}^+$  so that*

$$-\epsilon < \frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} - \frac{2}{5} < \epsilon$$

*for all  $n > N$ . A little algebra shows that this is equivalent to*

$$-\epsilon < \frac{11n - 27}{5(5n^2 + 2n + 1)} < \epsilon.$$

*Now the fraction in the middle of the above set of inequalities is always positive as long as  $n > 2$ , so as long as we make sure in the end that  $N > 2$ , we don't have to worry about the first inequality. On the other hand, solving for  $n$  in the second inequality is much more complicated than it was in the previous example. Fortunately, we don't need to find  $n$  in terms of  $\epsilon$  exactly, we just need an approximation that will guarantee the inequality we need. Here is how to proceed: First, notice*

$$\begin{aligned} \frac{11n - 27}{5(5n^2 + 2n + 1)} &= \frac{n(11 - \frac{27}{n})}{5n^2(5 + \frac{2}{n} + \frac{1}{n^2})} \\ &= \frac{1}{n} \left( \frac{11 - \frac{27}{n}}{5(5 + \frac{2}{n} + \frac{1}{n^2})} \right). \end{aligned}$$

*Now, since  $n > 0$ , it is easy to see that  $11 - \frac{27}{n} < 11$  and that  $5 + \frac{2}{n} + \frac{1}{n^2} > 5$ . This second inequality implies that  $\frac{1}{5 + \frac{2}{n} + \frac{1}{n^2}} < \frac{1}{5}$  (both terms are positive), and so we have that*

$$\frac{(11 - \frac{27}{n})}{5(5 + \frac{2}{n} + \frac{1}{n^2})} < \frac{11}{5 \cdot 5} = \frac{11}{25}.$$

*Therefore we see that*

$$\frac{1}{n} \left( \frac{11 - \frac{27}{n}}{5(5 + \frac{2}{n} + \frac{1}{n^2})} \right) < \frac{11}{25n},$$

and so, if we choose  $N$  large enough to guarantee that  $\frac{11}{25n} < \epsilon$  we will certainly have that  $\frac{1}{n} \left( \frac{11 - \frac{27}{n}}{5(5 + \frac{2}{n} + \frac{1}{n^2})} \right) < \epsilon$  as well. But this is easy enough, just choose  $N$  to be greater than  $\frac{11}{25\epsilon}$ . Don't forget that we also need to make sure  $N > 2$ . This is done by just saying that we choose  $N \in \mathbb{Z}^+$  so that  $N > \max(2, \frac{11}{25\epsilon})$ . Ok, now let's put this all together in a formal proof:

**Proof:** Let  $\epsilon > 0$  be given. Choose  $N \in \mathbb{Z}^+$  so that  $N > \max(2, \frac{11}{25\epsilon})$ . If  $n > N$ , it follows that

$$\frac{11}{25n} < \epsilon.$$

However, since  $n > 0$ , we know that  $11 > 11 - \frac{27}{n}$  and  $5 < 5 + \frac{2}{n} + \frac{1}{n^2}$ . Hence, we can conclude that

$$\frac{1}{n} \left( \frac{11 - \frac{27}{n}}{5(5 + \frac{2}{n} + \frac{1}{n^2})} \right) < \frac{11}{25n} < \epsilon.$$

Multiply the left hand side of the above inequality by  $\frac{n}{n}$  to get

$$\frac{11n - 27}{5(5n^2 + 2n + 1)} < \epsilon.$$

Algebra shows that

$$\frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} - \frac{2}{5} = \frac{11n - 27}{5(5n^2 + 2n + 1)},$$

so we have that if  $n > N$ , then

$$\frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} - \frac{2}{5} < \epsilon.$$

Also, since  $n > N > 2$ , it is clear that

$$\frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} - \frac{2}{5} = \frac{11n - 27}{5(5n^2 + 2n + 1)} > 0 > -\epsilon.$$

Thus, for  $n > N$ , we have shown

$$\left| \frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} - \frac{2}{5} \right| < \epsilon.$$

To conclude this section we will discuss a few general properties of convergence that can help to determine convergence or divergence of a sequence. First, it is useful to notice that if we change the first  $K$  terms of a sequence it won't affect the convergence of the sequence. We leave the proof of this to the exercises.

**Proposition 1.4.9.** *Suppose that  $\{a_n\}$  and  $\{b_n\}$  are sequences and that there is a  $K \in \mathbb{Z}^+$  so that  $a_n = b_n$  for all  $n > K$ . Assume also that  $\{a_n\}$  converges to the real number  $L$ , then the sequence  $\{b_n\}$  also converges to  $L$ .*

The next proposition gives an easy first check for convergence since its negation says that if a sequence is not bounded, then it must diverge.

**Proposition 1.4.10.** *Every convergent sequence is bounded.*

*Proof.* Let  $\{a_n\}$  be a convergent sequence and let  $L$  be its limit. Then we know that for any  $\epsilon > 0$  we can find an  $N \in \mathbb{Z}^+$  so that  $|a_n - L| < \epsilon$  for all  $n > N$ . In particular, we can take  $\epsilon$  to be equal to 1, then we know that there is some natural number  $N_1$  so that  $|a_n - L| < 1$  for all  $n > N_1$ . We can now turn this into a bound for  $|a_n|$  by noting that  $|a_n| = |a_n - L + L| \leq |a_n - L| + |L|$  by the triangle inequality, hence we conclude

$$|a_n| < |L| + 1$$

for all  $n > N_1$ .

Now, to get a bound on  $|a_n|$  for all  $n \in \mathbb{Z}^+$ , we let  $M$  be the maximal element of the set

$$\{|a_1|, |a_2|, \dots, |a_{N_1}|, |L| + 1\}.$$

Then certainly  $|a_n| \leq M$  when  $1 \leq n \leq N_1$ , and also  $|a_n| < |L| + 1 \leq M$  for  $n > N_1$ . Thus we have that  $|a_n| \leq M$  for all  $n$ .  $\square$

**Example 1.4.11.** *The sequence  $\{a_n = n\}$  is unbounded (Archimedean Property), hence divergent.*

The next proposition and its corollary give us another good method for showing that a sequence diverges.

**Proposition 1.4.12.** *Let  $\{a_n\}$  be real sequence converging to  $L \in \mathbb{R}$ . Then every subsequence  $\{a_{n_k}\}$  of  $\{a_n\}$  also converges to  $L$ .*

*Proof.* Let  $\epsilon > 0$ . Since  $a_n \rightarrow L$ , there is  $N \in \mathbb{Z}^+$  such that  $|a_n - L| < \epsilon$  for all  $n > N$ . If  $k > N$  then  $n_k \geq k > N$ , so  $|a_{n_k} - L| < \epsilon$ .  $\square$

**Corollary 1.4.13.** *Suppose that  $\{a_n\}$  is a real sequence with two convergent subsequences,  $\{b_n\}$  and  $\{c_n\}$ . If  $\lim_{n \rightarrow \infty} c_n \neq \lim_{n \rightarrow \infty} b_n$ , then  $\{a_n\}$  must be divergent.*

**Example 1.4.14.** *The sequence given by  $a_n = (-1)^n$  has even terms all equal to 1 and odd terms all equal to  $-1$ . Thus the subsequence of even terms converges to 1 and the subsequence of odd terms converges to  $-1$ . From the corollary, we conclude that the sequence  $\{a_n\}$  diverges.*

**Infinite Limits:** Some sequences diverge. The implication so far is that we should simply not have commerce with a divergent sequence, but this is not really accurate or desirable. On the contrary, in many cases it is desirable or necessary to understand the “limiting behavior” of a divergence sequence in some sense. Whereas any two convergent sequences “look the same” in some sense, divergent sequences may look different.<sup>6</sup> Well, if that is our metaphor let us now meet two unhappy families.

A real sequence  $\{a_n\}$  **diverges to infinity** if for all  $M \geq 0$  there is  $N \in \mathbb{N}$  such that for all  $n > N$  we have  $a_n \geq M$ . Otherwise put, for any fixed  $M$ , the terms of the sequence are *eventually* as large as  $M$ . Symbolically we write  $a_n \rightarrow \infty$ , though we hasten to add that  $\infty$  is not a real number and this indicates not convergence but a certain kind of divergence.

Similarly, a real sequence  $\{a_n\}$  **diverges to negative infinity** if for all  $m \leq 0$  there is  $N \in \mathbb{N}$  such that for all  $n > N$  we have  $a_n \leq m$ . Otherwise put, for any fixed  $m$ , the terms are *eventually* no larger than  $m$ . Symbolically we write  $a_n \rightarrow -\infty$  (and give the same warning as above).

**Example 1.4.15.** *a) If  $a_n = n$  for all  $n \in \mathbb{Z}^+$ , then  $a_n \rightarrow \infty$ . This may sound like a tautology (“as  $n$  approaches infinity,  $n$  approaches infinity”) but it isn’t. It is a restatement of the Archimedean Property: for any  $M \geq 0$ , there is  $N \in \mathbb{Z}^+$  such that  $N \geq M$ . So for all  $n > N$  we have  $n \geq M$ .*

*b) Suppose  $\{a_n\}$  and  $\{b_n\}$  are two sequences such that  $a_n \leq b_n$  for all  $n \in \mathbb{Z}^+$ . If  $a_n \rightarrow \infty$ , then also  $b_n \rightarrow \infty$ . Indeed, for all  $M \geq 0$  there is  $N \in \mathbb{N}$  such*

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<sup>6</sup>“Happy families are all alike; every unhappy family is unhappy in its own way.” Tolstoy, *Anna Karenina*

that  $a_n \geq M$  for all  $n > N$ . So for all  $n > N$  we have  $M \leq a_n \leq b_n$ .

c) Combining part a) and b) we see that for all  $\alpha > 1$ , the sequence  $a_n = n^\alpha$  diverges to infinity.

**Proposition 1.4.16.** Let  $r \in \mathbb{R}$  and consider the geometric sequence  $\{a_n\}$  given by  $a_n = r^n$  for all  $n \in \mathbb{N}$ . Then:

a) If  $r = 1$ , then  $r^n \rightarrow 1$ .

b) If  $r = -1$  then  $\{r^n\}$  is bounded but divergent.

c) If  $r > 1$ , then  $r^n \rightarrow \infty$ .

d) If  $r < -1$ , then  $\{r^n\}$  is unbounded both above and below, hence divergent.

e) If  $|r| < 1$ , then  $r^n \rightarrow 0$ .

*Proof.* Parts a) and b) have been established above.

For part c), we first establish **Bernoulli's Inequality**: for all  $x \geq -1$  and all  $n \in \mathbb{N}$ , we have

$$(1+x)^n \geq 1+nx. \quad (1.4)$$

This is a straightforward induction proof: when  $n = 1$  (base case) both sides of (1.4) are  $1+x$ . Now let  $n \in \mathbb{Z}^+$ ,  $x \geq -1$  and suppose that  $(1+x)^n \geq 1+nx$ . Since  $x \geq -1$  we have  $1+x \geq 0$ , so multiplying by  $(1+x)$  preserves inequalities. Thus, using our induction hypothesis we get

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1+(n+1)x+nx^2 \geq 1+(n+1)x,$$

completing the induction step.

Now let  $r > 1$ , so  $c = r - 1 > 0$ . For  $n \in \mathbb{Z}^+$ , Bernoulli's Inequality gives

$$r^n = (1+(r-1))^n = (1+c)^n \geq 1+nc.$$

Let  $M \geq 0$  and take  $N = \frac{M-1}{c}$ ; then if  $n > N$ , we have

$$r^n \geq 1+nc \geq M.$$

So  $r^n \rightarrow \infty$ .

d) If  $r < -1$ , then  $r^n = (-1)^n|r|^n$ . From this it follows that the subsequence of odd-numbered terms diverges to  $-\infty$  while the subsequence of even-numbered terms diverges to  $\infty$ .

e) Suppose  $|r| < 1$ . Since  $a_n \rightarrow 0$  iff  $|a_n| \rightarrow 0$ , we may replace  $r$  with  $|r|$  and thus assume  $0 < r < 1$ . Then  $R = \frac{1}{r} > 1$ . Fix  $\epsilon > 0$ . Part c),  $R^n \rightarrow \infty$ , so there is  $N \in \mathbb{N}$  such that for all  $n > N$  we have  $R^n > \frac{1}{\epsilon}$ . Taking reciprocals, we get  $|r^n| = r^n < \epsilon$  for all  $n > N$ .  $\square$



In proposition 1.4.10 we proved that every convergent sequence is bounded. We will need a few further results along these lines for section 1.6. We state them here and leave the proofs for you to do as exercises.

**Proposition 1.4.17.** *Let  $\{a_n\}$  be a convergent sequence with  $\lim a_n = \alpha$  and assume that  $M$  is an upper bound for  $\{a_n\}$ , i.e.,  $a_n \leq M$  for all  $n \in \mathbb{Z}^+$ . Then  $\alpha \leq M$ .*

**Proof Hint:** Suppose on the contrary that  $\alpha > M$  and let  $\epsilon = (\alpha - M)/2$ . Use the definition of the limit to show how to find a particular  $N \in \mathbb{Z}^+$  with  $a_N > M$ . This contradicts the assumption. (See figure 1.4.3.)

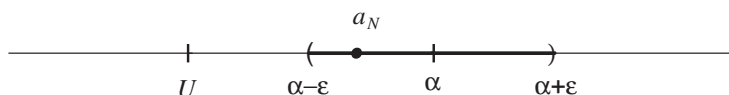


figure 1.4.3

**Remark:** Of course there is a similar proposition for lower bounds.

The next proposition says that if  $\{a_n\}$  is an *increasing* convergent sequence, then the limit,  $\lim a_n$ , gives an upper bound for  $\{a_n\}$ . Of course, there is a similar proposition for decreasing sequences.

**Proposition 1.4.18.** *Let  $\{a_n\}$  be an increasing sequence with  $\lim_{n \rightarrow \infty} a_n = \alpha$ , then  $a_n \leq \alpha$  for all  $n \in \mathbb{Z}^+$ .*

**Proof Hint:** Suppose on the contrary that there is some particular  $a_k$  with  $a_k > \alpha$ . Let  $\epsilon = (a_k - \alpha)/2$  and use the definition of the limit to show that there is some  $m > k$  with  $a_m < a_k$ , contradicting the fact that the sequence is increasing. (See figure 1.4.4.)

**Remark:** Combining these two propositions for the case of an increasing convergent sequence  $\{a_n\}$  we see that  $\lim_{n \rightarrow \infty} a_n$  is an upper bound for this sequence and that it is less than or equal to all upper bounds for this sequence, i.e., it is the least of all of the upper bounds. Thus it is called the *least upper bound* of the sequence  $\{a_n\}$ . Similarly, if  $\{b_n\}$  is a decreasing convergent sequence, then  $\lim_{n \rightarrow \infty} b_n$  gives the *greatest lower bound* of  $\{b_n\}$ .

## EXERCISES 1.4

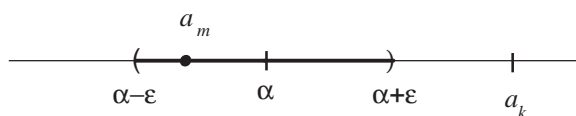


figure 1.4.4

1. Let  $a_n = \frac{2n+4}{3n-2}$ . Find  $N \in \mathbb{Z}^+$  so that  $|a_n - 2/3| < .01$  for all  $n > N$ . Justify your work.
2. Let  $a_n = \frac{3n+2}{2n-15}$ . Find  $N \in \mathbb{Z}^+$  so that  $|a_n - 3/2| < .05$  for all  $n > N$ . Justify your work.
3. Let  $a_n = \frac{n^2+2}{5n^2+1}$ . Find  $N \in \mathbb{Z}^+$  so that  $|a_n - 1/5| < .02$  for all  $n > N$ . Justify your work.
4. Let  $a_n = 3 - 1/n$ . Using the definition of the limit, prove that  $\lim_{n \rightarrow \infty} a_n = 3$ .
5. Let  $a_n = \frac{2n+4}{3n+1}$ . Using the definition of the limit, prove that  $\lim_{n \rightarrow \infty} a_n = 2/3$ .
6. Let  $a_n = \frac{5-n}{3n-7}$ . Using the definition of the limit, prove that  $\lim_{n \rightarrow \infty} a_n = -1/3$ .
7. Let  $a_n = \frac{5n^2+4}{3n^2+4}$ . Using the definition of the limit, prove that  $\lim_{n \rightarrow \infty} a_n = 5/3$ .
8. Let  $a_n = \frac{2n^2+4n-3}{3n^2+2n+1}$ . Using the definition of the limit, prove that  $\lim_{n \rightarrow \infty} a_n = 2/3$ .
9. Let  $a_n = 1/n$ . Using the definition of the limit, prove that  $a_n$  does not converge to  $1/4$ .
10. Edward wrote on his midterm exam that the definition of the limit is the following: The sequence  $\{a_n\}$  converges to the real number  $L$  if there exists an  $N \in \mathbb{Z}^+$  so that for every  $\epsilon > 0$  we have  $|a_n - L| < \epsilon$  for all  $n > N$ . Show Edward why he is wrong by demonstrating that if this were the definition of the limit then it would not be true that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ . (Hint: What does it mean if  $|a - b| < \epsilon$  for every  $\epsilon > 0$ ?)

11. Jacob wrote on his midterm exam that the definition of the limit is the following: The sequence  $\{a_n\}$  converges to the real number  $L$  if for every  $\epsilon > 0$  we have  $|a_n - L| < \epsilon$  for all  $n \in \mathbb{Z}^+$ . Show Jacob why he is wrong by demonstrating that if this were the definition of the limit then it would not be true that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .
12. Bella wrote on her midterm exam that the definition of the limit is the following: The sequence  $\{a_n\}$  converges to the real number  $L$  if for every  $\epsilon > 0$  there is  $N \in \mathbb{Z}^+$  such that we have  $|a_n - L| < \epsilon$  for all  $n \geq N$ .
  - a) What Bella wrote is not the definition we gave above. Why not?
  - b) Show that nevertheless Bella is correct: the sequence  $\{a_n\}$  converges to  $L$  according to our definition if and only if it converges to  $L$  according to her definition.
13. Give the explicit conditions for a sequence to diverge, i.e., give a “positive” version of the negation of the definition of convergence (definition 1.4.2 and 1.4.3).
14. Explicitly write out the negation of the incorrect definition given by Edward in exercise 10.
15. Explicitly write out the negation of the incorrect definition given by Jacob in exercise 11.
16. Prove Proposition 1.4.9.
17. Prove that  $\lim_{n \rightarrow \infty} |a_n| = 0$  if and only if  $\lim_{n \rightarrow \infty} a_n = 0$ .
18. Suppose that  $\lim_{n \rightarrow \infty} a_n = L$ .
  - a) Prove that if  $a_n \geq 0$  for all  $n \in \mathbb{Z}^+$ , then  $L \geq 0$ .
  - b) Give an example to show that  $a_n > 0$  for all  $n \in \mathbb{Z}^+$  does not imply that  $L > 0$ .
19. Suppose that  $\lim_{n \rightarrow \infty} a_n = L$  and that  $r$  is a real number. Prove that  $\lim_{n \rightarrow \infty} r a_n = r L$ .
20. Prove from the definition of divergence that the sequence  $a_n = (-1)^n$  is divergent.

21. Evaluate the following limits.
- $\lim_{n \rightarrow \infty} (3/2)^n$ .
  - $\lim_{n \rightarrow \infty} (1/2)^n$ .
22. a) Fill in the details of the proof of Proposition 1.4.17.  
b) Fill in the details of the proof of Proposition 1.4.18.
23. Let  $\{a_n\}$  be a real sequence.
- Show: if  $a_n \rightarrow \infty$ , then  $\{a_n\}$  is unbounded above.
  - Show: if  $a_n \rightarrow -\infty$ , then  $\{a_n\}$  is unbounded below.
  - Exhibit a sequence which is unbounded above and below but does not diverge to  $\infty$  nor to  $-\infty$ .
  - Show: if  $\{a_n\}$  is increasing, then  $a_n \rightarrow \infty$  if and only if  $\{a_n\}$  is unbounded above.
  - Show: if  $\{a_n\}$  is decreasing, then  $a_n \rightarrow -\infty$  if and only if  $\{a_n\}$  is unbounded below.
24. Let  $\{a_n\}$  be a real sequence.
- Show that the following are equivalent:
    - $\{a_n\}$  is unbounded above.
    - $\{a_n\}$  admits a subsequence diverging to infinity.
  - Show that the following are equivalent:
    - $\{a_n\}$  is unbounded below.
    - $\{a_n\}$  admits a subsequence diverging to negative infinity.
25. Show that there is a real sequence  $\{a_n\}$  that satisfies all of the following:<sup>7</sup>
- For all  $L \in \mathbb{R}$ , there is a subsequence  $a_{n_k} \rightarrow L$ .
  - There is a subsequence diverging to infinity.
  - There is a subsequence diverging to negative infinity.

## 1.5 Tools for Computing Limits

In this section we will develop a number of tools that will allow us to compute some limits of sequences without going back to the definition.

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<sup>7</sup>Here it is important that the same sequence must do all of these things at once!

**Proposition 1.5.1.** *Let  $\{a_n\}$  and  $\{b_n\}$  be real sequences such that  $a_n \rightarrow 0$  and  $\{b_n\}$  is bounded. Then  $a_n b_n \rightarrow 0$ .*

*Proof.* Since  $\{b_n\}$  is bounded there is some number  $M > 0$  with  $|b_n| < M$  for all  $n \in \mathbb{Z}^+$ . Also, since  $\lim_{n \rightarrow \infty} a_n = 0$ , when  $\epsilon > 0$  is given we can find an  $N \in \mathbb{Z}^+$  so that  $|a_n| < \frac{\epsilon}{M}$  for all  $n > N$ . But then

$$|a_n b_n| = |a_n| |b_n| < \frac{\epsilon}{M} M = \epsilon$$

whenever  $n > N$ . □

**Example 1.5.2.** *The above proposition can be applied, using  $a_n = 1/n$ , to show that the following limits are zero.*

- a)  $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$ .
- b)  $\lim_{n \rightarrow \infty} \frac{1}{n^k} = 0$ , for any  $k \in \mathbb{Z}^+$ .
- c)  $\lim_{n \rightarrow \infty} \frac{\cos(n)}{n} = 0$ .

**Remark:**

From the definition of the limit we know that if  $\lim_{n \rightarrow \infty} a_n = L$  then when  $\epsilon > 0$  is given, we can find an  $N \in \mathbb{Z}^+$  so that  $|a_n - L| < \epsilon$  for all  $n > N$ . In the above proof we have made a slight variation on this sentence without explanation. Here is what is going on: if some  $\epsilon > 0$  is given to us, and  $M$  is some fixed, positive number, then we can think of the quotient  $\epsilon/M$  as a “new  $\epsilon$ ”, let’s call it  $\tilde{\epsilon}$ , that is, let  $\tilde{\epsilon} = \epsilon/M$ . Then, since  $\lim_{n \rightarrow \infty} a_n = L$ , there is some number  $\tilde{N} \in \mathbb{Z}^+$  so that  $|a_n - L| < \tilde{\epsilon}$  for all  $n > \tilde{N}$ . We will use this trick over and over again in the proof of the next proposition.

**Lemma 1.5.3.** *If  $\lim_{n \rightarrow \infty} a_n = L$  and  $L \neq 0$  then there is a natural number  $N$  such that  $|a_n| > \frac{|L|}{2}$  for all  $n > N$ .*

We leave the proof of the lemma to the exercises, but here is a hint: Choose  $\epsilon = \frac{|L|}{2}$  and look at figure 1.5.1 below.

The next theorem allows us to compute new limits by algebraically manipulating limits which we already understand.

**Theorem 1.5.4.** *Suppose that  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = M$ . Then we have:*

- a) For all  $\alpha \in \mathbb{R}$ , we have  $\lim_{n \rightarrow \infty} \alpha a_n = \alpha L$ .
- b)  $\lim_{n \rightarrow \infty} (a_n + b_n) = L + M$ .



figure 1.5.1

- c)  $\lim_{n \rightarrow \infty} (a_n b_n) = LM$ .  
 d) If  $M \neq 0$  then  $\lim_{n \rightarrow \infty} \frac{1}{b_n} = \frac{1}{M}$ .  
 e) If  $M \neq 0$  then  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{M}$ .

*Proof.* a) The basic idea is that since  $|a_n - L|$  can be made arbitrarily small, so can  $|\alpha a_n - \alpha L| = |\alpha| |a_n - L|$ . More formally, let  $\epsilon > 0$  be given. If  $\alpha = 0$  then  $\alpha a_n - \alpha L = 0$  for all  $n$ , so any  $N$  works for all  $\epsilon > 0$  (this holds for any constant sequence, as we've seen before). So assume  $\alpha \neq 0$ . Then, since  $a_n \rightarrow L$ , there is  $N \in \mathbb{N}$  such that for all  $n > N$  we have  $|a_n - L| < \frac{\epsilon}{|\alpha|}$ . Then, for all  $n > N$  we have

$$|\alpha a_n - \alpha L| = |\alpha| |a_n - L| < |\alpha| \frac{\epsilon}{|\alpha|} = \epsilon.$$

b) The basic idea is

$$|(a_n + b_n) - (L + M)| = |(a_n - L) + (b_n - M)| \leq |a_n - L| + |b_n - M|.$$

Then, since  $|a_n - L|$  and  $|b_n - M|$  can each be made arbitrarily small, so can their sum, and we win. Now more formally: let  $\epsilon > 0$  be given. Since  $a_n \rightarrow L$ , there is  $N_1 \in \mathbb{N}$  such that for all  $n > N_1$  we have  $|a_n - L| < \frac{\epsilon}{2}$ . Similarly, since  $b_n \rightarrow M$ , there is  $N_2 \in \mathbb{N}$  such that for all  $n > N_2$  we have  $|b_n - M| < \frac{\epsilon}{2}$ . Taking  $N = \max(N_1, N_2)$ , we have for all  $n > N$  that

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So  $a_n + b_n \rightarrow L + M$ .

c) In order to use the assumptions that  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = M$

we must separate out terms like  $|a_n - L|$  and  $|b_n - M|$  from the expression  $|a_n b_n - LM|$ . To do this we use the old algebraic trick of adding and subtracting the same thing, namely

$$|a_n b_n - LM| = |a_n b_n - Lb_n + Lb_n - LM|.$$

This yields

$$\begin{aligned} |a_n b_n - LM| &= |a_n b_n - Lb_n + Lb_n - LM| \\ &\leq |a_n b_n - Lb_n| + |Lb_n - LM| \\ &\leq |a_n - L||b_n| + |L||b_n - M|. \end{aligned}$$

Now, if  $\epsilon > 0$  is given, we know we can pick  $N_1 \in \mathbb{N}$  so that  $|b_n - M| < \epsilon/2|L|$  for  $n > N_1$ , so the second term above is no problem. For the first term, we have to recall that since the sequence  $\{b_n\}$  is convergent, we know it is bounded. Hence, there is a number  $B > 0$  such that  $|b_n| < B$  for all  $n \in \mathbb{Z}^+$ , so the first term above (i.e.,  $|a_n - L||b_n|$ ) is less than  $|a_n - L|B$  for all  $n$ . Now we can pick  $N_2 \in \mathbb{N}$  so that  $|a_n - L| < \epsilon/2B$  for all  $n > N_2$ . Hence, if we let  $N = \max\{N_1, N_2\}$ , then whenever  $n > N$  we have both

$$|a_n - L||b_n| < \frac{\epsilon}{2B}B = \frac{\epsilon}{2}$$

and

$$|L||b_n - M| < \frac{\epsilon}{2|L|}|L| = \frac{\epsilon}{2}.$$

Putting these inequalities together we see that when  $n > N$  we have

$$|a_n b_n - LM| \leq |a_n - L||b_n| + |L||b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

d) First we observe that the statement, strictly speaking, doesn't make sense. Namely, the expression  $\frac{a_n}{b_n}$  is undefined when  $b_n = 0$ . We have assumed that  $M \neq 0$  and  $b_n \rightarrow M$ . By Lemma 1.5.3, there is  $N_1 \in \mathbb{N}$  such that  $|b_n| > \frac{|M|}{2}$  for all  $n > N_1$ . Having established that, the meaning of part d) is that we should only consider the sequence  $\{\frac{a_n}{b_n}\}$  for sufficiently large  $n$ , say  $n > N_1$ , in which case all of the terms are defined. Having established that, consider

$$\left| \frac{1}{b_n} - \frac{1}{M} \right| = \left| \frac{M - b_n}{b_n M} \right| = \frac{|b_n - M|}{|b_n||M|}.$$

And now we are in good shape because of the above estimate: namely, for all  $n > N_1$ , since  $|b_n| > \frac{|M|}{2}$  we have

$$\frac{1}{|b_n|} < \frac{2}{|M|}$$

and thus

$$\frac{|b_n - M|}{|b_n||M|} \leq |b_n - M| \cdot \frac{2}{|M|^2}.$$

Great: since  $b_n \rightarrow M$ , there is  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$  we have  $|b_n - M| < \frac{\epsilon}{2|M|^2}$ . Take  $N = \max(n_1, n_2)$ ; then for all  $n > N$ , we have

$$\left| \frac{1}{b_n} - \frac{1}{M} \right| \leq |b_n - M| \cdot \frac{2}{|M|^2} < \frac{\epsilon}{2|M|^2} \frac{2}{|M|^2} = \epsilon.$$

e) The same provisos as part d) apply, of course: we may have  $b_n = 0$  for finitely many  $n$ , and we agree to start the sequence after the last such value. Having said that, since  $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$ , the result follows immediately by combining parts c) and d).  $\square$

**Example 1.5.5.** *An easy application of part b) of Proposition 1.5.4 yields the following result:*

*If  $\lim_{n \rightarrow \infty} a_n = L$  and  $r$  is a real number, then  $\lim_{n \rightarrow \infty} ra_n = rL$ .*

*To see how this follows from part b) above, just let  $\{b_n\}$  be the sequence given by  $b_n = r$  for all  $n$ .*

**Example 1.5.6.** *Proposition 1.5.4 also helps us to evaluate limits of sequences of rational expressions of  $n$ . For example if  $\{a_n\}$  is given by*

$$a_n = \frac{3n^2 + 2n - 2}{2n^2 + 3n + 5}$$

*we can rewrite this expression (by dividing top and bottom by  $n^2$ ) to get*

$$a_n = \frac{3 + \frac{2}{n} - \frac{2}{n^2}}{2 + \frac{3}{n} + \frac{5}{n^2}}.$$

*Applying parts a) and b) of Proposition 1.5.4 repeatedly (and using example 1.5.5) we see that*

$$\lim_{n \rightarrow \infty} \left( 3 + \frac{2}{n} - \frac{2}{n^2} \right) = 3$$



and

$$\lim_{n \rightarrow \infty} \left( 2 + \frac{3}{n} + \frac{5}{n^2} \right) = 2,$$

and so, by applying part c) of Proposition 1.5.4, we have

$$\lim_{n \rightarrow \infty} a_n = \frac{3}{2}.$$

**Proposition 1.5.7.** (*Three Sequence Principle*) Let  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  be real sequences, and let  $\alpha, \beta, \gamma$  be nonzero real numbers. Suppose that for all  $n \in \mathbb{Z}^+$  we have

$$\alpha a_n + \beta b_n + \gamma c_n = 0.$$

Then if any two of the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  converge, so does the third.

*Proof.* Because of the symmetry in the statement, we may suppose that  $a_n \rightarrow L$  and  $b_n \rightarrow M$ . Since  $c_n = \frac{-\alpha}{\gamma} a_n - \frac{\beta}{\gamma} b_n$  for all  $n \in \mathbb{Z}^+$ , we have

$$c_n \rightarrow \frac{-\alpha}{\gamma} L - \frac{\beta}{\gamma} M. \quad \square$$

The following result says roughly that if a sequence behaves in the limit like a convergent geometric sequence, then it too must converge:

**Proposition 1.5.8.** Let  $\{a_n\}$  be a sequence satisfying  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = r$ , with  $r < 1$ . Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

*Proof.* Since  $r < 1$ , we can find a real number  $c$  with  $r < c < 1$  (e.g.  $c = (r + 1)/2$ ). Now let  $\epsilon = c - r$  (notice that  $\epsilon > 0$ ) and choose  $N \in \mathbb{Z}^+$  so that

$$\left| \frac{|a_{n+1}|}{|a_n|} - r \right| < \epsilon$$

whenever  $n > N$ . Then, in particular, we have

$$\frac{|a_{n+1}|}{|a_n|} < r + \epsilon = r + (c - r) = c$$

whenever  $n > N$ . Rewriting this, we have  $|a_{n+1}| < c|a_n|$ , when  $n > N$ .

Now, since the existence (and value) of a limit does not depend on the first  $N$  terms of the sequence, we can assume that  $|a_{n+1}| < c|a_n|$  for **all**  $n \in \mathbb{Z}^+$ . A

simple induction argument then leads to the conclusion that  $|a_{n+1}| < c^n|a_1|$  for all  $n \in \mathbb{Z}^+$ . The sequence  $\{c^n\}$  is geometric with  $0 < c < 1$ , so we know that  $\lim_{n \rightarrow \infty} c^n|a_1| = 0$ . Thus, given  $\epsilon > 0$ , we can find  $N \in \mathbb{Z}^+$  so that  $|c^n a_1| < \epsilon$  for all  $n > N$ . But since  $|a_{n+1}| < |c^n a_1|$ , we conclude that  $|a_{n+1}| < \epsilon$  for all  $n > N$ . Thus  $\lim_{n \rightarrow \infty} a_n = 0$ .  $\square$

**Example 1.5.9.** Consider the sequence given by  $a_n = (-1)^n \frac{n}{2^n}$  for  $n \in \mathbb{Z}^+$ .

We have

$$\frac{|a_{n+1}|}{|a_n|} = \left( \frac{n+1}{2^{n+1}} \right) \left( \frac{2^n}{n} \right) = \frac{n+1}{2n}.$$

Using Proposition 1.5.4 we can compute that  $\lim \frac{n+1}{2n} = \frac{1}{2}$ , and so we conclude that  $\lim a_n = 0$ .

For the remainder of this section we will discuss two results from calculus which help us to evaluate even more complicated limits. The first of these is a theorem about continuous functions.

**Definition 1.5.10.** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called continuous at  $c \in \mathbb{R}$  if for every  $\epsilon > 0$  there is a  $\delta > 0$  so that

$$|f(x) - f(c)| < \epsilon,$$

whenever  $|x - c| < \delta$ .

The first result is then:

**Proposition 1.5.11.** Let  $L \in \mathbb{R}$ . Suppose that  $a_n \rightarrow L$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous at  $l$ . Then  $f(a_n) \rightarrow f(L)$ .

*Proof.* Given some  $\epsilon > 0$ , we need to find  $N \in \mathbb{Z}^+$  so that  $|f(a_n) - f(L)| < \epsilon$  for all  $n > N$ . Since  $f$  is continuous at  $L$  we know that there is a  $\delta > 0$  so that  $|f(x) - f(L)| < \epsilon$  whenever  $x$  satisfies  $|x - L| < \delta$ . This tells us that, if we can find  $N$  so that  $|a_n - L| < \delta$  for all  $n > N$ , then we will have the desired result. But the fact that  $a_n \rightarrow L$  tells us exactly this (using  $\delta$  in place of  $\epsilon$  in the definition of  $\lim a_n$ ). We can write this out formally as follows:

Fix  $\epsilon > 0$ . Since  $f$  is continuous at  $L$ , there is  $\delta > 0$  so that  $|f(x) - f(L)| < \epsilon$  whenever  $|x - L| < \delta$ . Since  $a_n \rightarrow L$ , we can find  $N \in \mathbb{Z}^+$  so that for every  $n > N$ , we have  $|a_n - L| < \delta$ . Thus for  $n > N$  we have  $|f(a_n) - f(L)| < \epsilon$ .  $\square$

**Example 1.5.12.** In Example 1.5.6 we saw that  $\lim_{n \rightarrow \infty} a_n = 3/2$  when  $a_n = \frac{3n^2+2n-2}{2n^2+3n+5}$ . Now if we consider the continuous functions  $f(x) = \sqrt{|x|}$ ,  $g(x) = x^2$ , we get

$$\lim_{n \rightarrow \infty} \frac{3n^2 + 2n - 2}{2n^2 + 3n + 5} = \lim_{n \rightarrow \infty} \sqrt{\left| \frac{3n^2 + 2n - 2}{2n^2 + 3n + 5} \right|} = \sqrt{3/2},$$

$$\lim_{n \rightarrow \infty} \left( \frac{3n^2 + 2n - 2}{2n^2 + 3n + 5} \right)^2 = 9/4,$$

The first equality holds, because: since  $a_n \rightarrow \frac{3}{2}$ , certainly  $a_n \geq 0$  eventually. (It might also be worth noting that the original  $a_n$  is given by  $f(\frac{1}{n})$  where  $f(x) = \frac{3+2x-2x^2}{2+3x+5x^2}$ , which is continuous at  $x = 0$ .)

For our other application of calculus to sequences, we first remark that often the sequences whose limits we are trying to evaluate can be written in the form

$$a_n = f(n)$$

for some familiar function  $f(x)$  which is actually defined for *all real numbers*,  $x \in \mathbb{R}$ . In such a case it can be useful to study

$$\lim_{x \rightarrow \infty} f(x)$$

to get information about  $\lim_{n \rightarrow \infty} a_n$ .

Indeed, recall from calculus that we have

**Definition 1.5.13.** Let  $f(x)$  be a function defined for all real  $x \in \mathbb{R}$  (or at least for all  $x$  larger than some real number  $a$ ). Then we say  $f(x)$  converges to the real number  $L$ , as  $x$  goes to infinity, if the following condition is true:

For every  $\epsilon > 0$  there is an  $R \in \mathbb{R}$  so that  $|f(x) - L| < \epsilon$  whenever  $x > R$ .

In this case we write  $\lim_{x \rightarrow \infty} f(x) = L$ .

From this definition it is straightforward to prove (see exercise 9)

**Proposition 1.5.14.** Let  $f$  be defined on  $\mathbb{R}$  and assume  $\lim_{x \rightarrow \infty} f(x) = L$ . For  $n \in \mathbb{Z}^+$ , let  $a_n = f(n)$ . Then  $\lim_{n \rightarrow \infty} a_n = L$ .

Well, the above observation doesn't really help much until we recall that there is a theorem from calculus that helps us evaluate limits of the above type. First we need to discuss a special kind of divergence.

**Definition 1.5.15.** *Let  $f$  be a function defined for all real  $x \in \mathbb{R}$ . Then we say  $f(x)$  diverges to infinity, as  $x$  goes to infinity, if the following condition is true:*

*For every  $M > 0$  there is an  $R \in \mathbb{R}$  so that  $f(x) > M$  whenever  $x > R$ .*

*In this case we write  $\lim_{x \rightarrow \infty} f(x) = \infty$ .*

**Example 1.5.16.** *The function  $f(x) = x$  diverges to infinity as  $x$  goes to infinity, but the function  $g(x) = x \sin(x)$  does not.*

We now recall a result which delights many calculus students and dismays many calculus instructors.

**Theorem 1.5.17. L' Hôpital's Rule:** *Suppose  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are differentiable functions satisfying*

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty \quad \text{or} \quad \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = 0.$$

*Then*

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$$

*as long as the limit on the right hand side exists (or diverges to infinity).*

*Proof.* See e.g. [Cl-HC, Thm. 7.1]. □

**Example 1.5.18.** *Let  $a_n = \log(n)/n$  for all  $n$ . Then we can write  $a_n = f(n)/g(n)$  where  $f(x) = \log(x)$  and  $g(x) = x$  for all  $x > 0$ . Now  $\lim_{x \rightarrow \infty} f(x) = \infty$  and  $\lim_{x \rightarrow \infty} g(x) = \infty$  so we can apply L'Hôpital's Rule to get.*

$$\lim_{x \rightarrow \infty} \frac{\log(x)}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

*Finally, applying Proposition 1.5.14 we conclude that*

$$\lim_{n \rightarrow \infty} \frac{\log(n)}{n} = 0.$$

**Remark 1.5.19.** *L'Hôpital's rule applies only to functions defined on  $\mathbb{R}$  so we must first convert our questions about sequences into questions about functions on  $\mathbb{R}$  before we apply this theorem. It makes no sense to differentiate an expression like  $\log(n)$ , so L'Hôpital's rule doesn't even make sense here.*

Nevertheless, like many results from calculus, L'Hôpital's Rule has a discrete analogue. See the (unfairly difficult, to be sure) Exercise 12.

## EXERCISES 1.5

1. For each of the following limits either evaluate or explain why it is divergent.
 

|   |   |
|---|---|
| a.) $\lim_{n \rightarrow \infty} \frac{\log(n)}{n}$               | b.) $\lim_{n \rightarrow \infty} \frac{e^n}{n}$           |
| c.) $\lim_{n \rightarrow \infty} \frac{n}{e^n}$                   | d.) $\lim_{n \rightarrow \infty} \frac{n^2}{e^n}$         |
| e.) $\lim_{n \rightarrow \infty} \frac{n}{2^n}$                   | f.) $\lim_{n \rightarrow \infty} \frac{n}{\log(n)}$       |
| g.) $\lim_{n \rightarrow \infty} \frac{\log(n)}{n^2}$             | h.) $\lim_{n \rightarrow \infty} \frac{n^2+1}{n \log(n)}$ |
| i.) $\lim_{n \rightarrow \infty} \frac{\cos(n)}{n}$               | j.) $\lim_{n \rightarrow \infty} \frac{n}{\sin(n)}$       |
| k.) $\lim_{n \rightarrow \infty} (n \sin(1/n))$                   | l.) $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$     |
| m.) $\lim_{n \rightarrow \infty} (1 + n)^{\frac{1}{n}}$           | n.) $\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{n}$   |
| o.) $\lim_{n \rightarrow \infty} \sqrt{n^2 + n} - \sqrt{n^2 - n}$ |   |
  
2. Fill in the details of part a) of Proposition 1.5.4, i.e., assume that  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = M$  and prove that  $\lim_{n \rightarrow \infty} (a_n + b_n) = L + M$ . (**Comment:** This problem is now obsolete since the result is proved in the text. It is left in to preserve the numbering.)
  
3. Prove that if the sequence  $\{a_n\}$  converges and the sequence  $\{b_n\}$  diverges, then the sequence  $\{a_n + b_n\}$  diverges.
  
4. Use the properties of limits to explain why  $\lim_{n \rightarrow \infty} \frac{3n+1}{2n+5} = 3/2$ . State clearly where you are using each of the properties.
  
5. Evaluate the following limits.
 

|  |
|--|
| a.) $\lim_{n \rightarrow \infty} \frac{3n^2-2n+7}{6n^2+3n+1}$                                    |
| b.) $\lim_{n \rightarrow \infty} \left( \frac{3n^2-2n+7}{6n^2+3n+1} \right)^3$                   |
| c.) $\lim_{n \rightarrow \infty} \sqrt{\frac{3n^2-2n+7}{6n^2+3n+1}}$                             |
| d.) $\lim_{n \rightarrow \infty} \left( \exp \left( \frac{3n^2-2n+7}{6n^2+3n+1} \right) \right)$ |
  
6. a) Prove that if  $\lim_{n \rightarrow \infty} a_n = \infty$  then  $\lim_{n \rightarrow \infty} 1/a_n = 0$ .

b) Assume that  $a_n \neq 0$  for all  $n \in \mathbb{Z}^+$  and that  $\lim_{n \rightarrow \infty} a_n = 0$ , then prove that  $\lim_{n \rightarrow \infty} |1/a_n| = \infty$ .

c) Provide an example of a sequence  $\{a_n\}$  of nonzero real numbers for which  $\lim_{n \rightarrow \infty} a_n = 0$  but  $\lim_{n \rightarrow \infty} 1/a_n = \infty$  is not true.

7. In this exercise we prove the *Squeeze Lemma*: Assume that  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  are sequences with  $a_n \leq b_n \leq c_n$  for all  $n \in \mathbb{Z}^+$ . Assume also that  $\lim a_n = \lim c_n = L$ . Then  $\lim b_n = L$ .

We break the proof into two pieces:

a) Assume  $\{r_n\}$  and  $\{s_n\}$  are sequences with  $0 \leq r_n \leq s_n$  and  $\lim s_n = 0$ . Use the definition of the limit to show that  $\lim r_n = 0$ .

b) Now consider sequences  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  with  $a_n \leq b_n \leq c_n$  and  $\lim a_n = \lim c_n = L$ . Let  $r_n = b_n - a_n$  and  $s_n = c_n - a_n$  and show that part a) can be applied to prove that  $\lim r_n = 0$ . Now use Proposition 1.5.4 to prove that  $\lim b_n = L$ .

8. Provide the details of the proof of Lemma 1.5.3.

9. Provide a proof for Proposition 1.5.14.

10. Prove that if  $a_n = f(n)$  and  $\lim_{x \rightarrow \infty} f(x) = \infty$ , then  $\lim_{n \rightarrow \infty} a_n = \infty$ .

11. Give the details of the proof for part c) of Proposition 1.5.4 in the case that  $M < 0$  and  $a_n = 1$  for all  $n$ . (Careful, you will need to state and prove a new version of Lemma 1.5.3.)

12. Prove the following theorem of **Stolz-Cesaro**:

Let  $\{a_n\}$  and  $\{b_n\}$  be real sequences. Suppose:

(i) The sequence  $\{b_n\}$  is strictly increasing and  $b_n \rightarrow \infty$ .

(ii) We have  $\lim_{n \rightarrow \infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = L$ .

Then  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$ .

## 1.6 Abstract Theorems on Convergence of Sequences: What is Reality?

### 1.6.1 O'Connor's Property of Completeness

So far, whenever we have shown that a sequence converges we have also determined the limit. In fact, it is not possible to use the definition of convergence without stating explicitly what the limiting value is. But it would be useful to have criteria for convergence that don't require knowledge of the limiting value. A very intuitive example of this is the idea that if a sequence is decreasing and bounded below then it should converge to some number which is no less than the lower bound for the sequence.

**Example 1.6.1.** (See exercise 1.3.20 and example 1.4.1.) Let  $a_n$  be the sequence given recursively by

$$a_1 = 2; \forall n \geq 1, a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}.$$

From this definition, it is clear that each  $a_n$  is a positive rational number. Almost nothing else is immediately clear! So let's compute the first few terms:

$$a_1 = 2,$$

$$a_2 = 3/2 = 1.5,$$

$$a_3 = 17/12 \approx 1.416666\dots,$$

$$a_4 = 577/408 \approx 1.41421568\dots$$

$$a_5 = 665857/470832 \approx 1.41421356237468\dots$$

$$a_6 = 886731088897/627013566048 \approx 1.41421356237309\dots$$

These computations suggest that the sequence is decreasing and that it converges to a limit  $L \approx 1.4142135623\dots$ . How do we prove that?

Let us show that the sequence is decreasing. First we observe that

$$a_n^2 = \frac{a_{n-1}^2}{4} + \frac{1}{a_{n-1}^2} + 1,$$

so

$$a_n^2 - 2 = \frac{a_{n-1}^2}{4} + \frac{1}{a_{n-1}^2} - 1 = \left( \frac{a_{n-1}}{2} - \frac{1}{a_{n-1}} \right)^2 \geq 0$$



for all  $n \geq 2$ . Thus we have

$$\forall n \in \mathbb{Z}^+, a_n - a_{n+1} = \frac{a_n}{2} - \frac{1}{a_n} = \frac{a_n^2 - 2}{2a_n} \geq 0,$$

so the sequence is decreasing.

Notice that we are in a new situation: we think our sequence converges because we can see approximately what (we think) it converges to. In terms of proving the sequence converges, this is of no help whatsoever: our definition of convergence requires us to know the limit exactly. Or rather, to know it exactly or correctly guess it exactly. Maybe that decimal expansion is familiar to you. If not, here is another approach. Suppose  $a_n \rightarrow L \in \mathbb{R}$ . Then

$$L = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{a_n}{2} + \frac{1}{a_n} = \frac{L}{2} + \frac{1}{L},$$

which leads to the equation  $L^2 = 2$ . We know by now that the limit of a sequence of positive numbers cannot be negative, so this argument shows that if the sequence converges, it must converge to  $L = \sqrt{2}$ .

Can we show the sequence converges now? In fact we will be distracted for a little while by an even more basic question: what is  $\sqrt{2}$ ??

The rational numbers  $\mathbb{Q}$  are fairly easy to comprehend. We begin with the positive numbers and then tack on 0 and the negative integers.<sup>8</sup> Next, to get the rationals, either we think of dividing our natural numbers into equal portions or we think of comparing whole numbers by their ratios. The ancient Greeks were happy with this concept, but they also knew that there are numbers that cannot be represented as a ratio of whole numbers. For example, although a segment of length  $\sqrt{2}$  is easily constructed as the hypotenuse of a right triangle with legs of length 1, the Greeks had proven the following:

**Proposition 1.6.2.** *There are no natural numbers  $a$  and  $b$  so that  $\sqrt{2} = a/b$ , i.e.,  $\sqrt{2}$  is not rational.*

*Proof.* Let's suppose on the contrary that we have natural numbers  $a$  and  $b$  with  $\sqrt{2} = a/b$ . Of course, we can assume that  $a$  and  $b$  have no common divisors, since if they did we could "cancel" those divisors to get an equivalent

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<sup>8</sup>That 0 is a number was not "natural" in many parts of the world until relatively recently.

fraction. That is, we assume that the fraction  $a/b$  is in “lowest terms”. Now, if  $a/b = \sqrt{2}$ , then we have  $a = \sqrt{2}b$ , and so  $a^2 = 2b^2$ . Since  $2b^2$  is an even number, we conclude that  $a^2$  is even. But, the square of an odd number is odd, so if  $a^2$  is even it must be the case that  $a$  is even too. Thus  $a$  can be written as  $a = 2l$  for some natural number  $l$ . Squaring this we see that  $a^2 = 4l^2$ . Now we substitute back into the equation  $a^2 = 2b^2$  to get  $4l^2 = 2b^2$ . Cancelling a factor of 2 yields  $b^2 = 2l^2$ . Now we can repeat the above argument to conclude that  $b$  is even too (since  $b^2$  is even). Thus we have shown that both  $a$  and  $b$  are even, contradicting the assumption that  $a/b$  is in “lowest terms”. Since this argument leads to a contradiction, we must conclude that the original assumption that  $\sqrt{2} = a/b$  is incorrect.  $\square$

So, then, if  $\sqrt{2}$  is not rational, what sort of number is it? If we punch  $\sqrt{2}$  into our calculator we get a numerical readout like 1.414213562. But this is a rational number (represented in fraction form by  $\frac{1,414,213,562}{1,000,000,000}$ ) so it can't really be  $\sqrt{2}$ ! In fact, any finite decimal represents a rational number, so irrational numbers cannot have a finite decimal representation. Well then, as we have been taught in high school, the decimal representation for irrationals like  $\sqrt{2}$  must “go on for ever”. What is the meaning of decimals that “go on for ever”? One reasonable interpretation is that the decimal represents a sequence of better and better approximations of the actual number  $\sqrt{2}$ . If we define a sequence with

$$a_1 = 1, a_2 = 1.4, a_3 = 1.41, a_4 = 1.414, \text{ etc.}$$

then we might say that  $\sqrt{2}$  is *defined* to be the limit of this sequence. But why does this sequence have a limit if the thing it converges to is defined by the limit? This all seems pretty circular but, in fact, if one is careful one can use these ideas to give a precise definition of the real numbers.

Rather than constructing the real numbers from scratch, our approach for this course (and for most undergraduate courses; the alternative would be very unpleasant to many students) is to come up with a list of properties that we expect the real numbers to have, describe a set which has these properties, and then prove that the properties uniquely define the set up to a change of name. (It doesn't matter what symbols we use for our numbers, as long as we can provide a dictionary from one set of symbols to the other.) The list of properties include the many properties of numbers which we are familiar with from grade school. There are the algebraic properties such as commutativity, associativity and distributivity as well as the order properties

i.e., the properties of inequalities (some are mentioned in section 1.3). Of course when you start listing properties of numbers you will notice that there are infinitely many things that you could write down but that many of the properties can actually be proven from some of the other properties – so the game is to try to find a list of properties from which all of the other ones follow.

However, after writing the list of algebraic and order properties of the real numbers it becomes clear that there must be some further property that distinguishes the real numbers from other sets of numbers, since, for instance, the rational numbers have all of the same algebraic and order properties as the real numbers. To put a finer point on it, in Example 1.6.1 we met a decreasing sequence of positive rational numbers which looks like it is convergent. But we showed that if it converges to a number  $L$ , then  $L^2 = 2$ , so if we worked within the rational numbers only, that sequence would not be convergent! This means that we need some additional axiom that applies to the real numbers and not to the rational numbers, from which we can deduce that this sequence indeed converges in  $\mathbb{R}$  to  $\sqrt{2}$ . The missing ingredient is some kind of axiom of *completeness*. In fact there are several forms such a property may take. We begin with O’Connor’s<sup>9</sup> Property of Completeness, which is most immediately adapted to our present sequential approach.

**O’Connor’s Property of Completeness:**

Every bounded increasing real sequence is convergent.

It is easy to see that O’Connor’s Property of Completeness also implies that a bounded *decreasing* real sequence converges. Indeed, if  $\{b_n\}$  is a decreasing sequence bounded below by  $m$ , then  $\{-b_n\}$  is an increasing sequence bounded above by  $-m$ . It follows from O’Connor’s Property of Completeness that  $\{-b_n\}$  converges, say to  $L$ . Then  $\{b_n\}$  converges to  $-L$ .

In particular, O’Connor’s Property of Completeness finally allows us to complete our discussion of the sequence studied in Example 1.6.1: being a decreasing sequence of positive real numbers, it is also bounded, and therefore it converges by O’Connor’s Property of Completeness. We already showed that if it converges to any real number, it converges to  $\sqrt{2}$ , so indeed it

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<sup>9</sup>Mary Flannery O’Connor (1925–1964) was a key figure in the Southern Gothic literary school. She is (according to Pete L. Clark) Georgia’s greatest writer. Her last short story collection, published posthumously, takes its name from one of its stories, *Everything that rises must converge*. Though she omitted “boundedness,” we have decided to name this version of completeness after her.

converges to  $\sqrt{2}$ . Before we go on with our grand general discussion, let us complete the story of this particular sequence. This sequence corresponds to an algorithm used by the ancient Babylonians to compute  $\sqrt{2}$ . Moreover, it is precisely the sequence of approximations to the root of  $f(x) = x^2 - 2$  given by Newton's method starting at  $x_1 = 1$ : see Example 1.4.1. (Thus recognizing the terms of the sequence as numerical approximations to  $\sqrt{2}$  hits a little too close to home: you have to ask yourself how your calculator / software package / iphone app is computing  $\sqrt{2}$ . There is a good chance that it is using this sequence!) This places the sequence in a much more general context but does not immediately give a proof of convergence: rather it raises the much more general and interesting issue of when Newton's method converges. There are general results on this – see e.g. [Cl-HC, Thm. 7.11] – but we will not discuss them here.

## 1.6.2 Bolzano-Weierstrass

We can now prove the single most important result of the first chapter.

**Theorem 1.6.3** (Bolzano-Weierstrass). *Every bounded real sequence has a convergent subsequence.*

*Proof.* Let  $\{a_n\}$  be a bounded real sequence. By the Rising Sun Lemma, there is a monotone subsequence  $\{a_{n_k}\}$ , which is of course still bounded. By O'Connor's Property of Completeness, the sequence  $\{a_{n_k}\}$  converges.  $\square$

**Example 1.6.4.** *The sequence given by  $a_n = \sin(n)$  is bounded between -1 and 1 but is certainly not convergent. As a challenge try to find an explicit subsequence which does converge.*

## 1.6.3 Dedekind's Property of Completeness

We come now to an all-important definition. Let  $S$  be a nonempty subset of  $\mathbb{R}$ . We know what it means for  $S$  to be bounded above: there must exist  $M \in \mathbb{R}$  such that  $x \leq M$  for all  $M \in S$ . Notice that by starting with an upper bound for  $S$  and making it larger, we still get an upper bound. In particular, if  $M$  is an upper bound for  $S$  then so is  $M + n$  for all  $n \in \mathbb{Z}^+$  and thus *the set of upper bounds of  $S$*  is not itself bounded above: there is no largest upper bound.

What if we try to make the upper bound smaller? That is a much more

interesting question! We may or may not still get an upper bound. For example, if  $S = [0, 1]$  and  $M_1 = 2$ , then  $M_2 = \frac{3}{2}$  is smaller and still an upper bound, but  $M_3 = \frac{1}{2}$  is not: it is smaller than the element 1 of  $S$ . In this case there is a clear choice of a “best upper bound” for  $S$ , namely  $M = 1$ . This is a best choice because on the one hand it is an upper bound for  $S$  and on the other hand it is an element of  $S$ . An upper bound for a set which is also an element of the set is precisely a *maximum*, a concept we have seen before. Certainly if a set  $S$  has a maximum  $M$ , then that maximum is the *least* upper bound: if  $M' < M$ , then.... $M' < M$  (and  $M \in S!$ ), so  $M'$  is not an upper bound.

However, in general an infinite subset of  $\mathbb{R}$  which is bounded above need not have a maximum. For instance, consider  $S = [0, 1)$ . Then for all  $x \in S$ ,  $\frac{x+1}{2}$  still lies in  $S$  and is greater than  $x$ . On the other hand, 1 is still an upper bound for  $S$ , and moreover it is the *least upper bound*: if we push any farther to the left than 1, then we have pushed past elements of  $S$ . (The argument for this is very close to the one we just gave. Think about it.)

We pause for a point of terminology: a least upper bound for a subset  $S \subset \mathbb{R}$  is also called a *supremum* for  $S$ , and sometimes written  $\sup(S)$ . (Clearly the supremum, when it exists, is unique: two different numbers cannot both be least!) This leads to the question: does every nonempty subset  $S \subset \mathbb{R}$  which is bounded above have a least upper bound? The more you think about this question, the more plausible it becomes that it should have an affirmative answer. It basically means this: if you start out “entirely to the right” of a nonempty subset  $S$  and start walking to the left, keeping yourself entirely to the right of  $S$ , then there is a definite point at which your journey stops: you have not walked *past* any element of  $S$  (though you may be standing precisely on top of an element of  $S$ ), and if you walk any farther, no matter how small a distance, then you will walk past elements of  $S$ .

We assert that this is a true, and fundamental, property of  $\mathbb{R}$ .

**Dedekind’s Property of Completeness:**

Every nonempty subset of the real numbers which is bounded above has a least upper bound (or supremum).

**Proposition 1.6.5.** *O’Connor’s Property of Completeness implies the Archimedean*

*Property.*

*Proof.* Seeking a contradiction, we suppose the Archimedean Property does not hold. This means precisely that the sequence in which  $a_n = n$  for all  $n \in \mathbb{Z}^+$  is bounded, so according to O'Connor's Property of Completeness, it would therefore converge, say to  $L$ . This means that there is some  $N \in \mathbb{Z}^+$  such that for all  $n > N$  we have  $|a_n - L| < \frac{1}{2}$ . In particular then, taking  $n = N + 1$  and  $n = N + 2$  we get

$$|n + 1 - L| < \frac{1}{2}, |n + 2 - L| < \frac{1}{2},$$

so

$$1 = |(n+1) - (n+2)| = |(n+1-L) - (n+2-L)| \leq |n+1-L| + |n+2-L| < \frac{1}{2} + \frac{1}{2} = 1.$$

Contradiction! □

**Proposition 1.6.6.** *Dedekind's Property of Completeness implies O'Connor's Property of Completeness.*

*Proof.* Suppose  $\{a_n\}$  is an increasing real sequence which is bounded above. Then the image of the sequence is a nonempty subset of  $\mathbb{R}$  which is bounded above, so by Dedekind's Property of Completeness, it has a supremum, say  $L$ . To spell it out, there is  $L \in \mathbb{R}$  such that  $a_n \leq L$  for all  $n \in \mathbb{Z}^+$  and if  $L' < L$  then  $a_n > L'$  for at least one  $n \in \mathbb{Z}^+$ . We claim  $a_n \rightarrow L$ . To see this, let  $\epsilon > 0$ . Because the sequence is increasing, it's enough to show that for some  $N \in \mathbb{Z}^+$  we have  $a_N > L - \epsilon$ ; then for all  $n \geq N$  we will have

$$L - \epsilon < a_N \leq a_n \leq L < L + \epsilon.$$

If this were not the case then we'd have  $a_n \leq L - \epsilon$  for all  $n$ ...but then  $L - \epsilon$  would be an upper bound for the image of  $\{a_n\}$  which is smaller than the least upper bound  $L$ : contradiction. □

And conversely:

**Proposition 1.6.7.** *O'Connor's Property of Completeness implies Dedekind's Property of Completeness.*

*Proof.* (Based on [K04, Exc. 3.9, p.31].) Let  $S$  be a nonempty subset of  $\mathbb{R}$  which is bounded above. We may assume that  $S$  has no maximum, since a supremum exists in that case. Let  $a_0 \in S$  and let  $b_0$  be an upper bound for  $S$ , so  $a_0 \leq b_0$ . In fact,  $a_0 < b_0$ : if  $a_0 = b_0$ , then  $a_0$  is a maximum for  $S$ .

Now we subdivide the interval  $[a_0, b_0]$  into two intervals of equal length, namely  $[a_0, \frac{a_0+b_0}{2}]$  and  $[\frac{a_0+b_0}{2}, b_0]$ . If there are any elements of  $S$  in the right subinterval  $[\frac{a_0+b_0}{2}, b_0]$  then we “choose” the right subinterval: formally, we take  $a_1$  to be its left endpoint  $\frac{a_0+b_0}{2}$  and  $b_1$  to be its right endpoint  $b_0$ . Otherwise, we “choose” the left subinterval: we take  $a_1$  to be its left endpoint  $a_0$  and  $b_1$  to be its right endpoint  $\frac{a_0+b_0}{2}$ .

Now we subdivide the interval  $[a_1, b_1]$  into two subintervals of equal length, according to exactly the same procedure: namely, we choose the right subinterval if any element of  $S$  lies in it, and otherwise choose the the left subinterval, either way getting an interval  $[a_2, b_2]$ .

We proceed in this way for all  $n \in \mathbb{Z}^+$ , getting a sequence of nested closed subintervals

$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots \supset [a_n, b_n] \supset \dots,$$

such that  $b_n - a_n = \frac{b_0 - a_0}{2^n}$ . The sequence  $\{b_n\}$  is decreasing and bounded below by  $a_0$ , so by O’Connor’s Property of Completeness it converges to  $b$  (say). We claim that  $b$  is the supremum of  $S$ . First we show that  $b$  is an upper bound for  $S$ : if not, there is  $x \in S$  such that  $b < x$ . Since  $\{b_n\}$  is decreasing and converges to  $b$ , there is some  $n$  such that  $b < b_n < x$  and thus  $b_n$  is not an upper bound for  $S$ : this contradicts our construction. Now suppose that there is a  $c < b$  which is an upper bound for  $S$ . Because  $c$  is not a maximum, we have  $b - x > b - c$  for all  $x \in S$ . Because O’Connor’s Property of Completeness implies the Archimedean Property, we have  $b_n - a_n = \frac{b_0 - a_0}{2^n} \rightarrow 0$ , and thus there is  $N \in \mathbb{Z}^+$  such that  $b_n - a_n < b - c$ . By construction there is  $x \in S \cap [a_n, b_n]$ , and thus

$$b - x \leq b_n - x \leq b_n - a_n < b - c,$$

contradiction. Thus we have  $b = \sup(S)$ . □

### 1.6.4 Cauchy’s Property of Completeness

**Definition 1.6.8.** *A real sequence  $\{a_n\}$  is a Cauchy sequence if it satisfies the following: for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that  $|a_n - a_m| < \epsilon$  for all  $m, n \geq N$ .*

**Remark 1.6.9.** *By now I hope it's clear that if we had written "for all  $m, n > N$ " then this would be an equivalent definition.*

**Proposition 1.6.10.** *Every convergent sequence in  $\mathbb{R}$  is a Cauchy sequence.*

*Proof.* Suppose  $a_n \rightarrow L$ , and fix  $\epsilon > 0$ . Let  $N \in \mathbb{N}$  be such that for all  $n > N$  we have  $|a_n - L| < \frac{\epsilon}{2}$ . So, if  $m, n \geq N$ , we have

$$|a_n - a_m| = |(a_n - L) + (L - a_m)| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square$$

This proof has a moral: two quantities which can each be made arbitrarily close to a third quantity can be made arbitrarily close to each other.

**Theorem 1.6.11.** *Every Cauchy sequence of real numbers converges to some real number.*

*Proof.* Let  $\{a_n\}$  be a Cauchy sequence in  $\mathbb{R}$ . The strategy is to show that  $\{a_n\}$  is bounded, then apply Bolzano-Weierstrass to get a subsequence converging to some  $L \in \mathbb{R}$ , then finally to show that  $\{a_n\}$  itself converges to  $L$ .

Step 1: Since  $\{a_n\}$  is Cauchy, there is  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ , we have  $|a_m - a_n| < 1$ . Taking  $m = N$ , we get that for all  $n > N$  that  $|a_N - a_n| < 1$ , so by the Reverse Triangle Inequality we have  $||a_n| - |a_N|| < 1$  and thus  $|a_n| < |a_N| + 1$ . Therefore we have for all  $n \in \mathbb{Z}^+$  that

$$|a_n| \leq \max(|a_1|, \dots, |a_N|, |a_N| + 1).$$

Step 2: By the Bolzano-Weierstrass Theorem, there is a subsequence  $\{a_{n_k}\}$  converging to some  $L \in \mathbb{R}$ . (We spoiled this a bit, huh?)

Step 3: Let  $\epsilon > 0$ . Choose  $N_1 \in \mathbb{N}$  such that  $|a_{n_k} - L| < \frac{\epsilon}{2}$  for all  $k > N$  and  $N_2 \in \mathbb{N}$  such that for all  $m, n > N_2$  we have  $|a_m - a_n| < \frac{\epsilon}{2}$ , and let  $N = \max(N_1, N_2)$ . Let  $n > N$ , and choose  $k$  such that  $n_k > N$ . Then

$$\begin{aligned} |a_n - L| &= |(a_n - a_{n_k}) + (a_{n_k} - L)| \\ &\leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square \end{aligned}$$

In view of Proposition 1.6.10 and Theorem 1.6.11, a real sequence is a Cauchy sequence if and only if it is convergent. So one may ask: why introduce Cauchy sequences at all, if they turn out to be precisely the convergent



sequences? The answer is that this equivalence is very useful in proving that a sequence converges. Our definition of convergence is “converges to  $L$  for some  $L \in \mathbb{R}$ .” In order to prove that a sequence converges, we need to know (in some sense) the limit. O’Connor’s Property of Completeness gets around this by asserting that a bounded monotone sequence convergence without saying what its limit is: as we saw, this is extremely useful, but it applies only to monotone sequences. Knowing that convergent real sequences are precisely the Cauchy sequences gives us a *general* method of showing the convergence of a real sequence without having any idea what the limit may be: we just need to show that – in Cauchy’s precise sense – the terms of the sequence eventually get sufficiently close together.

**Lemma 1.6.12.** *Let  $\{a_n\}$  be an increasing real sequence. The following are equivalent:*

(i) *The sequence  $\{a_n\}$  is Cauchy.*

(ii) *For all  $\epsilon > 0$  there is an upper bound  $B$  for the sequence and  $N \in \mathbb{Z}^+$  such that  $B - a_N \leq \epsilon$ .*

*Proof.* (i)  $\implies$  (ii): Assume (i). Then for all  $\epsilon > 0$ , there is  $N \in \mathbb{Z}^+$  such that for all  $m, n > N$  we have  $|a_m - a_n| < \epsilon$ . So for all  $n > N$  we have  $a_n < a_N + \epsilon$  and thus  $B = a_N + \epsilon$  works. Conversely, assume t

$$a_N \leq a_n < a_N + \epsilon.$$

(ii)  $\implies$  (i): Assume (ii), fix  $\epsilon > 0$ , and let  $B$  be an upper bound for the sequence and  $N \in \mathbb{Z}^+$  such that  $B - a_N \leq \frac{\epsilon}{4}$ . Then for all  $n > N$  we have

$$a_N \leq a_n \leq B \leq a_N + \frac{\epsilon}{4} < a_N + \frac{\epsilon}{2}.$$

Thus  $|a_n - a_N| < \frac{\epsilon}{2}$ . So if  $m, n > N$ , we have

$$|a_m - a_n| \leq |a_m - a_N| + |a_n - a_N| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square$$

**Theorem 1.6.13.** *a) The Archimedean Property is equivalent to the assertion that every bounded increasing sequence is a Cauchy sequence.*

*b) Cauchy’s Property of Completeness together with the Archimedean Property implies O’Connor’s Property of Completeness.*

*Proof.* a) Suppose the Archimedean Property holds, and let  $\{a_n\}$  be an increasing sequence bounded above by  $B_0$ . By Lemma 1.6.12, to show that  $\{a_n\}$  is Cauchy it is enough to find, for all  $\epsilon > 0$ , an upper bound  $B_\epsilon$  for the sequence and an  $N \in \mathbb{Z}^+$  such that  $B_\epsilon - a_N \leq \epsilon$ . Put  $d_0 = B_0 - a_1$ . Then  $B_0$  is an upper bound which differs from some term of the sequence by at most  $d_0$ . We claim that it is enough to show that whenever we have an upper bound  $B_n$  and  $N_n \in \mathbb{Z}^+$  such that  $B_n - a_{N_n} \leq D$  (for some  $D > 0$ ), then there is also an upper bound  $B_{n+1}$  and  $N_{n+1} \in \mathbb{Z}^+$  such that  $B_{n+1} - a_{N_{n+1}} \leq \frac{D}{2}$ . If so, then for all  $n$  we can get an upper bound and a term in the sequence which differ by at most  $\frac{d_0}{2^n}$ . By the Archimedean Property,  $\frac{d_0}{2^n} \rightarrow 0$ , so for sufficiently large  $n$  it is at most  $\epsilon$ , giving what we want. The proof of this is somewhat familiar: consider

$$m_n = \frac{a_{N_n} + B_n}{2},$$

the midpoint of the interval  $[a_{N_n}, B_n]$ . If  $m_n$  is an upper bound for  $\{a_n\}$ , we take  $B_{n+1} = m_n$  and  $N_{n+1} = N_n$ , and now we have an upper bound and a term of the sequence differing by  $\frac{1}{2}$  as much as the previous ones. If  $m_n$  is not an upper bound, then there is some  $N_{n+1}$  such that  $a_{N_{n+1}} > m_n$ , and taking  $B_{n+1} = B_n$ , again we have decreased the difference between the upper bound and a term in the sequence by a factor of 2.

Conversely, if the Archimedean Property failed, then  $a_n = n$  is a bounded increasing sequence, and it is certainly not Cauchy: indeed for all  $n \in \mathbb{Z}^+$  we have  $|a_{n+1} - a_n| = 1$ , so the Cauchy condition fails for  $\epsilon = 1$ .

b) Assuming the Archimedean Property and Cauchy's Property of Completeness, part a) implies that every bounded increasing sequence converges.  $\square$

After being subjected to a barrage of different completeness properties, the reader is probably wondering what's going on. The answer is that we are gaining some access into the foundations of the real number system. If we take it for granted that the real numbers satisfy Dedekind's Property of Completeness, then as we have seen the completeness properties of O'Connor and Cauchy follow rather easily. There remains the question of giving a rigorous definition/construction of the system of real numbers. One's first effort might be to define real numbers in terms of their decimal expansions. This turns out to be possible, but unrewardingly difficult. Dedekind himself gave the first satisfactory construction of the real numbers, using *Dedekind cuts*. These too are rather difficult and tedious to work with, and (worse)

a technical facility with Dedekind cuts is of little or no use to the majority of students of mathematics.<sup>10</sup> Arguably the quickest and most transparent construction of the real numbers is in terms of a certain equivalence relation on Cauchy sequences of rational numbers. This construction is given in complete detail (and in fact, in a bit more generality) in [Cl-HC, Ch. 16].<sup>11</sup> From this construction it comes out quite naturally that the real numbers satisfy the Archimedean property and Cauchy's Property of Completeness. Thus we took some (relatively small!) pains here to show that these imply the other completeness properties.

### EXERCISES 1.6

- Suppose that  $a$  is a natural number and that  $a^2$  is divisible by 3, show that  $a$  is divisible by 3. (Hint: show that if  $a$  is not divisible by 3, then neither is  $a^2$ .)
  - Show that  $\sqrt{3}$  is not rational.
- Show that the sequence of Example 1.6.1 is *strictly* decreasing. (Hint: use the irrationality of  $\sqrt{2}$ .)
- Use Newton's method to construct a sequence that converges to  $\sqrt{3}$ . Prove that your sequence is convergent to  $\sqrt{3}$ .
- Let  $\{a_n\}$  be the sequence defined by  $a_1 = 1$  and  $a_{n+1} = a_n + \frac{1}{(n+1)^2}$ . Prove that this sequence converges. (Hint: use induction to prove that  $a_n \leq 2 - 1/n$  for all  $n \in \mathbb{Z}^+$ .)
- Show that any real sequence admits a subsequence which is either convergent, divergent to  $\infty$  or divergent to  $-\infty$ .

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<sup>10</sup>One must read this at least in part as a statement of the cultural status quo. There is a branch of mathematics in which these ideas get an attractive development, namely the theory of partially ordered sets. It just happens that the majority of working mathematicians have no acquaintance with this theory.

<sup>11</sup>Honesty compels me to add that even this treatment is magnificently unpalatable for consumption by most real life undergraduates. There are things in mathematics that one is comforted to see that someone else has worked out but does not wish to explore on one's own.

(Hint: any real sequence is either bounded, unbounded above or unbounded below. Use previous results and exercises!)

6. Recall that a sequence  $\{A_n\}_{n=1}^{\infty}$  of sets is *nested* if we have

$$A_1 \supseteq A_2 \supseteq \dots A_n \supseteq A_{n+1} \supseteq \dots$$

a) (**Nested Interval Theorem**) Let  $\{A_n\}_{n=1}^{\infty}$  be a nested sequence of closed bounded intervals in  $\mathbb{R}$ : i.e., for all  $n \in \mathbb{Z}^+$  we have  $A_n = [a_n, b_n]$  with  $a_n \leq b_n$  such that  $A_{n+1} \subseteq A_n$ . Show that  $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ : that is, there is an  $x$  which lies in  $A_n$  for all  $n \in \mathbb{Z}^+$ .

(Suggestion: consider the sequences  $\{a_n\}$  and  $\{b_n\}$ .)

b) Under the hypotheses of part a), suppose that the intervals approach 0 in length:  $\lim_{n \rightarrow \infty} b_n - a_n = 0$ . Show that  $\bigcap_{n=1}^{\infty} A_n$  consists of *exactly* one point.

7. a) For each  $n \in \mathbb{Z}^+$  let  $A_n$  be the interval  $A_n = [n, \infty)$ . Show that  $A_{n+1} \subset A_n$  for all  $n \in \mathbb{Z}^+$  and yet there is no number  $\alpha \in \mathbb{R}$  with  $\alpha \in A_n$  for all  $n$ . Why does this not contradict the Nested Interval Theorem?  
 b) For each  $n \in \mathbb{Z}^+$  let  $A_n$  be the interval  $A_n = (0, 1/n]$ . Show that  $A_{n+1} \subset A_n$  for all  $n \in \mathbb{Z}^+$  and yet there is no number  $\alpha \in \mathbb{R}$  with  $\alpha \in A_n$  for all  $n$ . Why does this not contradict the Nested Interval Theorem?

8. Let  $\{a_n\}$  be a sequence which is bounded above. We define a new sequence  $\{b_n\}$  by  $b_n = \max\{a_1, a_2, \dots, a_n\}$ .

a) Show that the sequence  $\{b_n\}$  is increasing. (*Hint*: Notice that if  $A$  and  $B$  are finite sets of real numbers with  $A \subseteq B$ , then  $\max A \leq \max B$ .)

b) Assume that  $U$  is an upper bound for  $\{a_n\}$ , show that  $U$  is an upper bound for  $\{b_n\}$ .

c) From O'Connor's Property of Completeness we know that  $\{b_n\}$  must have a limit, call it  $L$ . Prove that  $a_n \leq L$  for all  $n \in \mathbb{Z}^+$ , i.e.,  $L$  is an upper bound for  $\{a_n\}$ . Also prove that if  $U$  is any upper bound for  $\{a_n\}$  then  $L \leq U$ . Thus  $L$  is the *least upper bound* of the sequence  $\{a_n\}$ .

Similarly, if the sequence  $\{a_n\}$  is bounded below, we can use O'Connor's Property of Completeness to prove that it has a *greatest lower bound*.

9. Let  $\{a_n\}_{n=1}^{\infty}$  be a bounded sequence. For each  $k \in \mathbb{Z}^+$  consider the subsequence which begins with  $a_k$  and continues on from there, i.e.,  $\{a_n\}_{n=k}^{\infty}$ .

Let  $L_k = \sup_{n \geq k} a_n$  denote the least upper bound of the  $k^{\text{th}}$  such subsequence. (This least upper bound is shown to exist in exercise 6.)

- a) Prove that the sequence  $\{L_k\}$  is decreasing.
- b) Prove that if  $M$  is a lower bound for the sequence  $\{a_n\}$  then it is also a lower bound for the sequence  $\{L_k\}$ .

From the O'Connor's Property of Completeness, we conclude that the sequence  $\{L_k\}$  has a limit,  $\inf L_k$ . This limit is called the *limsup* of the sequence  $\{a_n\}$ , and is denoted by  $\limsup a_n$ . Notice that

$$\limsup a_n = \inf_{k \geq 1} \sup_{n \geq k} a_n.$$

10. Let  $\{a_n\}$  be a bounded sequence and let  $L = \limsup a_n$ .
  - a) Show that there is a convergent subsequence,  $\{b_n\}$ , of  $\{a_n\}$  with  $\lim_{n \rightarrow \infty} b_n = L$ .  
(This gives an alternate proof of the Bolzano-Weierstrass Theorem.)
  - b) Prove that if  $\{c_n\}$  is a convergent subsequence of  $\{a_n\}$  then

$$\lim_{n \rightarrow \infty} c_n \leq L.$$

## 1.7 Applications to Calculus

Armed with a deeper knowledge of the real numbers, we are now able to prove some of the most important results about continuous functions on  $\mathbb{R}$  that are usually stated without proof in introductory calculus courses.

### 1.7.1 Continuous Functions

We need to mildly extend our definition of a continuous function. Let  $X \subset \mathbb{R}$ , let  $f : X \rightarrow \mathbb{R}$  be a function, and let  $c \in X$ . We say  $f$  is *continuous at*  $c$  if for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x \in X$ , if  $|x - c| < \delta$  then  $|f(x) - f(c)| < \epsilon$ . A function  $f : X \rightarrow \mathbb{R}$  is *continuous* if it is continuous at every  $c \in X$ .

Notice that when  $X = \mathbb{R}$  we recover our earlier definition of continuity. The only change in general is that we only consider values of  $x$  lying in  $X$ , i.e., values at which the function is defined. We emphasize that in this text we will concentrate on the case in which  $X$  is an interval.

**Example 1.7.1.** *A function  $f : X \subset \mathbb{R} \rightarrow \mathbb{R}$  is constant if for all  $x, y \in X$  we have  $f(x) = f(y)$ . In this case there is  $C \in \mathbb{R}$  such that  $f(x) = C$  for all  $x \in X$ . Constant functions are continuous at every  $c \in X$ . Indeed, for any  $\epsilon > 0$ , we may choose any  $\delta > 0$  – for the sake of definiteness, let’s take  $\delta = 1$ . Then for all  $x \in X$  we have  $|f(x) - f(c)| = |C - C| = 0 < \epsilon$  (no matter what  $\delta$  was).*

**Proposition 1.7.2.** *Let  $X \subset \mathbb{R}$ , let  $f, g : X \rightarrow \mathbb{R}$ , and let  $c \in X$ .*

a) *If  $f$  and  $g$  are both continuous at  $c$ , then the function*

$$f + g : X \rightarrow \mathbb{R}, \quad x \mapsto f(x) + g(x)$$

*is also continuous at  $c$ .*

b) *If  $f$  is continuous at  $c$  and  $A \in \mathbb{R}$  then the function*

$$Af : X \rightarrow \mathbb{R}, \quad x \mapsto Af(x)$$

*is also continuous at  $c$ .*

c) *If  $f$  and  $g$  are both continuous at  $c$ , then the function*

$$fg : X \rightarrow \mathbb{R}, \quad x \mapsto f(x)g(x)$$

*is also continuous at  $c$ .*

d) *If  $f$  and  $g$  are both continuous at  $c$  and  $0 \notin g(X)$ , then the function*

$$\frac{f}{g} : X \rightarrow \mathbb{R}, \quad x \mapsto f(x)/g(x)$$

*is also continuous at  $c$ .*

*Proof.* Throughout it will be implicit that we only consider  $x$  in the domain  $X$  of  $f$  and  $g$ . Thus when we write “for all  $x$  with  $|x - c| < \delta$ ”, we mean “for all  $x \in X$  with  $|x - c| < \delta$ .” In each part we work out the idea of the proof first and then translate it into a formal  $\epsilon$ - $\delta$  argument.

a) Fix  $\epsilon > 0$ . We must show that there exists  $\delta > 0$  such that  $|x - c| < \delta$  implies  $|f(x) + g(x) - (f(c) + g(c))| < \epsilon$ . Now

$$\begin{aligned} |f(x) + g(x) - (f(c) + g(c))| &= |(f(x) - f(c)) + (g(x) - g(c))| \\ &\leq |f(x) - f(c)| + |g(x) - g(c)|. \end{aligned}$$

This is good: since  $f$  and  $g$  are both continuous at  $c$ , we can make each of  $|f(x) - f(c)|$  and  $|g(x) - g(c)|$  as small as we like by taking  $x$  sufficiently close to  $c$ . The sum of two quantities which can each be made as small as we like can be made as small as we like.

Now formally: choose  $\delta_1 > 0$  such that  $|x - c| < \delta_1$  implies  $|f(x) - f(c)| < \frac{\epsilon}{2}$ . Choose  $\delta_2 > 0$  such that  $|x - c| < \delta_2$  implies  $|g(x) - g(c)| < \frac{\epsilon}{2}$ . Let  $\delta = \min(\delta_1, \delta_2)$ . Then  $|x - c| < \delta$  implies  $|x - c| < \delta_1$  and  $|x - c| < \delta_2$ , so

$$|f(x) + g(x) - (f(c) + g(c))| \leq |f(x) - f(c)| + |g(x) - g(c)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

b) Fix  $\epsilon > 0$ . We must show that there exists  $\delta > 0$  such that  $|x - c| < \delta$  implies  $|Af(x) - Af(c)| < \epsilon$ . But  $|Af(x) - Af(c)| = |A||f(x) - f(c)|$ . Moreover, precisely because  $f$  is continuous at  $c$  we may make the quantity  $|f(x) - f(c)|$  as small as we like by taking  $x$  sufficiently close to  $c$ . A quantity which we can make as small as we like times a constant can still be made as small as we like.

Now formally: we may assume  $A \neq 0$  because otherwise  $f$  is the constant function 0, which we saw above is continuous. For any  $\epsilon > 0$ , since  $f$  is continuous at  $c$  there exists  $\delta > 0$  such that  $|x - c| < \delta$  implies  $|f(x) - f(c)| < \frac{\epsilon}{|A|}$ . (Note what is being done here: by continuity, we can make  $|f(x) - f(c)|$  less than *any* positive number we choose. It is convenient for us to make it smaller than  $\frac{\epsilon}{|A|}$ , where  $\epsilon$  is a previously given positive number.) Then  $|x - c| < \delta$  implies

$$|Af(x) - Af(c)| = |A||f(x) - f(c)| < |A| \cdot \frac{\epsilon}{|A|} = \epsilon.$$

c) Fix  $\epsilon > 0$ . We must show that there exists  $\delta > 0$  such that  $|x - c| < \delta$  implies  $|f(x)g(x) - f(c)g(c)| < \epsilon$ . The situation here is a bit perplexing: we need to use the continuity of  $f$  and  $g$  at  $c$ , and to do this it stands to reason that we should be estimating  $|f(x)g(x) - f(c)g(c)|$  in terms of  $|f(x) - f(c)|$  and  $|g(x) - g(c)|$ , but unfortunately we don't see these latter two expressions. So *we force them to appear* by adding and subtracting  $f(x)g(c)$ :

$$|f(x)g(x) - f(c)g(c)| = |f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)|$$

$$\leq |f(x)||g(x) - g(c)| + |g(c)||f(x) - f(c)|.$$

This is much better:  $|g(c)||f(x) - f(c)|$  is a constant times something which can be made arbitrarily small, so it can be made arbitrarily small. Moreover, in the term  $|f(x)||g(x) - g(c)|$  we can make  $|g(x) - g(c)|$  arbitrarily small by taking  $x$  sufficiently close to  $c$  and then, by continuity of  $f$ ,  $|f(x)|$  gets arbitrarily close to  $f(c)$ . So  $|f(x)|$  is nonconstant but *bounded*, and something which is bounded times something which can be made arbitrarily small can be made arbitrarily small. Now formally: there is  $\delta_1 > 0$  such that if  $|x - c| < \delta_1$  we have  $|f(x) - f(c)| < 1$ . By the Reverse Triangle Inequality,

$$|f(x)| - |f(c)| \leq |f(x) - f(c)| < 1,$$

so

$$|f(x)| \leq |f(c)| + 1.$$

There exists  $\delta_2 > 0$  such that  $|x - c| < \delta_2$  implies  $|g(x) - g(c)| < \frac{\epsilon}{2(|f(c)|+1)}$ . Finally, there exists  $\delta_3$  such that  $|x - c| < \delta_3$  implies  $|f(x) - f(c)| < \frac{\epsilon}{2|g(c)|}$ . (Here we are assuming that  $g(c) \neq 0$ . If  $g(c) = 0$  then we don't have the  $|g(c)||f(x) - f(c)|$  term and the argument is easier.) Put  $\delta = \min(\delta_1, \delta_2, \delta_3)$ . Then if  $|x - c| < \delta$ , we have that  $|x - c|$  is less than  $\delta_1$ ,  $\delta_2$  and  $\delta_3$ , so

$$\begin{aligned} |f(x)g(x) - f(c)g(c)| &\leq |f(x)||g(x) - g(c)| + |g(c)||f(x) - f(c)| \\ &< (|f(c)| + 1) \cdot \frac{\epsilon}{2(|f(c)| + 1)} + |g(c)| \frac{\epsilon}{2|g(c)|} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

d) Since  $\frac{f}{g} = f \cdot \frac{1}{g}$ , in light of part c) it suffices to show that if  $g$  is continuous at  $c$  and  $g(x) \neq 0$  for all  $x \in X$ , then  $\frac{1}{g}$  is continuous at  $c$ . Fix  $\epsilon > 0$ . We must show that there exists  $\delta > 0$  such that  $|x - c| < \delta$  implies

$$\left| \frac{1}{g(x)} - \frac{1}{g(c)} \right| < \epsilon.$$

Now

$$\left| \frac{1}{g(x)} - \frac{1}{g(c)} \right| = \frac{|g(x) - g(c)|}{|g(x)||g(c)|} = \frac{|g(x) - g(c)|}{|g(x)||g(c)|}.$$

Since  $g$  is continuous at  $c$ , we can make the numerator  $|g(x) - g(c)|$  as small as we like by taking  $x$  sufficiently close to  $c$ . This will make the entire fraction as small as like provided the denominator is not getting arbitrarily small as  $x$  approaches  $c$ . But indeed, since  $g$  is continuous at  $g(c) \neq 0$ , the



denominator approaches  $|g(c)|^2 \neq 0$ . Thus again we have an arbitrarily small quantity times a bounded quantity, so it can be made arbitrarily small. Now formally: taking  $\epsilon = \frac{|g(c)|}{2}$ , there exists  $\delta_1 > 0$  such that if  $|x - c| < \delta_1$  then  $|g(x) - g(c)| < \frac{|g(c)|}{2}$ . The Reverse Triangle Inequality implies

$$|g(c)| - |g(x)| \leq |g(x) - g(c)| < \frac{|g(c)|}{2},$$

or

$$|g(x)| > |g(c)| - \frac{|g(c)|}{2} = \frac{|g(c)|}{2}.$$

Thus

$$\frac{1}{|g(x)||g(c)|} \leq \frac{2}{|g(c)|^2}.$$

Also there exists  $\delta_2 > 0$  such that  $|x - c| < \delta_2$  implies

$$|g(x) - g(c)| < \left(\frac{|g(c)|^2}{2}\right)\epsilon.$$

Put  $\delta = \min(\delta_1, \delta_2)$ . Then  $|x - c| < \delta$  implies

$$\left|\frac{1}{g(x)} - \frac{1}{g(c)}\right| = \left(\frac{1}{|g(x)||g(c)|}\right) |g(x) - g(c)| < \frac{2}{|g(c)|^2} \left(\frac{|g(c)|^2}{2}\right)\epsilon = \epsilon. \quad \square$$

**Proposition 1.7.3.** *Let  $X$  and  $Y$  be subsets of  $\mathbb{R}$ , let  $f : X \rightarrow \mathbb{R}$  and  $g : Y \rightarrow \mathbb{R}$  and suppose that  $f(X) \subseteq Y$  (so that the composition  $g \circ f$  is defined). Let  $c \in X$ . If  $f$  is continuous at  $c$  and  $g$  is continuous at  $f(c)$  then  $g \circ f$  is continuous at  $c$ .*

*Proof.* Fix  $\epsilon > 0$ . Because  $g$  is continuous at  $f(c)$ , there exists  $\gamma > 0$  such if  $|y - f(c)| < \gamma$ , then  $|g(y) - g(f(c))| < \epsilon$ . Because  $f$  is continuous at  $c$ , there exists  $\delta > 0$  such that if  $|x - c| < \delta$ , then  $|f(x) - f(c)| < \gamma$ . So, if  $|x - c| < \delta$ , then  $|f(x) - f(c)| < \gamma$ , so  $|g(f(x)) - g(f(c))| < \epsilon$ .  $\square$

**Proposition 1.7.4.** *Let  $X \subset \mathbb{R}$ , and let  $f : X \rightarrow \mathbb{R}$  be a function. Suppose that  $f$  is continuous at a point  $c \in X$ . Let  $L \in \mathbb{R}$ .*

a) *If  $f(c) < L$ , then there is  $\delta > 0$  such that for all  $x \in X$  such that  $|x - c| < \delta$ , then  $f(x) < L$ .*

b) *If  $f(c) > L$ , then there is  $\delta > 0$  such that for all  $x \in X$  such that  $|x - c| < \delta$ , then  $f(x) > L$ .*

*Proof.* We will prove part a). Routine modifications of this argument yield a proof of part b); we will leave this to the reader as an opportunity to check her understanding of the argument.

Take  $\epsilon = L - f(c) > 0$ . Because  $f$  is continuous at  $c$ , there exists  $\delta > 0$  such that for all  $x \in X$  such that  $|x - c| < \delta$  we have

$$f(x) - f(c) \leq |f(x) - f(c)| < L - f(c),$$

which implies that  $f(x) < L$ . □

**Theorem 1.7.5.** *Let  $X \subset \mathbb{R}$ , and let  $f : X \rightarrow \mathbb{R}$  be a function. Let  $c \in X$ . The following are equivalent (i.e., each implies the other):*

- (i) *The function  $f$  is continuous at  $c$ .*
- (ii) *For all real sequences  $\{a_n\}$  such that  $a_n \in X$  for all  $n \in \mathbb{Z}^+$  and  $a_n \rightarrow c$ , we have  $f(a_n) \rightarrow f(c)$ .*

*Proof.* (i)  $\implies$  (ii): Fix  $\epsilon > 0$ . Because  $f$  is continuous at  $c$ , there is  $\delta > 0$  such that for all  $x \in X$  with  $|x - c| < \delta$ , we have  $|f(x) - f(c)| < \epsilon$ . Because  $a_n \rightarrow c$ , there is  $N \in \mathbb{N}$  such that for all  $n > N$  we have  $|a_n - c| < \delta$ . Thus for all  $n > N$  we have  $|f(a_n) - f(c)| < \epsilon$ .

(ii)  $\implies$  (i): We will prove the contrapositive, so assume that  $f$  is *not* continuous at  $c$ . This means that there exists  $\epsilon > 0$  such that for all  $\delta > 0$  there is  $x \in X$  such that  $|x - c| < \delta$  and  $|f(x) - f(c)| \geq \epsilon$ . For any  $n \in \mathbb{Z}^+$ , taking  $\delta = \frac{1}{n}$  gives us an  $x_n \in X$  such that  $|x_n - c| < \frac{1}{n}$  and  $|f(x_n) - f(c)| \geq \epsilon$ . It then follows that  $x_n \rightarrow c$  and  $f(x_n) \not\rightarrow f(c)$ . □

Theorem 1.7.5 is an exciting result for students of the theory of infinite sequences: it means that the concept of continuity of a function can be entirely understood in terms of convergence of infinite sequences. This opens up the door to proving some famous theorems about continuous functions using what we've established about infinite sequences. That is what we will go on to do in the rest of this section.

## 1.7.2 The Intermediate Value Theorem

A function  $f : I \rightarrow \mathbb{R}$  has the *intermediate value property* if for all  $a, b \in I$  with  $a < b$  and all  $L$  in between  $f(a)$  and  $f(b)$  – i.e., with  $f(a) < L < f(b)$  or  $f(b) < L < f(a)$  – there exists some  $c \in (a, b)$  with  $f(c) = L$ . (You should think of this as saying that a function does not “skip” values.)

**Theorem 1.7.6.** (*Intermediate Value Theorem*) Let  $I$  be an interval, and let  $f : I \rightarrow \mathbb{R}$  be a continuous function. Then  $f$  satisfies the Intermediate Value Property.

*Proof.* Let  $a < b$  be two points of  $I$ . For the most of the proof we will assume that  $f(a) < f(b)$ . At the end we will explain how the other case can be reduced to this one. Let  $L \in \mathbb{R}$  be such that  $f(a) < L < f(b)$ . We must find  $c \in (a, b)$  such that  $f(c) = L$ . To do this, we consider the following set:

$$S = \{x \in [a, b] \mid f(x) \leq L\}.$$

Observe that  $a \in S$ , and that  $S \subset [a, b]$ . Thus  $S$  is a nonempty subset of  $\mathbb{R}$  which is bounded above. By the Dedekind Completeness of  $\mathbb{R}$ ,  $S$  has a supremum, say  $c$ . We claim that  $f(c) = L$ , and we will show this by ruling out the other two possibilities.

Case 1: Suppose  $f(c) < L$ . By Proposition 1.7.4a) there is  $\delta > 0$  such that for all  $x \in (c - \delta, c + \delta)$  we have  $f(x) < L$ . But this means that  $S$  contains elements greater than  $c$ , e.g.  $c + \frac{\delta}{2}$ , contradicting the fact that  $c$  is an upper bound for  $S$ .

Case 2: Suppose  $f(c) > L$ . By Proposition 1.7.4b) there is  $\delta > 0$  such that for all  $x \in (c - \delta, c + \delta)$  we have  $f(x) > L$ . Thus no point of  $[c - \frac{\delta}{2}, c]$  lies in  $S$ ; since  $c$  is an upper bound for  $S$ , this means that  $c - \frac{\delta}{2}$  is an upper bound for  $S$ , contradicting the fact that  $c$  is the supremum of  $S$ .

Finally, suppose  $f(a) > f(b)$ , and let  $L \in \mathbb{R}$  be such that  $f(a) > L > f(b)$ . We can modify the above argument to deal with this case; the reader may wish to do so as a check on her understanding. Alternately, we may replace  $f$  by  $-f$ : this is still continuous, and then  $-f(a) < -L < -f(b)$ , so the above argument applies to give  $c \in (a, b)$  such that  $-f(c) = -L$ ...so  $f(c) = L$ .  $\square$

### 1.7.3 The Extreme Value Theorem

Let  $X$  be a set, and let  $f : X \rightarrow \mathbb{R}$  be a function. We say that  $f$  is *bounded* if its image  $f(X)$  is a bounded subset of  $\mathbb{R}$ : explicitly, if there are  $a \leq b \in \mathbb{R}$  such that  $a \leq f(x) \leq b$  for all  $x \in X$ . (Notice that this generalizes our definition of a bounded sequence, in which the domain is  $\mathbb{Z}^+$ .)

Our next theorem states that every continuous function  $f$  on a closed interval must be bounded above. (Of course, by applying the theorem to  $-f$ , one can conclude that it is also bounded below as well.)

**Theorem 1.7.7.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function defined on a closed, bounded interval. Then  $f$  is bounded.*

*Proof.* Step 1: Seeking a contradiction, we suppose that  $f$  is not bounded above: then, for all  $n \in \mathbb{Z}^+$ , there is  $a_n \in [a, b]$  such that  $f(a_n) > n$ . This defines a bounded sequence  $\{a_n\}$ , so by Bolzano-Weierstrass there is a convergent subsequence, say  $a_{n_k} \rightarrow c \in [a, b]$ . By Proposition 1.7.5, since  $f$  is continuous at  $c$  we have  $f(a_{n_k}) \rightarrow f(c)$ . However,  $f(a_{n_k}) > n_k$  for all  $k \in \mathbb{Z}^+$ , so the sequence  $\{f(a_{n_k})\}$  is unbounded and thus divergent: contradiction. This shows that  $f$  is bounded above.

Step 2: Since  $f$  is continuous, so is  $-f$ . So Step 1 gives that  $-f$  is bounded above, say by  $M$ . This means that  $f$  is bounded below by  $-M$ .  $\square$

The final theorem of this section gives the theoretical foundation of the optimization problems from freshman calculus. In those problems you were given a function on a closed interval,  $f : [a, b] \rightarrow \mathbb{R}$ , and asked to find  $c \in [a, b]$  so that  $f(c)$  is the maximum value of  $f$  on  $[a, b]$ , i.e.,  $f(c) \geq f(x)$  for all  $x \in [a, b]$ . This theorem says that such a  $c$  must exist.

**Theorem 1.7.8.** (*Extreme Value Theorem*) *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function on the closed interval  $[a, b]$ . Then the image  $f([a, b])$  has a minimum and a maximum. Explicitly, there is  $c_m \in [a, b]$  such that  $f(c_m) \leq f(x)$  for all  $x \in [a, b]$  and  $c_M \in [a, b]$  such that  $f(c_M) \geq f(x)$  for all  $x \in [a, b]$ .*

*Proof.* Step 1: By Theorem 1.7.7, we know that  $f$  is bounded above: there is  $M \in \mathbb{R}$  such that  $f(x) \leq M$  for all  $x \in [a, b]$ . By Dedekind's Property of Completeness, the set  $f([a, b])$  therefore has a supremum  $s$ . Our goal is to show that  $s \in f([a, b])$ , for then it is a maximum.

Seeking a contradiction, we suppose that  $s \notin f([a, b])$ . Then we may define  $g : [a, b] \rightarrow \mathbb{R}$  by  $g(x) = \frac{1}{s-f(x)}$ . The function  $g$  is the reciprocal of a continuous function which is nowhere zero, so it too is a continuous function. Moreover, since  $s$  is the supremum of  $f([a, b])$  and not a maximum, for all  $n \in \mathbb{Z}^+$  there is  $x_n \in [a, b]$  such that  $f(x_n) > s - \frac{1}{n}$ . Thus  $s - f(x_n) < \frac{1}{n}$  so

$$g(x_n) = \frac{1}{f(x_n) - s} > n.$$

This means that the continuous function  $g : [a, b] \rightarrow \mathbb{R}$  is not bounded above, contradicting Theorem 1.7.7.

Step 2: Applying Step 1 to  $-f$  we find that  $-f$  has a minimum, say  $m$ , and thus  $-m$  is the maximum of  $f$ .  $\square$

### 1.7.4 The Uniform Continuity Theorem

Let  $X \subset \mathbb{R}$ . A function  $f : X \rightarrow \mathbb{R}$  is *uniformly continuous* if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x_1, x_2 \in X$ , if  $|x_1 - x_2| < \delta$ , then  $|f(x_1) - f(x_2)| < \epsilon$ .

The difference between the definition of uniform continuity and the definition of continuity lies entirely in the order of the quantifiers. A function  $f : X \rightarrow \mathbb{R}$  is continuous if it is continuous at each  $c \in X$ , and this means that for a given  $\epsilon > 0$ , the choice of  $\delta$  is allowed to depend not just on  $\epsilon$  but also on  $c$ . Uniform continuity means that having chosen  $\epsilon > 0$ , there must exist a  $\delta > 0$  which works for all  $c \in X$  simultaneously (or “uniformly”). Thus uniform continuity implies continuity. In general, it is much stronger.

**Example 1.7.9.** *The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $x \mapsto x^2$  is continuous. However, it is not uniformly continuous. To see this, take  $\epsilon = 1$ . Uniform continuity would imply that there is some  $\delta > 0$  such that for all  $x_1, x_2 \in \mathbb{R}$ , if  $|x_1 - x_2| < \delta$ , then  $|x_1^2 - x_2^2| < 1$ . But consider:*

$$(x + \delta/2)^2 - x^2 = x\delta + \frac{\delta^2}{4} > x\delta.$$

*Thus if  $x > \frac{1}{\delta}$  then  $(x + \delta/2)^2 - x^2 > 1$ . This shows that there is no  $\delta > 0$  such that any two real numbers with distance less than  $\delta$  part have the property that their squares are less than one unit apart.*

In the above example, the domain was not a closed, bounded interval  $[a, b]$ . This brings us to the following important result.

**Theorem 1.7.10.** *(Uniform Continuity Theorem) Every continuous function  $f : [a, b] \rightarrow \mathbb{R}$  is uniformly continuous.*

*Proof.* Suppose not: then there exists  $\epsilon > 0$  such that for all  $n \in \mathbb{Z}^+$ , there are  $x_n, y_n \in [a, b]$  with  $|x_n - y_n| < \frac{1}{n}$  but  $|f(x_n) - f(y_n)| \geq \epsilon$ . By the Bolzano-Weierstrass Theorem,  $\{x_n\}$  has a subsequence  $\{x_{n_k}\}$  which is convergent, say to  $L$ . Since  $x_{n_k} - y_{n_k} \rightarrow 0$  and  $y_{n_k} = x_{n_k} - (x_{n_k} - y_{n_k})$  for all  $k \in \mathbb{Z}^+$ , also  $y_{n_k} \rightarrow L$  (a case of the Three Sequence Principle). Since  $f$  is continuous, we have  $f(x_{n_k}) \rightarrow f(L)$  and  $f(y_{n_k}) \rightarrow f(L)$ . Since

$$|f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(L)| + |f(y_{n_k}) - f(L)|,$$

so for sufficiently large  $k$  we have  $|f(x_{n_k}) - f(y_{n_k})| < \epsilon$ : contradiction.  $\square$

We will not make further use of the Uniform Continuity Theorem in this course, but it plays a basic role in the theoretical underpinnings of calculus. Most of all, it appears in the (most standard) proof that every continuous function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable.

**EXERCISES 1.7**

1. Let  $f(x)$  be a continuous function on the closed interval  $[a, b]$  and let  $\alpha$  be a real number between  $f(a)$  and  $f(b)$ , i.e.,  $f(a) < \alpha < f(b)$  or  $f(b) < \alpha < f(a)$ . Show that there is some value,  $c \in (a, b)$  such that  $f(c) = \alpha$ . (Hint: if  $f(a) < f(b)$  apply Theorem 1.7.6 to  $f(x) - \alpha$  - make sure you check all of the hypotheses.)
2. Prove that there is a real number  $x$  so that  $\sin(x) = x - 1$ .
3. Use Theorem 1.7.6 to prove that every positive real number,  $r > 0$ , has a square root. (Hint: Consider  $f(x) = x^2 - r$  on a suitable interval.)
4. Assume that  $f$  is a continuous function on  $[0, 1]$  and that  $f(x) \in [0, 1]$  for each  $x$ . Show that there is a  $c \in [0, 1]$  so that  $f(c) = c$ .
5. Suppose that  $f$  and  $g$  are continuous on  $[a, b]$  and that  $f(a) > g(a)$  and  $g(b) > f(b)$ . Prove that there is some number  $c \in (a, b)$  so that  $f(c) = g(c)$ .
6.
  - a) Give an example of a continuous function,  $f$ , which is defined on the open interval  $(0, 1)$  which is not bounded above.
  - b) Give an example of a continuous function,  $f$ , which defined on the open interval  $(0, 1)$  and is bounded above, but does not achieve its maximum on  $(0, 1)$ . i.e., there is no number  $c \in (0, 1)$  satisfying  $f(c) \geq f(x)$  for all  $x \in (0, 1)$ .
7.
  - a) Give an example of a function,  $f$ , which is defined on the closed interval  $[0, 1]$  but is not bounded above.
  - b) Give an example of a function,  $f$ , which is defined on the closed interval  $[0, 1]$  and is bounded above, but does not achieve its maximum on  $[0, 1]$ .





# Chapter 2

## Series

### 2.1 Introduction to Series

In common parlance the words series and sequence are essentially synonymous. However, in mathematics the distinction between the two is that a series is the sum of the terms of a sequence.

**Definition 2.1.1.** Let  $\{a_n\}$  be a real sequence and define a new sequence  $\{S_n\}$  by the recursion relation

$$S_1 = a_1, \forall n \in \mathbb{Z}^+, S_{n+1} = S_n + a_{n+1}.$$

The sequence  $\{S_n\}$  is called the sequence of partial sums of  $\{a_n\}$ .

Another way to think about  $S_n$  is that it is the sum of the first  $n$  terms of the sequence  $\{a_n\}$ , namely

$$S_n = a_1 + a_2 + \dots + a_n.$$

A shorthand form of writing this sum is by using the *sigma notation*:

$$S_n = \sum_{j=1}^n a_j.$$

This is read as  $S_n$  equals the sum from  $j$  equals one to  $n$  of  $a_j$ . We use the subscript  $j$  on the terms  $a_j$  (instead of  $n$ ) because this is denoting an arbitrary term in the sequence while  $n$  is being used to denote how far we sum the sequence.

**Example 2.1.2.** Using sigma notation, the sum  $1 + 2 + 3 + 4 + 5$  can be written as  $\sum_{j=1}^5 j$ . It can also be denoted  $\sum_{n=1}^5 n$ , or  $\sum_{n=0}^4 (n+1)$ . Similarly, the sum

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

can be written as  $\sum_{j=2}^6 \frac{1}{j}$ , or  $\sum_{n=2}^6 \frac{1}{n}$ , or  $\sum_{n=1}^5 \frac{1}{n+1}$ . On the other hand, the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$$

can be written as  $\sum_{j=1}^n \frac{1}{j}$  or  $\sum_{k=1}^n \frac{1}{k}$  but can not be written as  $\sum_{n=1}^n \frac{1}{n}$ .

**Definition 2.1.3.** Let  $\{a_n\}$  be a sequence and let  $\{S_n\}$  be the sequence of partial sums of  $\{a_n\}$ . If  $\{S_n\}$  converges we say that  $\{a_n\}$  is summable. In this case, we denote the  $\lim_{n \rightarrow \infty} S_n$  by

$$\sum_{j=1}^{\infty} a_j.$$

**Definition 2.1.4.** The expression  $\sum_{j=1}^{\infty} a_j$  is called an infinite series (whether or not the sequence  $\{a_n\}$  is summable). When we are given an infinite series  $\sum_{j=1}^{\infty} a_j$  the sequence  $\{a_n\}$  is called the sequence of terms. If the sequence of terms is summable, the infinite series is said to be convergent. If it is not convergent it is said to diverge.

**Remark 2.1.5.** There is an ambiguity built into this terminology. In practice, when we write “the infinite series  $\sum_{n=1}^{\infty} a_n$ ” we could mean either the sequence of partial sums  $\{S_n\}$  obtained from the sequence  $\{a_n\}$  or  $\lim_{n \rightarrow \infty} S_n$ . Although from a strictly logical perspective this is unacceptable, in practice it causes little to no confusion, and it is certainly not worth trying to adjust this extremely widely used notation in order to fix it.

**Example 2.1.6.** Consider the sequence of terms given by

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Then

$$\begin{aligned}
 S_1 &= a_1 = 1 - \frac{1}{2} = \frac{1}{2}, \\
 S_2 &= a_1 + a_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) \\
 &= 1 + \left(\frac{1}{2} - \frac{1}{2}\right) - \frac{1}{3} \\
 &= \frac{2}{3}, \\
 S_3 &= a_1 + a_2 + a_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \\
 &= 1 + \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) - \frac{1}{4} \\
 &= \frac{3}{4},
 \end{aligned}$$

and so forth. Continuing this regrouping, we see that

$$\begin{aligned}
 S_n &= a_1 + a_2 + \cdots + a_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
 &= 1 + \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n}\right) - \frac{1}{n+1} \\
 &= \frac{n}{n+1}.
 \end{aligned}$$

Therefore  $\lim_{n \rightarrow \infty} S_n = 1$  and so  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ .

**Example 2.1.7.** Let  $a_n = \frac{1}{2^n}$ . Then

$$\begin{aligned}
 S_1 &= a_1 = \frac{1}{2} \\
 S_2 &= a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
 S_3 &= a_1 + a_2 + a_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8},
 \end{aligned}$$

and so forth. A straightforward induction argument shows that, in general,

$$S_n = 1 - \frac{1}{2^n}.$$

Thus  $\lim S_n = 1$ , and so  $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ .

**Example 2.1.8.** Let  $\{b_n\}$  be a given sequence. We can build a sequence  $\{a_n\}$  whose sequence of partial sums is given by  $\{b_n\}$  in the following way: Let  $a_1 = b_1$  and for  $n > 1$  let  $a_n = b_n - b_{n-1}$ . Then we have  $a_1 + a_2 = b_1 + (b_2 - b_1) = b_2$ ,  $a_1 + a_2 + a_3 = b_1 + (b_2 - b_1) + (b_3 - b_2) = b_3$ , etc. Thus the sequence  $\{b_n\}$  converges if and only if the sequence  $\{a_n\}$  is summable.

The above example shows that a sequence is convergent if and only if a related sequence is summable. Similarly, a sequence is summable if and only if the sequence of partial sums converges. However it should be kept in mind that the sequence given by  $a_n = 1/n$  converges but is *not* summable. On the other hand we have

**Proposition 2.1.9.** (*Nth Term Test*)

If the sequence  $\{a_n\}$  is summable then  $a_n \rightarrow 0$ .

*Proof.* Let  $\{S_n\}$  be the sequence of partial sums. We will give two proofs.

**First Proof:** By assumption there is  $S \in \mathbb{R}$  such that  $S_n \rightarrow S$ . Since  $a_n = S_n - S_{n-1}$ , the algebra of limits gives

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} = S - S = 0.$$

**Second Proof:** Since  $\{S_n\}$  is convergent, it is Cauchy: for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $m, n \geq N$  we have  $|S_m - S_n| < \epsilon$ . In particular, let  $n \geq N$  and take  $m = n + 1$ : thus

$$|a_{n+1}| = |S_{n+1} - S_n| < \epsilon.$$

Thus we have  $|a_n| < \epsilon$  for all  $n > N$ , so  $a_n \rightarrow 0$ . □

**Remark 2.1.10.** *The first proof seems easier, but the second proof is more penetrating: we know that the Cauchy criterion is necessary and sufficient for the sequence  $\{S_n\}$  to converge, but in passing from “for all  $m, n \geq N$  we have...” to “for  $n \geq N$  and  $m = n + 1$  we have...” we have discarded a lot, so it is not terribly surprising that this weaker condition is necessary but no longer sufficient for the sequence of partial sums to converge.*

As with sequences, convergent series behave well with respect to sums and multiplication by a fixed real number (scalar multiplication). Of course multiplication of two series is more complicated, since even for finite sums, the product of two sums is not simply the sum of the products. We will return to a discussion of products of series in section 2.3. For now we state the result for sums and scalar multiplication of series, leaving the proofs to Exercises 2.1.12 and 2.1.13.

**Proposition 2.1.11.** *Let  $\{a_n\}$  and  $\{b_n\}$  be summable sequences, and let  $r \in \mathbb{R}$ . Define two new sequences by  $c_n = a_n + b_n$  and  $d_n = ra_n$ . Then  $\{c_n\}$  and  $\{d_n\}$  are both summable and*

$$\begin{aligned}\sum_{n=1}^{\infty} c_n &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \\ \sum_{n=1}^{\infty} d_n &= r \sum_{n=1}^{\infty} a_n.\end{aligned}$$

We conclude this section with an uber-example: namely, an example which we will use to build large parts of the general theory.

A sequence  $\{a_n\}$  of nonzero real numbers is *geometric* if:

$$\forall n \in \mathbb{Z}^+, \frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1}}{a_n}.$$

Although we should (as ever) be flexible about it, it is convenient to start geometric sequences at  $n = 0$ . In Exercise 17 you are asked to show that for a geometric sequence there are unique real numbers  $C$  and  $r$  such that

$$\forall n \in \mathbb{N}, a_n = Cr^n.$$

Indeed, we will let slip now that  $r$  is the ratio of any two consecutive terms of the sequence; we will call it the *geometric ratio*. The convergence of geometric sequences was discussed in Proposition 1.4.16. We are now interested in the convergence of geometric series  $\sum_{n=0}^{\infty} Cr^n$ . This is a different question, but in fact Proposition 1.4.16 is highly relevant.

**Proposition 2.1.12.** *The geometric series  $\sum_{n=0}^{\infty} r^n$  converges if and only if  $|r| < 1$ . In the case that  $|r| < 1$  we have*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

*Proof.* Step 1: By the Nth Term Test (Proposition 2.1.9), if the series  $\sum_{n=0}^{\infty} r^n$  converges then  $r^n \rightarrow 0$ . By Proposition 1.4.16 we have  $r^n \rightarrow 0$  if and only if  $|r| < 1$ . So if  $|r| \geq 1$  then the series diverges.

Step 2: Let  $r \in \mathbb{R} \setminus \{1\}$ ; we will give an explicit formula for

$$S_n = 1 + r + \dots + r^n.$$

Indeed, we have

$$rS_n = r + r^2 + \dots + r^n + r^{n+1},$$

and when we subtract most of the terms cancel, leaving

$$(r - 1)S_n = r^{n+1} - 1.$$

Since  $r \neq 1$ ,  $r - 1 \neq 0$ , and we may divide by it to get

$$S_n = \frac{r^{n+1} - 1}{r - 1}.$$

(By the way, no problem about  $r = 1$ :  $1 + 1^1 + \dots + 1^n = n + 1$ .)

Step 3: Now suppose  $|r| < 1$ . Again, Proposition 1.4.16 gives  $r^n \rightarrow 0$ . Thus

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{r^{n+1} - 1}{r - 1} = \frac{0 - 1}{r - 1} = \frac{1}{1 - r}. \quad \square$$

**Example 2.1.13.**

- a) The sum  $\sum_{n=1}^{\infty} \frac{1}{5^n}$  equals  $\frac{1/5}{1 - 1/5} = \frac{1/5}{4/5} = 1/4$ .
- b) The sum  $\sum_{n=1}^{\infty} \frac{3}{5^n}$  equals  $3 \sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{3}{4}$ .
- c) The sum  $\sum_{n=3}^{\infty} \frac{1}{5^n}$  equals

$$\left( \sum_{n=1}^{\infty} \frac{1}{5^n} \right) - \left( \frac{1}{5} + \frac{1}{5^2} \right) = 1/4 - 1/5 - 1/25 = 1/100.$$

Exercise 18 gives a formula for the sum of a more general geometric series  $\sum_{n=m}^{\infty} Cr^n$ . (The method of the above example may be helpful there...)

**EXERCISES 2.1**

1. Consider the sequence given by  $a_n = \frac{1}{2^n}$ . Compute the first five partial sums of this sequence.
2. Rewrite the following sums using sigma notation:
  - a)  $(1 + 4 + 9 + 16 + 25 + 36 + 49)$
  - b)  $(5 + 6 + 7 + 8 + 9 + 10)$
  - c)  $(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots + \frac{1}{28})$
  - d)  $(2 + 2 + 2 + 2 + 2 + 2 + 2 + 2)$

3. Evaluate the following finite sums:

$$\begin{array}{lll} \text{a)} \sum_{k=1}^n 1 & \text{b)} \sum_{k=1}^n 1/n & \text{c)} \sum_{k=1}^n k \\ \text{d)} \sum_{k=1}^{2n} k & \text{e.)} \sum_{k=1}^n k^2 & \end{array}$$

4. Consider the sequence whose  $n^{\text{th}}$  term is  $a_n = \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2}, n \in \mathbb{Z}^+$ . Compute the first five partial sums of this sequence. What is the general formula for the  $n^{\text{th}}$  partial sum? Prove this formula by induction and prove that the sequence is summable.
5. Consider the sequence whose  $n^{\text{th}}$  term is  $a_n = \frac{1}{(n+2)} - \frac{1}{(n+3)}, n \in \mathbb{Z}^+$ . Compute the first five partial sums of this sequence. What is the general formula for the  $n^{\text{th}}$  partial sum? Prove this formula by induction and prove that the sequence is summable.
6. Consider the sequence whose  $n^{\text{th}}$  term is  $a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}, n \in \mathbb{Z}^+$ . Prove that this sequence is summable.
7. Let  $S_n = \sum_{j=1}^n \frac{1}{j^2}$ . Prove by induction that  $S_n \leq 2 - \frac{1}{n}$ . (Hint: Prove that  $-\frac{1}{k} + \frac{1}{(k+1)^2} = -\frac{1}{k+1} - \frac{1}{k(k+1)^2}$ .)
8. Let  $S_n = \sum_{j=1}^n \frac{1}{j^3}$ . Prove by induction that  $S_n \leq 2 - \frac{1}{n^2}$ . Conclude that

$$\sum_{j=1}^{\infty} \frac{1}{j^3} \text{ converges.}$$

9. Prove that the sequence given by  $a_n = \frac{n}{n+1}$  is not summable.
10. Find a sequence  $\{a_n\}$  whose  $n^{\text{th}}$  partial sum is  $\frac{n-1}{n+1}$ .
11. Find a sequence  $\{a_n\}$  whose  $n^{\text{th}}$  partial sum is the  $n^{\text{th}}$  term in the Fibonacci sequence.
12. Let  $\{a_n\}$  be a summable sequence and  $r$  a real number. Define a new sequence  $\{b_n\}$  by  $b_n = ra_n$ . Prove that  $\{b_n\}$  is summable and that

$$\sum_{n=1}^{\infty} b_n = r \sum_{n=1}^{\infty} a_n.$$

13. Let  $\{a_n\}$  and  $\{b_n\}$  be summable sequences. Define a new sequence by  $c_n = a_n + b_n$ . Prove that the sequence  $\{c_n\}$  is summable and that

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

(See exercise 1.5.2.)

14. Assume that the sequence  $\{a_n\}$  is summable and that the sequence  $\{b_n\}$  is not summable. Prove that the sequence given by  $c_n = a_n + b_n$  is not summable. (See exercise 1.5.3.)
15. Evaluate the following sums:
  - a)  $\sum_{n=1}^{\infty} \frac{1}{3^n}$
  - b)  $\sum_{n=3}^{\infty} \frac{1}{3^n}$
  - c)  $\sum_{n=1}^{\infty} \frac{1}{3^{n+2}}$
  - d)  $\sum_{n=1}^{\infty} \frac{2}{3^n}$
  - e.)  $\sum_{n=1}^{\infty} \frac{9}{10^n}$
  - f.)  $\sum_{n=1}^{\infty} \left( \frac{1}{3^n} + \frac{3}{5^n} \right)$



16. A rubber ball bounces  $1/3$  of the height from which it falls. If it is dropped from 10 feet and allowed to continue bouncing, how far does it travel?
17. Let  $\{a_n\}_{n=0}^{\infty}$  be a geometric sequence: recall this means that for all  $n \in \mathbb{N}$  we have  $a_n \neq 0$  and  $\frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1}}{a_n}$ . Show that for every geometric sequence there are unique real numbers  $r$  and  $C$  such that for all  $n \in \mathbb{N}$  we have

$$a_n = Cr^n.$$

18. Let  $C \in \mathbb{R}$  and  $r \in \mathbb{R} \setminus \{1\}$ .  
a) For  $0 \leq m \leq n$ , show that

$$\sum_{k=m}^n Cr^k = \frac{C(r^m - r^{n+1})}{1 - r}.$$

- b) Suppose  $|r| < 1$  and  $m \in \mathbb{N}$ . Show:

$$\sum_{n=m}^{\infty} Cr^n = \frac{Cr^m}{1 - r}.$$

- c) In part b), would it make sense to take  $m$  to be a negative integer?

19. For  $n \in \mathbb{Z}^+$ , let  $H_n = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}$ , the  **$n$ th harmonic number**.  
a) Observe that  $H_1 = 1 \in \mathbb{Z}$ , and compute that for all  $2 \leq n \leq 10$  we have  $H_n \in \mathbb{Q} \setminus \mathbb{Z}$ .  
b) Show that for all  $n \geq 2$ , we have  $H_n \in \mathbb{Q} \setminus \mathbb{Z}$ .

20. S. Abbott calls a real sequence  $\{a_n\}_{n=1}^{\infty}$  **pseudo-Cauchy** if

$$\lim_{n \rightarrow \infty} a_{n+1} - a_n = 0.$$

- a) Show: every Cauchy sequence is pseudo-Cauchy.  
b) Show: the sequence  $a_n = \log n$  is pseudo-Cauchy, increasing and not Cauchy.  
c) Prove or disprove: a bounded pseudo-Cauchy sequence is a Cauchy sequence.
21. Let  $\frac{P(x)}{Q(x)}$  be a rational function. The polynomial  $Q(x)$  has only finitely many roots, so we may choose  $N \in \mathbb{Z}^+$  such that  $Q(n) \neq 0$  for all  $n \geq N$ . Show: if the degree of  $P$  is greater than or equal to the degree of  $Q$ , then  $\sum_{n=N}^{\infty} \frac{P(n)}{Q(n)}$  diverges.

## 2.2 Series with Nonnegative Terms

In this section we will discuss tests for convergence of series with nonnegative terms. Similar results are true for sequences with nonpositive terms, but we won't dwell on that here. In the next section we will study series which may have some negative and some positive terms.

### 2.2.1 O'Connor on Series

When all the terms  $a_n$  are non-negative, there is no oscillation in the sequence of partial sums. As we saw in Chapter 1, this simplifies things considerably.

**Proposition 2.2.1.** *Let  $\{a_n\}$  be a sequence of non-negative real numbers.*

- a) *If the sequence of partial sums  $S_n = a_1 + \dots + a_n$  is bounded, then the series  $\sum_{n=1}^{\infty} a_n$  converges.*  
 b) *If the sequence of partial sums  $S_n = a_1 + \dots + a_n$  is unbounded, then the series  $\sum_{n=1}^{\infty} a_n$  diverges to  $\infty$ .*

*Proof.* Indeed, since  $a_n \geq 0$  for all  $n$  the sequence  $S_n = a_1 + \dots + a_n$  is increasing. As we saw in Chapter 1 (using Cauchy's Property of Completeness), a bounded increasing sequence converges, and an unbounded increasing sequence diverges to infinity.  $\square$

### 2.2.2 Comparison and Limit Comparison

**Proposition 2.2.2.** *(Comparison Test) Let  $\{a_n\}$  and  $\{b_n\}$  be sequences such that  $0 \leq a_n \leq b_n$  for all  $n \in \mathbb{Z}^+$ .*

- a) *We have*

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

*In particular:*

- b) *If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.*  
 c) *If  $\sum_{n=1}^{\infty} a_n$  diverges, then  $\sum_{n=1}^{\infty} b_n$  diverges.*

*Proof.* First we observe that in writing  $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$  for series that may not converge, we are being slightly abusive with notation...however in a way which is very convenient. Namely, we introduce the convention that  $x \leq \infty$  for all  $x \in \mathbb{R}$  and also  $\infty \leq \infty$ .

a) For  $n \in \mathbb{Z}^+$ , let  $S_n = a_1 + \dots + a_n$  and  $T_n = b_1 + \dots + b_n$ . Since  $a_n \leq b_n$  for all  $n$ , we have  $S_n \leq T_n$  for all  $n$ . Suppose that  $\lim_{n \rightarrow \infty} T_n = T < \infty$ , then  $T$  is an upper bound for each  $S_n$ , so  $\lim_{n \rightarrow \infty} S_n \leq T < \infty$ . If  $\lim_{n \rightarrow \infty} T_n = \infty$ , then  $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$  is the content-free assertion that  $\sum_{n=1}^{\infty} a_n \leq \infty$ .

b) This follows immediately from part a).

c) This is the contrapositive of part a), so is logically equivalent to it.  $\square$

**Example 2.2.3.** Consider the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ . A bit of experimentation suggests (correctly!) that there is no simple formula for the  $n$ th partial sum

$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}.$$

However, we can replace each  $a_n = \frac{1}{n^2}$  with a slightly larger  $b_n$  such that there is a simple formula for

$$T_n = b_1 + \dots + b_n$$

and  $\lim_{n \rightarrow \infty} T_n < \infty$ . Then we'll get  $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ . Here goes: take

$$b_1 = 1, \quad \forall n \geq 2, \quad b_n = \frac{1}{(n-1)n}.$$

Then  $a_1 = 1$ , while for all  $n \geq 2$ , since  $(n-1)n < n^2$ , we have

$$b_n = \frac{1}{(n-1)n} < \frac{1}{n^2} = a_n.$$

Moreover, for all  $n \geq 2$  we have

$$\begin{aligned} T_n &= 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{(n-1)n} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 + 1 - \frac{1}{n} = 2 - \frac{1}{n}. \end{aligned}$$

Therefore we have

$$\sum_{n=1}^{\infty} b_n = \lim_{n \rightarrow \infty} 2 - \frac{1}{n} = 2,$$

so by the Comparison Test we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} b_n = 2.$$

**Remark 2.2.4.** Note well that this argument shows the convergence of  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  without computing the sum. However, it does give us some estimates on the sum: it lies between 0 and 2. In fact it is possible to adapt the method to compute the sum of the series to arbitrary accuracy: e.g. we have

$$\begin{aligned} S_{100} &= \sum_{n=1}^{100} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq S_{100} + \sum_{n=101}^{\infty} b_n \\ &= S_{100} + \frac{1}{100 \cdot 101} + \frac{1}{101 \cdot 102} + \dots = S_{100} + \left( \frac{1}{100} - \frac{1}{101} \right) + \left( \frac{1}{101} - \frac{1}{102} \right) + \dots \\ &= S_{100} + \frac{1}{100}. \end{aligned}$$

We may compute that

$$S_{100} = 1.63498 \dots$$

so

$$1.63498 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1.64498 \dots$$

and we have computed the infinite series to (roughly) two decimal places. We could continue in this manner if we wanted.

However, these sort of approximations will not tell us the exact sum of the series. The 18th century mathematician L. Euler figured this out:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.644934 \dots$$

Over the centuries many proofs have been given, most using somewhat more advanced mathematics (e.g. complex variables or Fourier series). An unusually simple, elementary derivation was given rather recently by D. Daners [Da12]. An exposition of Daners's proof may also be found in [Cl-HC, §14.1].

**Remark 2.2.5.** Altering a finite number of terms of a series does not affect whether or not the series converges, thus the above tests for convergence or divergence are valid as long as the inequalities hold eventually. Thus, to apply the above theorem, it is enough to check that there is an  $N \in \mathbb{Z}^+$  so that the inequality  $0 \leq a_n \leq b_n$  holds for all  $n > N$ .

**Theorem 2.2.6.** (*Limit Comparison Test*) Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  two series. Suppose that there exists  $N \in \mathbb{Z}^+$  and  $M \in \mathbb{R}^{\geq 0}$  such that for all  $n \geq N$ ,  $0 \leq a_n \leq Mb_n$ . Then if  $\sum_{n=1}^{\infty} b_n$  converges, so does  $\sum_{n=1}^{\infty} a_n$ .

*Proof.* Since we may add, remove or change finitely many terms without disturbing the convergence of an infinite series, we may as well assume that  $0 \leq a_n \leq Mb_n$  holds for all  $n \in \mathbb{Z}^+$ . Now apply the Comparison Test:

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} Mb_n = M \sum_{n=1}^{\infty} b_n < \infty. \quad \square$$

**Corollary 2.2.7.** (*Calculus Student's Limit Comparison Test*)

Let  $\sum_n a_n$  and  $\sum_n b_n$  be two series. Suppose that for all sufficiently large  $n$  both  $a_n$  and  $b_n$  are positive and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \in [0, \infty]$ .

a) If  $0 < L < \infty$ , the series  $\sum_n a_n$  and  $\sum_n b_n$  converge or diverge together (i.e., either both converge or both diverge).

b) If  $L = \infty$  and  $\sum_n a_n$  converges, then  $\sum_n b_n$  converges.

c) If  $L = 0$  and  $\sum_n b_n$  converges, then  $\sum_n a_n$  converges.

*Proof.* a) If  $0 < L < \infty$ , then there exists  $N \in \mathbb{Z}^+$  such that  $0 < \frac{L}{2}b_n \leq a_n \leq (2L)b_n$ . Applying Theorem 2.2.6 to the second inequality, we get that if  $\sum_n b_n$  converges, then  $\sum_n a_n$  converges. The first inequality is equivalent to  $0 < b_n \leq \frac{2}{L}a_n$  for all  $n \geq N$ , and applying Theorem 2.2.6 to this we get that if  $\sum_n a_n$  converges, then  $\sum_n b_n$  converges. So the two series  $\sum_n a_n$ ,  $\sum_n b_n$  converge or diverge together.

b) If  $L = \infty$ , then there exists  $N \in \mathbb{Z}^+$  such that for all  $n \geq N$ ,  $a_n \geq b_n \geq 0$ . Applying Theorem 2.2.6 to this we get that if  $\sum_n a_n$  converges, then  $\sum_n b_n$  converges.

c) This case is left to the reader as an exercise.  $\square$

The reason behind the nomenclature here is that calculus students are (somehow) conversant with limits without being conversant with the inequalities that underlie them. This gives the Limit Comparison Test a certain allure in calculus class: in very basic situations, it is easier to apply. For instance, it allows us to see that all three series<sup>1</sup>

$$\sum_{n=2}^{\infty} \frac{1}{n^2 + 1}, \quad \sum_{n=2}^{\infty} \frac{1}{n^2}, \quad \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

<sup>1</sup>We start the summation at  $n = 2$  this time because  $\frac{1}{n^2 - 1}$  is not defined when  $n = 1$ .

converge or diverge together. Since

$$n^2 - 1 < n^2 < n^2 + 1,$$

we have

$$\frac{1}{n^2 + 1} < \frac{1}{n^2} < \frac{1}{n^2 - 1},$$

and the Comparison Test tells us

$$\sum_{n=2}^{\infty} \frac{1}{n^2 + 1} \leq \sum_{n=2}^{\infty} \frac{1}{n^2} \leq \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}.$$

By Example 2.2.3 we know that the middle series,  $\sum_{n=2}^{\infty} \frac{1}{n^2}$ , converges, so *straight out of the box* the Comparison Test tells us that  $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$  converges but it doesn't tell us that  $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$  converges because the inequality goes the wrong way. Since

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2-1}} = 1,$$

the calculus student is very happy to apply the Limit Comparison test to deduce the convergence of  $\sum_{n=2}^{\infty} \frac{1}{n^2-n}$  from the convergence of  $\sum_{n=2}^{\infty} \frac{1}{n^2}$ .

However, the above proofs make clear that if you look inside the shiny box of the Limit Comparison Test, you find the dusty, but trusty, Comparison Test. Indeed, to someone who has worked with inequalities enough to show convergence of sequences from the definition, it is almost obvious that  $n^2 - 1 > \frac{n^2}{2}$  eventually, so  $\frac{1}{n^2-1} < \frac{2}{n^2}$  eventually, so the Comparison Test really does work here provided you introduce the "slack" of a constant multiple. As we saw above, this is what's happening in general: when we know that a series with non-negative terms grows no faster than a constant times a convergent series with non-negative terms, it too converges.

### 2.2.3 *P*-series, Condensation and Integration

**Example 2.2.8.** We will now consider an important family of series involving a parameter. Namely, for  $p \in \mathbb{R}$ , we consider the ***p*-series**  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ . To be sure, for each value of  $p$  we get a different series, and we are interested in how the convergence/divergence of the series depends on  $p$ .

a) Notice that if  $p_1 < p_2$ , then for all  $n \in \mathbb{Z}^+$  we have  $n^{p_1} \leq n^{p_2}$ . (For  $n = 1$ ,

we have  $1^p = 1$  for all  $p$ , so equality holds. We use the fact that for all  $a > 1$ , the exponential function  $x \mapsto a^x$  is increasing: indeed, its derivative is  $(\log a)a^x$  is positive for all  $x \in \mathbb{R}$ .) It follows that for all  $n \in \mathbb{Z}^+$  we have

$$p_1 < p_2 \implies 0 \leq \frac{1}{n^{p_2}} \leq \frac{1}{n^{p_1}}$$

and thus

$$\sum_{n=1}^{\infty} \frac{1}{n^{p_2}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{p_1}}.$$

It follows that if the  $p$ -series converges for a given value of  $p$ , it also converges for all greater values. Contrapositively (thus equivalently), if the  $p$ -series diverges for a given value of  $p$ , it also diverges for all smaller values.

b) If  $p = 0$  then  $\frac{1}{n^p} = 1$  for all  $n$ , so the  $p$ -series is just  $\sum_{n=1}^{\infty} 1 = \infty$ . Along with part a), we see that the  $p$ -series diverges for all  $p \leq 0$ . In this range we have  $\frac{1}{n^p} \not\rightarrow 0$ , so divergence also follows from the  $N$ th Term Test.

c) Example 2.2.3 treats the case  $p = 2$  and shows that  $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2 < \infty$ . By part a), it follows that for all  $p > 2$  we have

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq 2 < \infty.$$

This leaves us in doubt of the behavior of the  $p$ -series for  $p \in (0, 2)$ .

The most famous  $p$ -series is when  $p = 1$ . We call the series  $\sum_{n=1}^{\infty} \frac{1}{n}$  the **harmonic series**.<sup>2</sup>

**Proposition 2.2.9.** *The harmonic series diverges:*

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

*Proof.* The following brilliant and elementary argument due to Cauchy.

Consider the terms arranged as follows:

$$\left(\frac{1}{1}\right) + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \dots,$$

---

<sup>2</sup>Some people call a  $p$ -series a **hyper-harmonic series**. But if that means anything in particular, I am not aware of it.

i.e., we group the terms in blocks of length  $2^k$ . Now observe that the power of  $\frac{1}{2}$  which beings each block is larger than every term in the preceding block, so if we replaced every term in the current block the the first term in the next block, we would only decrease the sum of the series. But this latter sum is much easier to deal with:

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty.$$

Therefore the harmonic series  $\sum_{n=1}^{\infty}$  diverges.  $\square$

**Example 2.2.10.** *Combining Proposition 2.2.9 with our comparison of  $p$ -series, we now know that  $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$  for all  $p \leq 1$ . Thus we have whittled down the range of  $p$  for which we do not yet know the convergence of the  $p$ -series to  $p \in (1, 2)$ .*

The apparently *ad hoc* argument used to prove the divergence of the harmonic series can be adapted to give the following useful test.

**Theorem 2.2.11.** *(Cauchy Condensation Test) Let  $\sum_{n=1}^{\infty} a_n$  be an infinite series such that  $a_n \geq a_{n+1} \geq 0$  for all  $n \in \mathbb{N}$ . Then:*

- a) *We have  $\sum_{n=1}^{\infty} a_n \leq \sum_{n=0}^{\infty} 2^n a_{2^n} \leq 2 \sum_{n=1}^{\infty} a_n$ .*
- b) *Thus the series  $\sum_n a_n$  converges iff the **condensed series**  $\sum_n 2^n a_{2^n}$  converges.*

*Proof.* We have

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots \\ &\leq a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + 8a_8 + \dots = \sum_{n=0}^{\infty} 2^n a_{2^n} \\ &= (a_1 + a_2) + (a_2 + a_4 + a_4 + a_4) + (a_4 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8) + (a_8 + \dots) \\ &\leq (a_1 + a_1) + (a_2 + a_2 + a_3 + a_3) + (a_4 + a_4 + a_5 + a_5 + a_6 + a_6 + a_7 + a_7) + (a_8 + \dots) \\ &= 2 \sum_{n=1}^{\infty} a_n. \end{aligned}$$

This establishes part a), and part b) follows immediately.  $\square$



The Cauchy Condensation Test is perhaps my favorite convergence test. First, it is an interesting result in its own right: it shows that under the given hypotheses, in order to determine whether a series converges we need to know only a very sparse subsequence of the sequence of terms – whatever is happening in between  $a_{2^n}$  and  $a_{2^{n+1}}$  is immaterial, *so long as the sequence remains decreasing*. Of course without the monotonicity hypothesis, nothing like this could hold.

On the other hand, it may be less clear that the Condensation Test is of any practical use: isn't the condensed series  $\sum_n 2^n a_{2^n}$  more complicated than the original series  $\sum_n a_n$ ? Strangely, though the notation looks that way, in fact quite often the opposite is the case: passing from the given series to the condensed series preserves the convergence/divergence but tends to exchange subtly convergent/divergent series for more obviously (and more rapidly) converging/diverging series.

As a first example of this, we will finish off the  $p$ -series.

**Theorem 2.2.12.** *The  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if and only if  $p > 1$ .*

*Proof.* Step 1: First we recall that if  $p < 0$ , then  $\frac{1}{n^p} \rightarrow \infty$ . Infinity is not zero, so  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  diverges by the  $N$ th term test.

Step 2: Now suppose that  $p \geq 0$ , so the sequence of terms  $\frac{1}{n^p}$  is positive and decreasing. By the Cauchy Condensation Test, the  $p$ -series converges iff the following series converges:

$$\sum_{n=1}^{\infty} 2^n (2^n)^{-p} = \sum_{n=1}^{\infty} (2^{1-p})^n.$$

Thus condensation has replaced the  $p$ -series by our best friend: a geometric series with  $r = 2^{1-p}$ . We have convergence if  $2^{1-p} = |2^{1-p}| < 1$ ; this happens if and only if  $1 - p < 0$ , so if and only if  $p > 1$ . Done!  $\square$

**Example 2.2.13.** *Consider the series  $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ . Since  $a_n = \frac{1}{n \log n}$  is positive and decreasing, the Condensation Test applies, and the convergence of the series is equivalent to the convergence of*

$$\sum_n \frac{2^n}{2^n \log 2^n} = \frac{1}{\log 2} \sum_n \frac{1}{n} = \infty.$$

*Thus  $\sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty$ . A few comments:*

(i) *Since our proof the divergence of the harmonic series itself used the Condensation Test, really we applied the Condensation Test twice, simplifying*

the series each time.

(ii) This example shows that the issue of convergence/divergence can be quite subtle: for any  $\epsilon > 0$ , we know that  $\sum_n \frac{1}{n^{1+\epsilon}} < \infty$ , since it is a  $p$ -series with  $p = 1 + \epsilon > 1$ . But  $\log n$  grows more slowly than  $n^\epsilon$ , and indeed slowly enough so that replacing  $n^\epsilon$  with  $\log n$  converts a convergent series to a divergent one.

For the next test we need to recall the concept of *improper integrals* from calculus. Let  $f : [1, \infty) \rightarrow \mathbb{R}^+$  be continuous. Recall that we say that the *improper integral*  $\int_1^\infty f(x) dx$  converges if  $\lim_{b \rightarrow \infty} \int_1^b f(x) dx$  converges. You should think of an improper integral as being a *continuous analogue* of an infinite series. More precisely, for an infinite series we start with a function  $a : \mathbb{Z}^+ \rightarrow \mathbb{R}$ , and then  $\sum_{n=1}^\infty a_n$  “adds up the values of the function”, whereas for an improper integral we start with a function  $f : [1, \infty) \rightarrow \mathbb{R}$  and then the improper integral  $\int_1^\infty f(x) dx$  “adds up the values of the function” (in a way which is somewhat more complicated to define).

“Pushing an analogy” is one of the things that mathematicians really like to do. When you know a result for improper integrals, you can ask whether it is true for infinite series. When you know a result for infinite series, you can ask whether it is true for improper integrals. The answer will not always be yes, but it will be interesting either way.

As one instance of this, we observe that if  $f : [1, \infty) \rightarrow [0, \infty)$  then

$$\int_1^\infty f(t) dt = \lim_{x \rightarrow \infty} F(x), \text{ where } F(x) = \int_1^x f(t) dt.$$

As is familiar from calculus, the function  $F(x)$  is increasing – since the function never dips below the  $x$ -axis, the integral represents area rather than signed area, and so as we push the upper limit of integration  $x$  from left to right we accumulate more area. In the setting of infinite series, when  $a_n \geq 0$  for all  $n \in \mathbb{Z}^+$  we have that  $\sum_{n=1}^\infty a_n$  is either convergent or divergent to infinity, and this followed from O’Connor’s Property of Completeness. So this raises the question: if  $F : [1, \infty) \rightarrow \mathbb{R}$  is an increasing function, is it true that  $\lim_{x \rightarrow \infty} F(x)$  exists or  $\lim_{x \rightarrow \infty} F(x) = \infty$ ? Yes, and the natural proof uses Dedekind’s Property of Completeness. We leave this as an exercise.

The following result is *more than an analogy* between infinite series and improper integrals: it gives a situation in which one can directly deduce convergence of series from integrals and conversely.

**Theorem 2.2.14.** (*Integral Test*) Let  $f : [1, \infty) \rightarrow (0, \infty)$  be a decreasing

function, and for  $n \in \mathbb{Z}^+$  put  $a_n = f(n)$ . Then

$$\sum_{n=2}^{\infty} a_n \leq \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} a_n.$$

Thus we have  $\sum_n a_n < \infty$  if and only if  $\int_1^{\infty} f(x)dx < \infty$ .

*Proof.* They say (don't they?) that in mathematics, a picture is never a proof. I agree...but sometimes the picture immediately generates a proof, and this is one of those times. So please draw a picture of a function  $y = f(x)$  over the interval  $[1, N+1]$  and then subdivide  $[1, N+1]$  into unit subintervals  $[n, n+1]$ . Now compare the integral  $\int_1^N f(x)dx$  with the upper and lower Riemann sums associated to the partition  $\{1, 2, \dots, N, N+1\}$ . Since  $f$  is decreasing, we see immediately that the lower sum is  $\sum_{n=2}^{N+1} a_n$  and the upper sum is  $\sum_{n=1}^N a_n$ , so that

$$\sum_{n=2}^{N+1} a_n \leq \int_1^N f(x)dx \leq \sum_{n=1}^N a_n.$$

Taking limits as  $N \rightarrow \infty$ , the result follows.  $\square$

**Remark 2.2.15.** a) *The Integral Test is due to Maclaurin<sup>3</sup> [Ma42] and later in more modern form to A.L. Cauchy [Ca89]. I don't know why it is traditional to attach Cauchy's name to the Condensation Test and not the Integral Test, but I have preserved the tradition nevertheless.*

b) *In calculus one mostly considers integrals of continuous functions. In Theorem 2.2.14 we assumed only that  $f$  was positive and decreasing. In practice we will apply the Integral Test when we can use the Fundamental Theorem of Calculus to evaluate the integral, and for this we will need  $f$  to be continuous. Moreover, in fact it is the case that every monotone function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable [Cl-HC, Thm. 8.13]. So Theorem 2.2.14 is correctly stated.*

c) *It happens that, at least among the series which arise naturally in calculus and undergraduate analysis, it is usually the case that the Condensation Test can be successfully applied to determine convergence / divergence of a series iff the Integral Test can be successfully applied. At some point I decided to look for a direct explanation, i.e., a general link between condensation and improper integrals. I haven't found one!*

---

<sup>3</sup>Colin Maclaurin, 1698-1746

**Example 2.2.16.** Let us use the Integral Test to determine the set of  $p > 0$  such that  $\sum_n \frac{1}{n^p}$  converges. Indeed the series converges iff the improper integral  $\int_1^\infty \frac{dx}{x^p}$  is finite. If  $p \neq 1$ , then we have

$$\int_1^\infty \frac{dx}{x^p} = \frac{x^{1-p}}{1-p} \Big|_{x=1}^{x=\infty}.$$

The upper limit is 0 if  $p - 1 < 0 \iff p > 1$  and is  $\infty$  if  $p < 1$ . Finally,

$$\int_1^\infty \frac{dx}{x} = \log x \Big|_{x=1}^\infty = \infty.$$

So, once again, the  $p$ -series diverges iff  $p > 1$ .

**Example 2.2.17.** We revisit the series  $\sum_{n=2}^\infty \frac{1}{n \log n}$ . We have

$$\int_2^b \frac{1}{x \log x} dx = \log \log x \Big|_2^b = \log \log b - \log \log 2,$$

we have

$$\int_2^\infty \frac{dx}{x \log x} = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \log x} = \lim_{b \rightarrow \infty} \log \log b - \log \log 2 = \infty.$$

Thus by the Integral Test we deduce once again that

$$\sum_{n=2}^\infty \frac{1}{n \log n} = \infty.$$

## 2.2.4 Euler Constants

**Theorem 2.2.18.** (Maclaurin-Cauchy) Let  $f : [1, \infty) \rightarrow \mathbb{R}$  be positive, continuous and decreasing, with  $\lim_{x \rightarrow \infty} f(x) = 0$ . Then we may define the **Euler constant**

$$\gamma_f := \lim_{N \rightarrow \infty} \left( \sum_{n=1}^N f(n) - \int_1^N f(x) dx \right).$$

In other words, the above limit exists.

*Proof.* Put

$$a_N = \sum_{n=1}^N f(n) - \int_1^N f(x)dx,$$

so our task is to show that the sequence  $\{a_N\}$  converges. As in the integral test we have that for all  $n \in \mathbb{Z}^+$

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n). \quad (2.1)$$

Using the second inequality in (2.1) we get

$$a_N = f(N) + \sum_{n=1}^{N-1} f(n) - \int_1^N f(x)dx \geq \sum_{n=1}^N (f(n) - \int_n^{n+1} f(x)dx) \geq 0,$$

and the first inequality in (2.1) we get

$$a_{N+1} - a_N = f(N+1) - \int_N^{N+1} f(x)dx \leq 0.$$

Thus  $\{a_N\}$  is decreasing and bounded below by 0, so it converges.  $\square$

**Example 2.2.19.** Let  $f(x) = \frac{1}{x}$ . Then

$$\gamma = \lim_{n \rightarrow \infty} \gamma_N = \lim_{N \rightarrow \infty} \sum_{n=1}^N f(n) - \int_1^N f(x)dx$$

is the **Euler-Mascheroni constant**. In the notation of the proof one has  $a_1 = 1 > a_2 = 1 + 1/2 - \log 2 \approx 0.806 > a_3 = 1 + 1/2 + 1/3 - \log 3 \approx 0.734$ , and so forth. My laptop computer took a couple of minutes to calculate (by sheer brute force) that

$$a_{5 \times 10^4} = 0.5772256648681995272745120903 \dots$$

This shows well the limits of brute force calculation even with modern computing power since this is correct only to the first nine decimal places: in fact the tenth decimal digit of  $\gamma$  is 9. In fact in 1736 Euler correctly calculated the first 15 decimal digits of  $\gamma$ , whereas in 1790 Lorenzo Mascheroni correctly calculated the first 19 decimal digits (while incorrectly calculating several more). As of 2009, 29,844,489,545 digits of  $\gamma$  have been computed,

by Alexander J. Yee and Raymond Chan.

The constant  $\gamma$  in fact plays a prominent role in classical analysis and number theory: it tends to show in asymptotic formulas in the darndest places. For instance, for a positive integer  $n$ , let  $\varphi(n)$  be the number of integers  $k$  with  $1 \leq k \leq n$  such that no prime number  $p$  simultaneously divides both  $k$  and  $n$ . (The classical name for  $\varphi$  is the **totient function**, but nowadays most people seem to call it the “Euler phi function”.) It is not so hard to see that  $\lim_{n \rightarrow \infty} \varphi(n)/n = 6/\pi^2$ , but function is somewhat irregular (i.e., far from being monotone) and it is of great interest to give precise lower bounds. The best lower bound I know is that for all  $n > 2$ ,

$$\varphi(n) > \frac{n}{e^\gamma \log \log n + \frac{3}{\log \log n}}. \quad (2.2)$$

Note that directly from the definition we have  $\varphi(n) \leq n$ . On the other hand, taking  $n$  to be a product of increasingly many distinct primes, one sees that  $\liminf_{n \rightarrow \infty} \frac{\varphi(n)}{n} = 0$ , i.e.,  $\varphi$  cannot be bounded below by  $Cn$  for any positive constant  $n$ . Given these two facts, (2.2) shows that the discrepancy between  $\varphi(n)$  and  $n$  is very subtle indeed.

**Remark 2.2.20.** Whether  $\gamma$  is rational or irrational is unknown. It might be interesting to explore ir/rationality results of other Euler constants  $\gamma_f$ .

## EXERCISES 2.2

1. Determine whether the series converges or diverges. Give reasons.

- |  |  |
|--|--|
| a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$                 | b) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$                 |
| c) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$               | d) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$               |
| e.) $\sum_{n=1}^{\infty} \left(\frac{1}{n^2+1}\right)^n$ | f.) $\sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^n$     |
| g.) $\sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^{-n}$  | h.) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$         |
| i.) $\sum_{n=1}^{\infty} \frac{\log n}{n^2+1}$           | j.) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/2}$ |
| k.) $\sum_{n=1}^{\infty} n2^{-n}$                        | l.) $\sum_{n=1}^{\infty} (n+1)^{-1/5}$                   |

2. Let  $\sum_n a_n$  be a series of nonnegative terms. Show that if  $\sum_n a_n$  converges then  $\sum_n a_n^2$  converges also.

3. Give an example of a divergent series whose sequence of partial sums is bounded.

4. a) Show that any series of the form  $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ , with  $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ , is convergent. (Hint:  $\frac{d_n}{10^n} \leq \frac{9}{10^n}$ .)
- b) Given  $x \in [0, 1)$  define  $d_n$  recursively as follows: First, notice that  $0 \leq 10x < 10$  so we can choose an integer  $d_1 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  with

$$d_1 \leq 10x < d_1 + 1.$$

Notice that this is equivalent to

$$0 \leq 10\left(x - \frac{d_1}{10}\right) < 1.$$

Now, as in the first step, since  $10\left(x - \frac{d_1}{10}\right)$  is in the interval  $[0, 1)$ , we can choose  $d_2 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  so that

$$d_2 \leq 10\left(10\left(x - \frac{d_1}{10}\right)\right) < d_2 + 1.$$

This can be rewritten as

$$0 \leq 100\left(x - \frac{d_1}{10} - \frac{d_2}{100}\right) < 1.$$

Continuing this process, show by induction that we can find  $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  so that

$$0 \leq 10^n \left( x - \frac{d_1}{10} - \frac{d_2}{100} - \cdots - \frac{d_n}{10^n} \right) < 1.$$

c) With the  $d_n$ 's defined as in part b) show that the series  $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$  converges to  $x$ .

**Remark:** This exercise shows that all real numbers have a decimal expansion.

5. This exercise presents a method to use the Condensation Test to approximate  $p$ -series to arbitrary accuracy.
- a) Let  $N \in \mathbb{N}$ . Show that under the hypotheses of the Condensation Test we have

$$\sum_{n=2^{N+1}}^{\infty} a_n \leq \sum_{n=0}^{\infty} 2^n a_{2^{n+N}}.$$

b) Use part a) to show that for all  $p > 1$  we have

$$\sum_{n=1}^{2^N} \frac{1}{n^p} \leq \sum_{n=1}^{\infty} \frac{1}{n^p} \leq \sum_{n=1}^{2^N} \frac{1}{n^p} + \frac{1}{2^{Np}(1 - 2^{1-p})}.$$

6. Let  $F : [1, \infty) \rightarrow \mathbb{R}$  be increasing. Show: either  $\lim_{x \rightarrow \infty} F(x) = L \in \mathbb{R}$  or  $\lim_{x \rightarrow \infty} F(x) = \infty$ .  
(Hint: use Dedekind's Property of Completeness.)
7. Let  $P(x)$  be a polynomial of degree  $A$  and  $Q(x)$  be a polynomial of degree  $B$ . Let  $N \in \mathbb{Z}^+$  be such that for all  $n \geq N$  we have  $Q(n) \neq 0$ . Show: the series  $\sum_{n=N}^{\infty} \frac{P(n)}{Q(n)}$  converges iff  $B - A \geq 2$ .
8. Determine whether each of the following series converges or diverges.
- a)  $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ .
- b)  $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$ .



## 2.3 Series with Terms of Both Signs

The methods that we developed for studying series with non-negative terms – especially, the use of O’Conor’s Property of Completeness that reduce the question to a simple dichotomy: is the sequence of partial sums bounded or not? – carry over immediately to series  $\sum_n a_n$  with  $a_n \leq 0$  for all  $n$ . Moreover, if we have only finitely many positive terms or only finitely many negative terms, then we may reduce the question of convergence to that of a series with non-negative terms or a series with non-positive terms.

So the essentially new case is that in which we have infinitely many positive terms and infinitely many negative terms. In the most important application – to power series – this will in fact occur very frequently. As a simple example, later we will show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \dots = \frac{1}{e}. \quad (2.3)$$

### 2.3.1 Absolute Convergence

Certainly the most simple-minded way to deal with terms of varying signs is to make all the terms non-negative by applying the absolute value. Namely, instead of considering the given series

$$\sum_n a_n$$

we may consider the **absolute series**

$$\sum_n |a_n|.$$

In (2.3) above, this would have us replace  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$  by  $\sum_{n=0}^{\infty} \frac{1}{n!}$ . It’s no problem to see that the absolute series converges: we have

$$\forall n \geq 2, n! = n(n-1) \cdots 3 \cdot 2 \cdot 1 \geq 2 \cdot 2 \cdots 2 \cdot 1 = 2^{n-1}$$

and

$$0! = 1 \geq 2^{0-1}, \quad 1! = 1 = 2^{1-1},$$

so  $\frac{1}{n!} \leq \frac{1}{2^{n-1}} = \frac{2}{2^n}$  for all  $n \in \mathbb{N}$  and thus

$$\sum_{n=0}^{\infty} \frac{1}{n!} \leq \sum_{n=0}^{\infty} \frac{2}{2^n} = 3.$$

The question of course is: what does the convergence of the absolute series  $\sum_{n=0}^{\infty} \frac{1}{n!}$  have to do with the convergence of the original series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$  ??? The triangle inequality shows that the partial sums of the original series are bounded in magnitude by the partial sums of the absolute series...but since we need to have non-negative terms in order to apply the Comparison Test, this is certainly not convincing. The following result comes to the rescue.

**Proposition 2.3.1.** *Let  $\sum_n a_n$  be a real series. If the absolute series  $\sum_n |a_n|$  converges, then so does  $\sum_n a_n$ .*

*Proof.* We shall give *two proofs* of this important result.

**First Proof:** Consider the three series  $\sum_n a_n$ ,  $\sum_n |a_n|$  and  $\sum_n a_n + |a_n|$ . By hypothesis,  $\sum_n |a_n|$  converges. But we claim that this implies that  $\sum_n a_n + |a_n|$  converges as well. Indeed, consider the expression  $a_n + |a_n|$ : it is equal to  $2a_n = 2|a_n|$  when  $a_n$  is non-negative and 0 when  $a_n$  is negative. So the series  $\sum_n a_n + |a_n|$  has non-negative terms and

$$\sum_n a_n + |a_n| \leq \sum_n 2|a_n| < \infty,$$

so  $\sum_n a_n + |a_n|$  converges by the Comparison Test. By the Three Series Principle,  $\sum_n a_n$  converges.

**Second Proof:** The above argument is clever – maybe too clever! Let’s try something more fundamental. By Cauchy’s Property of Completeness, the sequence of partial sums of the absolute series is a Cauchy sequence. Just by applying the triangle inequality, we’ll see that the sequence of partial sums of the original series is also Cauchy, hence convergent. Here goes: since  $\sum_n |a_n|$  converges, for every  $\epsilon > 0$  there exists  $N \in \mathbb{Z}^+$  such that for all  $n \geq m \geq N$ , we have  $\sum_{k=m}^n |a_k| < \epsilon$ . Therefore

$$\left| \sum_{k=m}^n a_k \right| \leq \sum_{k=m}^n |a_k| < \epsilon,$$

and  $\sum_n a_n$  converges by the Cauchy criterion. □

Proposition 2.3.1 justifies the following new terminology: we say that a series  $\sum_n a_n$  is **absolutely convergent** if the absolute series  $\sum_n |a_n|$  converges. Thus Proposition 2.3.1 gets rephrased as “An absolutely convergent series is convergent.” (Thank goodness – otherwise things would be very confusing!) By the way, do you hear the terminology pushing absolute convergence as

a platinum-eagle-membership-has-its-privileges kind of convergence? In fact this is true, and when presented with a series of terms of varying sign, one very much hopes it is not just convergent but absolutely convergent.

We say that a series is **nonabsolutely convergent** if it is convergent but not absolutely convergent.<sup>4</sup> Thus our simple dichotomy of convergence / divergence has been expanded to a trichotomy: for every infinite series  $\sum_n a_n$ , exactly one of the following holds:

- $\sum_n a_n$  is absolutely convergent.
- $\sum_n a_n$  is nonabsolutely convergent.
- $\sum_n a_n$  is divergent.

In the remainder of this section we consider absolute convergence only. In the next section we treat the one aspect of nonabsolute convergence that is really necessary for our purposes: the Alternating Series Test.

**Example 2.3.2.** For  $p \in \mathbb{R}$ , consider the **alternating  $p$ -series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots$$

The associated absolute series is just the usual  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ , which we know converges iff  $p > 1$ . Therefore the alternating  $p$ -series is absolutely convergent iff  $p > 1$ . In particular, the series converges for these values.

When  $p \leq 0$ ,  $\frac{(-1)^{n+1}}{n^p} \not\rightarrow 0$ , so the alternating  $p$ -series diverges by the  $N$ th Term Test. (We may generalize this observation as follows: since for any real sequence  $\{a_n\}$  we have  $a_n \rightarrow 0$  iff  $|a_n| \rightarrow 0$ , the  $N$ th term test applies to show the divergence of the absolute series  $\sum_n |a_n|$  exactly when it applies to show the divergence of the given series  $\sum_n a_n$ .) When  $p \in (0, 1)$ , the alternating  $p$ -series is not absolutely convergent, but it may conceivably still be convergent. Notice that we have not yet seen an example of a series which is convergent but not absolutely convergent, and we will defer the issue until later, but for now you might want to use a software package to compute

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<sup>4</sup>The more common terminology here is **conditionally convergent**, but I don't like it for various reasons.

various partial sums of the alternating harmonic series, e.g.

$$\sum_{n=1}^{10} \frac{(-1)^{n+1}}{n}, \quad \sum_{n=1}^{100} \frac{(-1)^{n+1}}{n}, \quad \sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} \dots$$

You cannot prove that a series converges or diverges by looking at finitely many partial sums, but you might acquire some intuition as to what the answer should be.

The following convergence test will be used more than all the others from now until the end of the course.

**Theorem 2.3.3.** (Ratio and Root Tests) Let  $\sum_n a_n$  be a real series.

a) **Ratio Test Part I:**

Suppose that  $a_n \neq 0$  for all  $n$ . If there is  $N \in \mathbb{Z}^+$  and  $r \in (0, 1)$  such that

$$\forall n \geq N, \quad \left| \frac{a_{n+1}}{a_n} \right| \leq r,$$

then the series  $\sum_n a_n$  is absolutely convergent.

b) **Ratio Test Part II:**

Suppose that  $a_n \neq 0$  for all  $n$ . If there is  $N \in \mathbb{Z}^+$  and  $r \geq 1$  such that

$$\forall n \geq N, \quad \left| \frac{a_{n+1}}{a_n} \right| \geq r,$$

the series  $\sum_n a_n$  diverges.

c) **Root Test Part I:**

If there is  $N \in \mathbb{Z}^+$  and  $r \in [0, 1)$  such that

$$\forall n \geq N, \quad \left| a_n \right|^{\frac{1}{n}} \leq r,$$

then the series  $\sum_n a_n$  is absolutely convergent.

d) **Root Test Part II:**

If  $|a_n|^{\frac{1}{n}} \geq 1$  for infinitely many  $n$ , then the series  $\sum_n a_n$  diverges.

*Proof.* Though to prove any four part theorem takes some space and time, the basic idea in all cases is to compare our series with a geometric series.

a) Starting at  $n = N$ , each ratio of absolute values of successive terms is at

most  $r$ : thus  $|a_{N+1}| \leq r|a_N|$ ,  $|a_{N+2}| \leq r|a_{N+1}| \leq r(r|a_N|) = r^2|a_N|$ , and so forth: an easy inductive argument shows that for all  $n \geq N$  we have

$$|a_n| \leq r^{n-N}|a_N|.$$

Therefore by comparison we get

$$\sum_{n \geq N} |a_n| \leq \sum_{n \geq N} r^{n-N}|a_N| = \frac{|a_N|}{r^N} \sum_n r^n < \infty,$$

since  $r \in (0, 1)$ . It follows that  $\sum_n |a_n| < \infty$ : absolute convergence.

b) For all  $n \geq N$  we have

$$\frac{|a_{n+1}|}{|a_n|} \geq r \geq 1,$$

so  $|a_{n+1}| \geq |a_n|$  for all  $n \geq N$ , and thus

$$0 \leq |a_N| \leq |a_{N+1}| \leq |a_{N+2}| \leq \dots$$

It follows that  $|a_n| \not\rightarrow 0$ , hence also  $a_n \not\rightarrow 0$ , so the series  $\sum_n a_n$  diverges by the  $N$ th Term Test.

c) This is similar, but easier: for all  $n \geq N$ , we have  $|a_n| \leq r^n$ , and since  $r \in [0, 1)$  we have

$$\sum_{n \geq N} |a_n| \leq \sum_{n \geq N} r^n < \infty.$$

d) If  $|a_n|^{\frac{1}{n}} \geq 1$ , then  $|a_n| \geq 1$ . If this holds for infinitely many  $n$ , then  $|a_n| \not\rightarrow 0$  and thus  $a_n \not\rightarrow 0$  and the series diverges by the  $N$ th Term Test.  $\square$

The above Ratio and Root Tests are not quite the ones you meet in calculus: by working with inequalities instead of limits, we get somewhat simpler, stronger statements. In fact, it is easy to deduce the more familiar version:

**Corollary 2.3.4.** (*Calculus Student's Ratio and Root Test*)

a) Let  $\sum_n a_n$  be a series with  $a_n \neq 0$  for all  $n$ . Suppose the ratio test limit

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists. Then: if  $\rho < 1$ , the series  $\sum_n a_n$  converges absolutely, while if  $\rho > 1$  the series  $\sum_n a_n$  diverges.

b) Let  $\sum_n a_n$  be a series. Suppose the root test limit

$$\theta = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$

exists. Then: if  $\theta < 1$ , the series  $\sum_n a_n$  converges absolutely, while if  $\theta > 1$  the series  $\sum_n a_n$  diverges.

*Proof.* We will prove the first assertion of part a) and leave the others as exercises. Suppose

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Choose  $r$  such that  $\rho < r < 1$ . Taking  $\epsilon = r - \rho$ , there is  $N \in \mathbb{Z}^+$  such that for all  $n \geq N$  we have

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - \rho \right| < r - \rho$$

and thus

$$\left| \frac{a_{n+1}}{a_n} \right| < r.$$

So the series  $\sum_n a_n$  is absolutely convergent by the Ratio Test.  $\square$

**Remark 2.3.5.** *The Ratio and Root Tests are closely related.*

(i) *In theory the Root Test is more powerful than the Ratio Test: it can be shown that if the Ratio Test limit  $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exists, then also the Root Test limit  $\theta = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$  exists and  $\rho = \theta$ . However, it is possible for the Root Test limit to exist when the Ratio Test limit does not. For proofs of these assertions see [Cl-HC, §11.5.3].*

(ii) *However, at least for the types of series one generally meets in undergraduate courses, it is usually the case that both limits exist and the Ratio Test limit is easier to evaluate. Thus in practice one should usually reach for the Ratio Test rather than the Root Test. The exception is when the terms are given in the form  $|a_n| = b_n^n$ , in which case  $|a_n|^{\frac{1}{n}} = b_n$ .*

(iii) *It may well be the case that neither the Ratio Test nor Root Test limits exist. There are aspects of the theory in which this becomes annoying. The remedy is to use  $\limsup$  and  $\liminf$  instead (concepts which are only briefly touched on in several exercises in this text). For versions of the tests which use these upper and lower limits, again see [Cl-HC, §11.5.3].*

**Example 2.3.6.** a) *Applying the Ratio Test to the series  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$  we find*

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \frac{2^n(n+1)^2}{2^{n+1}n^2} = \frac{(n+1)^2}{2n^2}.$$

And so  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = 1/2$ . Since this is less than 1, we conclude that the series converges.

b) Applying the Ratio Test to the series  $\sum_{n=1}^{\infty} \frac{n^2 a^n}{3^n}$  we see that

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|(n+1)^2 a^{n+1} / 3^{n+1}|}{|n^2 a^n / 3^n|} \\ &= \frac{|a|(n+1)^2}{3n^2}. \end{aligned}$$

And so  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = |a|/3$ . Thus we see that this series converges if  $-3 < a < 3$ , and it diverges if  $|a| > 3$ . The ratio test tells us nothing about the cases that  $a = 3$  or  $a = -3$ , but we can easily see that the series  $\sum n^2$  and  $\sum (-1)^n n^2$  are divergent since the terms do not converge to zero.

c) Consider the series  $\sum_{n=1}^{\infty} \frac{1}{n^n}$ . Since

$$|a_n|^{1/n} = \left(\frac{1}{n^n}\right)^{1/n} = \frac{1}{n},$$

this is a case where the Root Test will be easier to apply: we have

$$\theta = \lim_{n \rightarrow \infty} |a_n|^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0,$$

so the series converges.

What happens if we try to apply the Ratio Test here? We get

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^{n+1}} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n \frac{1}{n+1}.$$

Let us separately evaluate

$$L = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n.$$

First we apply the logarithm, getting

$$\log L = \lim_{n \rightarrow \infty} \log \left( \frac{n}{n+1} \right)^n = n \log \frac{n}{n+1} = \lim_{n \rightarrow \infty} -\frac{\log(1+1/n)}{1/n}.$$

We may apply L'Hôpital's Rule, getting

$$\log L = -\lim_{n \rightarrow \infty} \frac{-1/n^2 \cdot \frac{1}{1+1/n}}{-1/n^2} = -\lim_{n \rightarrow \infty} \frac{1}{1+1/n} = -1.$$

Therefore  $L = \frac{1}{e}$ , and

$$\rho = \frac{1}{e} \cdot \lim_{n \rightarrow \infty} \frac{1}{n+1} = \frac{1}{e} \cdot 0 = 0.$$

So, yes: the Root Test was easier!

**Example 2.3.7.** Consider the  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ . The Ratio Test limit is

$$\rho = \lim_{n \rightarrow \infty} \frac{n^p}{(n+1)^p} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^p = 1^p = 1.$$

The Root Test limit is

$$\theta = \lim_{n \rightarrow \infty} \frac{1}{n^{p/n}} = \lim_{n \rightarrow \infty} (n^{-1/n})^p = 1^p = 1.$$

(It takes a little work to see that  $n^{-1/n} \rightarrow 1$ : take the logarithm and apply L'Hôpital's Rule, as above.) This shows that no conclusion on convergence/divergence can be drawn when the Ratio Test or Root Test limit is 1: indeed, we know that for some values of  $p$  we have convergence and others we have divergence.

The more precise Theorem 2.3.3 does slightly better than the calculus version: it allows us to show that the  $p$ -series diverges when  $p \leq 0$ . This is still not very impressive, since if  $p \leq 0$  then  $\frac{1}{n^p} \not\rightarrow 0$ .

There is a clear moral here: the Ratio and Root Tests give a very useful general tool for determining convergence. In particular, in our later study of power series these will be the most important test by far. However, the Ratio and Root Tests are not capturing subtle behavior of convergence or divergence: when the either test works to show convergence, it works because the terms of the absolute series are eventually smaller than that of a convergent geometric series, and when either test works to show divergence, it works because the terms of the absolute series are eventually larger than those of a divergent geometric series...so in particular the sequence of terms does not converge to 0. (In fact when  $\rho > 1$  or  $\theta > 1$ , not only does the general term not tend to 0, it tends to infinity in absolute value with at least exponential speed!) When subtlety is needed, we should turn to things like the Comparison, Condensation and Integral Tests.



### 2.3.2 The Alternating Series Test

Consider the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots$$

Upon taking the absolute value of every term we get the usual harmonic series, which diverges, so the alternating harmonic series is *not* absolutely convergent. However, some computations with partial sums suggests that the alternating harmonic series *is* convergent, with sum  $\log 2$ . By looking more carefully at the partial sums, we can find a pattern that allows us to show that the series does indeed converge. (Whether it converges to  $\log 2$  is a different matter, of course, which we will revisit much later on.)

It will be convenient to write  $a_n = \frac{1}{n}$ , so that the alternating harmonic series is  $\sum_n \frac{(-1)^{n+1}}{n+1}$ . We draw the reader's attention to three properties of this series:

(AST1) The terms alternate in sign.

(AST2) The  $n$ th term approaches 0.

(AST3) The sequence of absolute values of the terms is decreasing:

$$a_1 \geq a_2 \geq \dots \geq a_n \geq \dots$$

These are the clues from which we will make our case for convergence. Here it is: consider the process of passing from the first partial sum  $S_1 = 1$  to  $S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ . We have  $S_3 \leq 1$ , and this is no accident: since  $a_2 \geq a_3$ , subtracting  $a_2$  and then adding  $a_3$  leaves us no larger than where we started. But indeed this argument is valid in passing from any  $S_{2n-1}$  to  $S_{2n+1}$ :

$$S_{2n+1} = S_{2n-1} - a_{2n} + a_{2n+1} \leq S_{2n-1}.$$

It follows that the sequence of odd-numbered partial sums  $\{S_{2n-1}\}$  is decreasing. Moreover,

$$S_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2n-1} - a_{2n}) + a_{2n-1} \geq 0.$$

Therefore all the odd-numbered terms are bounded below by 0. By the Monotone Sequence Lemma, the sequence  $\{S_{2n+1}\}$  converges to its greatest lower bound, say  $S_{\text{odd}}$ . On the other hand, just the opposite sort of thing happens for the even-numbered partial sums:

$$S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2} \geq S_{2n}$$

and

$$S_{2n+2} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n} - a_{2n+1}) - a_{2n+2} \leq a_1.$$

Therefore the sequence of even-numbered partial sums  $\{S_{2n}\}$  is increasing and bounded above by  $a_1$ , so it converges to its least upper bound, say  $S_{\text{even}}$ . Thus we have split up our sequence of partial sums into two complementary subsequences and found that each of these series converges. The full sequence  $\{S_n\}$  converges iff  $S_{\text{odd}} = S_{\text{even}}$ . (You are asked to prove this in Exercise 1.) Now the inequalities

$$S_2 \leq S_4 \leq \dots \leq S_{2n} \leq S_{2n+1} \leq S_{2n-1} \leq \dots \leq S_3 \leq S_1$$

show that  $S_{\text{even}} \leq S_{\text{odd}}$ . Moreover, for any  $n \in \mathbb{Z}^+$  we have

$$S_{\text{odd}} - S_{\text{even}} \leq S_{2n+1} - S_{2n} = a_{2n+1}.$$

Since  $a_{2n+1} \rightarrow 0$ , we conclude  $S_{\text{odd}} = S_{\text{even}} = S$ , i.e., the series converges.

In fact these inequalities give something else: since for all  $n$  we have  $S_{2n} \leq S_{2n+2} \leq S \leq S_{2n+1}$ , it follows that

$$|S - S_{2n}| = S - S_{2n} \leq S_{2n+1} - S_{2n} = a_{2n+1}$$

and similarly

$$|S - S_{2n+1}| = S_{2n+1} - S \leq S_{2n+1} - S_{2n+2} = a_{2n+2}.$$

Thus the error in cutting off the infinite sum  $\sum_{n=1}^{\infty} (-1)^{n+1} |a_n|$  after  $N$  terms is in absolute value at most the absolute value of the next term:  $a_{N+1}$ .

Of course in all this we never used that  $a_n = \frac{1}{n}$  but only that we had a series satisfying (AST1) (i.e., an alternating series), (AST2) and (AST3). Therefore the preceding arguments have in fact proved the following more general result, due originally due to Leibniz.<sup>5</sup>

<sup>5</sup>Gottfried Wilhelm von Leibniz, 1646-1716

**Theorem 2.3.8.** (*Lebniz's Alternating Series Test*)

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of non-negative real numbers which is decreasing and such that  $\lim_{n \rightarrow \infty} a_n = 0$ . Then:

- a) The associated **alternating series**  $\sum_n (-1)^{n+1} a_n$  converges.  
 b) For  $N \in \mathbb{Z}^+$ , put

$$E_N = \left| \left( \sum_{n=1}^{\infty} (-1)^{n+1} a_n \right) - \left( \sum_{n=1}^N (-1)^{n+1} a_n \right) \right|, \quad (2.4)$$

the error obtained by cutting off the infinite sum after  $N$  terms. Then we have the **error estimate**

$$E_N \leq a_{N+1}.$$

**Example 2.3.9.** Let  $p \in \mathbb{R}$ . The **alternating  $p$ -series**  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$  is:

- (i) divergent if  $p \leq 0$ ,  
 (ii) nonabsolutely convergent if  $0 < p \leq 1$ , and  
 (iii) absolutely convergent if  $p > 1$ .

For any convergent series  $\sum_{n=1}^{\infty} a_n = S$ , we may define  $E_N$  as in (2.4) above:

$$E_N = \left| S - \sum_{n=1}^N a_n \right|.$$

Then because the series converges to  $S$ ,  $\lim_{N \rightarrow \infty} E_N = 0$ , and conversely: in other words, to say that the error goes to 0 is a rephrasing of the fact that the partial sums of the series converge to  $S$ . Each of these statements is (in the local jargon) **soft**: we assert that a quantity approaches 0 and  $N \rightarrow \infty$ , so that in theory, given any  $\epsilon > 0$ , we have  $E_N < \epsilon$  for all sufficiently large  $N$ . But as we have by now seen many times, it is often possible to show that  $E_N \rightarrow 0$  without coming up with an *explicit* expression for  $N$  in terms of  $\epsilon$ . But this stronger statement is exactly what we have given in Theorem 2.3.8b): we have given an *explicit* upper bound on  $E_N$  as a function of  $N$ . This type of statement is called a **hard** statement or an **explicit error estimate**: such statements tend to be more difficult to come by than soft statements, but also more useful to have. Here, as long as we can similarly make explicit how large  $N$  has to be in order for  $a_N$  to be less than a given  $\epsilon > 0$ , we get a completely explicit error estimate and can use this to actually compute the sum  $S$  to arbitrary accuracy.

**Example 2.3.10.** We compute  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  to three decimal place accuracy. (Let us agree that to “compute a number  $\alpha$  to  $k$  decimal place accuracy means to compute it with error less than  $10^{-k}$ . A little thought shows that this is not quite enough to guarantee that the first  $k$  decimal places of the approximation are equal to the first  $k$  decimal places of  $\alpha$ , but we do not want to occupy ourselves with such issues here.) By Theorem 2.3.8b), it is enough to find an  $N \in \mathbb{Z}^+$  such that  $a_{N+1} = \frac{1}{N+1} < \frac{1}{1000}$ . We may take  $N = 1000$ . Therefore

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} \right| \leq \frac{1}{1001}.$$

Using a software package, we find that

$$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} = 0.6926474305598203096672310589 \dots$$

Exercise 7 shows that the exact value of the sum is  $\log 2$ , which my software package tells me is

$$\log 2 = 0.6931471805599453094172321214.$$

Thus the *actual error* in cutting off the sum after 1000 terms is

$$E_{1000} = 0.0004997500001249997500010625033.$$

It is important to remember that this (and other) error estimates only give upper bounds on the error: the true error could be much smaller.<sup>6</sup> In this case we were guaranteed to have an error at most  $\frac{1}{1001}$ , and the true error is about half of that. Thus the estimate for the error is reasonably accurate.

Note well that although the error estimate of Theorem 2.3.8b) is very easy to apply, if  $a_n$  tends to zero rather slowly (as in this example), it is not very efficient for computations. For instance, in order to compute the true sum of the alternating harmonic series to six decimal place accuracy using this method, we would need to add up the first million terms: that’s a lot of calculation. (Thus please be assured that this is *not* the way that a calculator or computer would compute  $\log 2$ .)

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<sup>6</sup>If you think about it for a little while, this is a fact of life: if we knew *exactly* what the error was, we would know the exact quantity!

**Example 2.3.11.** We compute  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$  to six decimal place accuracy. Thus we need to choose  $N$  such that  $a_{N+1} = \frac{1}{(N+1)!} < 10^{-6}$ , or equivalently such that  $(N+1)! > 10^6$ . A little calculation shows  $9! = 362,880$  and  $10! = 3,628,800$ , so that we may take  $N = 9$  (but not  $N = 8$ ). Therefore

$$\left| \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} - \sum_{n=0}^9 \frac{(-1)^n}{n!} \right| < \frac{1}{10!} < 10^{-6}.$$

Using a software package, we find

$$\sum_{n=0}^9 \frac{(-1)^n}{n!} = 0.3678791887125220458553791887.$$

In this case the exact value of the series is

$$\frac{1}{e} = 0.3678794411714423215955237701$$

so the true error is

$$E_9 = 0.0000002524589202757401445814516374,$$

which this time is only very slightly less than the guaranteed

$$\frac{1}{10!} = 0.0000002755731922398589065255731922.$$

Notice that in the above example we used the error estimate of the Alternating Series Test in a case where the series is absolutely convergent. Thus the one test we have given which can show nonabsolute convergence is arguably more useful to us than nonabsolute convergence itself.

### EXERCISES 2.3

- Let  $\{s_n\}$  be a sequence and let  $\{c_n\}$  and  $\{d_n\}$  be the subsequences given by  $c_n = s_{2n}$  and  $d_n = s_{2n+1}$ . Assume that  $\lim_{n \rightarrow \infty} c_n = L$  and that  $\lim_{n \rightarrow \infty} d_n = L$ . Prove that  $\lim_{n \rightarrow \infty} s_n = L$ .
- What does the Ratio Test tell you about the following series?

- a)  $\sum_n \frac{2^n}{n!}$   
 b)  $\sum_n \frac{n!}{2^n}$   
 c)  $\sum_n \frac{n}{3^n}$   
 d)  $\sum_n \frac{\log n}{n^2}$
3. Classify each of the following series is absolutely convergent, nonabsolutely convergent or divergent.
- a)  $\sum_n \frac{(-1)^n}{n^2}$   
 b)  $\sum_n \frac{(-1)^n}{\sqrt{n}}$   
 c)  $\sum_n \frac{(-1)^n n^2}{n!}$   
 d)  $\sum_n \frac{(-1)^n (n^2+3)}{3n^2+1}$   
 e.)  $\sum_n \frac{(-1)^n}{n \log n}$   
 f.)  $\sum_n \frac{(-1)^n n}{\sqrt{2^n+1}}$   
 g.)  $\sum_n n \left(-\frac{2}{5}\right)^n$
4. Corollary 2.3.4 contains four assertions, two for the Ratio Test and two for the Root Test. We proved the convergence part of the Ratio Test.
- a) Prove the divergence part of the Ratio Test.  
 b) Prove the convergence part of the Root Test.  
 c) Prove the divergence part of the Root Test.
5. For which values of  $a \in \mathbb{R}$  do the following series converge?
- a)  $\sum_n \frac{a^n}{2^n}$   
 b)  $\sum_n \frac{na^n}{2^n}$   
 c)  $\sum_n \left(\frac{4}{a^2}\right)^n$   
 d)  $\sum_n \frac{(-a)^n}{n!}$   
 e.)  $\sum_n \frac{(-a)^n}{n^2}$
6. Assume that the series  $\sum_{n=1}^{\infty} a_n$  converges absolutely. Prove that

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|.$$

7. One can use the convergence of the sequence  $\{\gamma_n\}$  given in Example 2.2.19 to evaluate the alternating harmonic series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$  as follows:

a) Let  $s_n = \sum_{k=1}^n \frac{1}{k}$  denote the  $n^{\text{th}}$  partial sum of the harmonic series and show that

$$1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2n-1} - \frac{1}{2n} = s_{2n} - s_n.$$

b) Use the fact that  $s_n = \gamma_n + \log(n+1)$  to write

$$1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2n-1} - \frac{1}{2n} = (\gamma_{2n} - \gamma_n) + \log\left(\frac{2n+1}{n+1}\right).$$

c) Taking the limit as  $n \rightarrow \infty$ , conclude that  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \log 2$ .

## 2.4 Cauchy Products

### 2.4.1 Cauchy products I: non-negative terms

Let  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  be two infinite series. Is there some notion of a **product** of these series?

In order to forestall possible confusion, let us point out that many students are tempted to consider the following “product” operation on series:

$$\left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right) \stackrel{??}{=} \sum_{n=0}^{\infty} a_n b_n.$$

In other words, given two sequences of terms  $\{a_n\}$ ,  $\{b_n\}$ , we form a new sequence of terms  $\{a_n b_n\}$  and then we form the associated series. In fact this

is not a very useful candidate for the product. What we surely want to happen is that if  $\sum_n a_n = A$  and  $\sum_n b_n = B$  then our “product series” should converge to  $AB$ . But for instance, take  $\{a_n\} = \{b_n\} = \frac{1}{2^n}$ . Then  $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \frac{1}{1-\frac{1}{2}} = 2$ , so  $AB = 4$ , whereas  $\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$ . Of course  $\frac{4}{3} < 4$ . What went wrong?

Plenty! We have ignored the laws of algebra for finite sums: e.g.

$$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = a_0b_0 + a_1b_1 + a_2b_2 + a_0b_1 + a_1b_0 + a_0b_2 + a_1b_1 + a_2b_0.$$

The product is different and more complicated – and indeed, if all the terms are positive, strictly larger – than just  $a_0b_0 + a_1b_1 + a_2b_2$ . We have forgotten about the cross-terms which show up when we multiply one expression involving several terms by another expression involving several terms.<sup>7</sup>

Let us try again at formally multiplying out a product of infinite series:

$$\begin{aligned} & (a_0 + a_1 + \dots + a_n + \dots)(b_0 + b_1 + \dots + b_n + \dots) \\ &= a_0b_0 + a_0b_1 + a_1b_0 + a_0b_2 + a_1b_1 + a_2b_0 + \dots + a_0b_n + a_1b_{n-1} + \dots + a_nb_0 + \dots \end{aligned}$$

So it is getting a bit notationally complicated. In order to shoehorn the right hand side into a single infinite series, we need to either (i) choose some particular ordering to take the terms  $a_k b_\ell$  on the right hand side, or (ii) collect some terms together into an  $n$ th term.

For the moment we choose the latter: we define for any  $n \in \mathbb{N}$

$$c_n = \sum_{k=0}^n a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$

and we define the **Cauchy product** of  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  to be the series

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right).$$

---

<sup>7</sup>To the readers who did not forget about the cross-terms: my apologies. But it is a common enough misconception that it had to be addressed.



**Theorem 2.4.1.** Let  $\{a_n\}_{n=0}^{\infty}$ ,  $\{b_n\}_{n=0}^{\infty}$  be two series with non-negative terms. Let  $\sum_{n=0}^{\infty} a_n = A$  and  $\sum_{n=0}^{\infty} b_n = B$ . Putting  $c_n = \sum_{k=0}^n a_k b_{n-k}$  we have that  $\sum_{n=0}^{\infty} c_n = AB$ . In particular, the Cauchy product series converges iff the two “factor series”  $\sum_n a_n$  and  $\sum_n b_n$  both converge.

*Proof.* It is instructive to define yet another sequence, the **box product**, as follows: for all  $N \in \mathbb{N}$ ,

$$\square_N = \sum_{0 \leq i, j \leq N} a_i b_j = (a_0 + \dots + a_N)(b_0 + \dots + b_N) = A_N B_N.$$

Thus by the usual product rule for sequences, we have

$$\lim_{N \rightarrow \infty} \square_N = \lim_{N \rightarrow \infty} A_N B_N = AB.$$

So the box product converges to the product of the sums of the two series. This suggests that we compare the Cauchy product to the box product. The entries of the box product can be arranged to form a square, viz:

$$\begin{aligned} \square_N &= a_0 b_0 + a_0 b_1 + \dots + a_0 b_N \\ &\quad + a_1 b_0 + a_1 b_1 + \dots + a_1 b_N \\ &\quad \vdots \\ &\quad + a_N b_0 + a_N b_1 + \dots + a_N b_N. \end{aligned}$$

On the other hand, the terms of the  $N$ th partial sum of the Cauchy product can naturally be arranged in a triangle:

$$\begin{aligned} C_N &= && a_0 b_0 \\ &&& + a_0 b_1 + a_1 b_0 \\ &&& + a_0 b_2 + a_1 b_1 + a_2 b_0 \\ &&& + a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \\ &&& \vdots \\ &&& + a_0 b_N + a_1 b_{N-1} + a_2 b_{N-2} + \dots + a_N b_0. \end{aligned}$$

Thus while  $\square_N$  is a sum of  $(N+1)^2$  terms,  $C_N$  is a sum of  $1 + 2 + \dots + N + 1 = \frac{(N+1)(N+2)}{2}$  terms: those lying on or below the diagonal of the square.

Thus in considerations involving the Cauchy product, the question is to what extent one can neglect the terms in the upper half of the square – i.e., those with  $a_i b_j$  with  $i + j > N$  – as  $N$  gets large.

Here, since all the  $a_i$ 's and  $b_j$ 's are non-negative and  $\square_N$  contains all the terms of  $C_N$  and others as well, we certainly have

$$C_N \leq \square_N = A_N B_N \leq AB.$$

Thus  $C = \lim_{N \rightarrow \infty} C_N \leq AB$ . For the converse, the key observation is that if we make the sides of the triangle twice as long, it will contain the box: that is, every term of  $\square_N$  is of the form  $a_i b_j$  with  $0 \leq i, j \leq N$ ; thus  $i + j \leq 2N$  so  $a_i b_j$  appears as a term in  $C_{2N}$ . It follows that  $C_{2N} \geq \square_N$  and thus

$$C = \lim_{N \rightarrow \infty} C_N = \lim_{N \rightarrow \infty} C_{2N} \geq \lim_{N \rightarrow \infty} \square_N = \lim_{N \rightarrow \infty} A_N B_N = AB.$$

Having shown both that  $C \leq AB$  and  $C \geq AB$ , we conclude

$$C = \sum_{n=0}^{\infty} a_n = AB = \left( \sum_{n=0}^{\infty} a_n \right) \left( \sum_{n=0}^{\infty} b_n \right) \quad \square.$$

## 2.4.2 Cauchy products II: when one series is absolutely convergent

**Theorem 2.4.2.** *Let  $\sum_{n=0}^{\infty} a_n = A$  and  $\sum_{n=0}^{\infty} b_n = B$  be two absolutely convergent series, and let  $c_n = \sum_{k=0}^n a_k b_{n-k}$ . Then the Cauchy product series  $\sum_{n=0}^{\infty} c_n$  is absolutely convergent, with sum  $AB$ .*

*Proof.* We have proved this result already when  $a_n, b_n \geq 0$  for all  $n$ . We wish, of course, to reduce to that case. As far as the convergence of the Cauchy product, this is completely straightforward: we have

$$\sum_{n=0}^{\infty} |c_n| = \sum_{n=0}^{\infty} \left| \sum_{k=0}^n a_k b_{n-k} \right| \leq \sum_{n=0}^{\infty} \sum_{k=0}^n |a_k| |b_{n-k}| < \infty,$$

the last inequality following from the fact that  $\sum_{n=0}^{\infty} \sum_{k=0}^n |a_k| |b_{n-k}|$  is the Cauchy product of the two non-negative series  $\sum_{n=0}^{\infty} |a_n|$  and  $\sum_{n=0}^{\infty} |b_n|$ , hence it converges. Therefore  $\sum_n |c_n|$  converges by comparison, so the Cauchy

product series  $\sum_n c_n$  converges.

We wish to show that  $\lim_{N \rightarrow \infty} C_N = \sum_{n=0}^{\infty} c_n = AB$ . Recall the notation

$$\square_N = \sum_{0 \leq i, j \leq N} a_i b_j = (a_0 + \dots + a_N)(b_0 + \dots + b_N) = A_N B_N.$$

We have

$$\begin{aligned} |C_N - AB| &\leq |\square_N - AB| + |\square_N - C_N| \\ &= |A_N B_N - AB| + |a_1 b_N| + |a_2 b_{N-1}| + |a_2 b_N| + \dots + |a_N b_1| + \dots + |a_N b_N| \\ &\leq |A_N B_N - AB| + \left( \sum_{n=0}^{\infty} |a_n| \right) \left( \sum_{n \geq N} |b_n| \right) + \left( \sum_{n=0}^{\infty} |b_n| \right) \left( \sum_{n \geq N} |a_n| \right). \end{aligned}$$

Fix  $\epsilon > 0$ ; since  $A_N B_N \rightarrow AB$ , for all sufficiently large  $N$  we have  $|A_N B_N - AB| < \frac{\epsilon}{3}$ . Put

$$\mathbb{A} = \sum_{n=0}^{\infty} |a_n|, \quad \mathbb{B} = \sum_{n=0}^{\infty} |b_n|.$$

By the Cauchy criterion, for sufficiently large  $N$  we have  $\sum_{n \geq N} |b_n| < \frac{\epsilon}{3\mathbb{A}}$  and  $\sum_{n \geq N} |a_n| < \frac{\epsilon}{3\mathbb{B}}$  and thus  $|C_N - AB| < \epsilon$ .  $\square$

While the proof of Theorem 2.4.2 may seem rather long, it is in fact a rather straightforward argument: one shows that the difference between the box product and the partial sums of the Cauchy product becomes negligible as  $N$  tends to infinity. In less space but with a bit more finesse, one can prove the following stronger result, a theorem of F. Mertens [Me72].<sup>8</sup>

**Theorem 2.4.3.** (*Mertens' Theorem*) Let  $\sum_{n=0}^{\infty} a_n = A$  be an absolutely convergent series and  $\sum_{n=0}^{\infty} b_n = B$  be a convergent series. Then the Cauchy product series  $\sum_{n=0}^{\infty} c_n$  converges to  $AB$ .

*Proof.* (Rudin [R, Thm. 3.50]): define (as usual)

$$A_N = \sum_{n=0}^N a_n, \quad B_N = \sum_{n=0}^N b_n, \quad C_N = \sum_{n=0}^N c_n$$

and also (for the first time)

$$\beta_n = B_n - B.$$

---

<sup>8</sup>Franz Carl Joseph Mertens, 1840-1927

Then for all  $N \in \mathbb{N}$ ,

$$\begin{aligned} C_N &= a_0b_0 + (a_0b_1 + a_1b_0) + \dots + (a_0b_N + \dots + a_Nb_0) \\ &= a_0B_N + a_1B_{N-1} + \dots + a_NB_0 \\ &= a_0(B + \beta_N) + a_1(B + \beta_{N-1}) + \dots + a_N(B + \beta_0) \\ &= A_NB + a_0\beta_N + a_1\beta_{N-1} + \dots + a_N\beta_0 = A_NB + \gamma_N, \end{aligned}$$

say, where  $\gamma_N = a_0\beta_N + a_1\beta_{N-1} + \dots + a_N\beta_0$ . Since our goal is to show that  $C_N \rightarrow AB$  and we know that  $A_NB \rightarrow AB$ , it suffices to show that  $\gamma_N \rightarrow 0$ . Now, put  $\alpha = \sum_{n=0}^{\infty} |a_n|$ . Since  $B_N \rightarrow B$ ,  $\beta_N \rightarrow 0$ , and thus for any  $\epsilon > 0$  we may choose  $N_0 \in \mathbb{N}$  such that for all  $n \geq N_0$  we have  $|\beta_n| \leq \frac{\epsilon}{2\alpha}$ . Put

$$M = \max_{0 \leq n \leq N_0} |\beta_n|.$$

By the Cauchy criterion, for all sufficiently large  $N$ , we have

$$M \sum_{n \geq N-N_0} |a_n| \leq \epsilon/2.$$

Then

$$\begin{aligned} |\gamma_N| &\leq |\beta_0a_N + \dots + \beta_{N_0}a_{N-N_0}| + |\beta_{N_0+1}a_{N-N_0-1} + \dots + \beta_Na_0| \\ &\leq |\beta_0a_N + \dots + \beta_{N_0}a_{N-N_0}| + \frac{\epsilon}{2} \\ &\leq M \left( \sum_{n \geq N-N_0} |a_n| \right) + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square \end{aligned}$$

### 2.4.3 Cauchy products III: a divergent Cauchy product

We end with an example – due to Cauchy! – of a Cauchy product of two nonabsolutely convergent series which fails to converge.

We will take  $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ . (The series is convergent by the Alternating Series Test.) The  $n$ th term in the Cauchy product is

$$c_n = \sum_{i+j=n} (-1)^i (-1)^j \frac{1}{\sqrt{i+1}} \frac{1}{\sqrt{j+1}}.$$

The first thing to notice is  $(-1)^i (-1)^j = (-1)^{i+j} = (-1)^n$ , so  $c_n$  is equal to  $(-1)^n$  times a sum of positive terms. We have  $i, j \leq n$  so  $\frac{1}{\sqrt{i+1}}, \frac{1}{\sqrt{j+1}} \geq \frac{1}{\sqrt{n+1}}$ , and thus each term in  $c_n$  has absolute value at least  $(\frac{1}{\sqrt{n+1}})^2 = \frac{1}{n+1}$ . Since we are summing from  $i = 0$  to  $n$  there are  $n + 1$  terms, all of the same size, we find  $|c_n| \geq 1$  for all  $n$ . Thus the general term of  $\sum_n c_n$  does not converge to 0, so the series diverges.

## 2.5 Rearrangements

We now consider some results on rearrangements of infinite series. We can view this as an investigation into the validity of the “commutative law” for infinite sums. The definition we gave for convergence of an infinite series

$$a_1 + a_2 + \dots + a_n + \dots$$

in terms of the limit of the sequence of partial sums  $A_1 = a_1 + \dots + a_n$  appears to make use of the *ordering* of the terms of the series. By way of comparison, we observe that this is not an issue in the convergence of infinite sequences. Indeed, the statement  $a_n \rightarrow L$  can be expressed as: for all  $\epsilon > 0$ , there are only finitely many terms of the sequence lying outside the interval  $(L - \epsilon, L + \epsilon)$ , a description which makes clear that convergence to  $L$  is unaffected by any reordering of the terms of the sequence.

Notice however that if we reorder the terms  $\{a_n\}$  of an infinite *series*  $\sum_n a_n$ , the corresponding change in the sequence  $\{A_n\}$  of partial sums is *not* simply a reordering, as one sees by looking at very simple examples. For instance, if we reorder

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots$$

as

$$\frac{1}{4} + \frac{1}{2} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots,$$

then the first partial sum of the new series is  $\frac{1}{4}$ , whereas every nonzero partial sum of the original series is at least  $\frac{1}{2}$ .

Thus we find at least some fuel for our suspicion that reordering the terms of an infinite series may not be so innocuous an operation as for that of an infinite sequence. All of this discussion is mainly justification for our setting up the “rearrangement problem” carefully, with a precision that might otherwise appear pedantic.

The formal notion of rearrangement of a series  $\sum_{n=1}^{\infty} a_n$  begins with a **permutation**  $\sigma$  of  $\mathbb{Z}^+$ , i.e., a bijective function  $\sigma : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ . We define the **rearrangement** of  $\sum_{n=1}^{\infty} a_n$  by  $\sigma$  to be the series

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = a_{\sigma(1)} + a_{\sigma(2)} + \dots$$

Our discussion of this issue will recapitulate in miniature our discussion of series as a whole, in that we will first consider the case of non-negative terms, then the case of absolute convergence and finally the case of nonabsolute convergence.

**Non-negative Terms:** Suppose  $a_n \geq 0$  for all  $n \in \mathbb{Z}^+$ . For  $n \in \mathbb{Z}^+$ , let  $A_n = \sum_{k=1}^n a_k$ . As we have already seen, in this case we have

$$\sum_{n=1}^{\infty} a_n = \sup_n A_n,$$

where we use the (overdue?) convention that  $\sup S = \infty$  when  $S \subset \mathbb{R}$  is nonempty and unbounded above. For a finite subset  $S \subset \mathbb{Z}^+$ , put

$$A_S = \sum_{n \in S} a_n.$$

**Proposition 2.5.1.** *If  $a_n \geq 0$  for all  $n \in \mathbb{Z}^+$ , we have*

$$\sum_{n=1}^{\infty} a_n = \sup\{A_S \mid S \text{ is a finite subset of } \mathbb{Z}^+\}.$$

*Proof.* Let  $A' = \sup\{A_S \mid S \text{ is a finite subset of } \mathbb{Z}^+\}$ . Above we recalled that  $\sum_{n=1}^{\infty} a_n = \sup_n A_n$ . Since  $A_n = A_{\{1, \dots, n\}}$ , this shows  $\sum_{n=1}^{\infty} a_n \leq A'$ . Conversely, every finite subset  $S$  of  $\mathbb{Z}^+$  is contained in  $\{1, \dots, n\}$  for some  $n$ , and thus  $A_S \leq A_n \leq \sum_{n=1}^{\infty} a_n$ , so  $A' \leq \sum_{n=1}^{\infty} a_n$ .  $\square$

The point of this simple result is that, as in the case of sequences above, it gives a description of the sum of a series of non-negative terms which is manifestly unchanged by rearrangement: if  $\sigma : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  is a bijection, then as  $S = \{n_1, \dots, n_k\}$  ranges over all finite subsets of  $\mathbb{Z}^+$ , so does  $\sigma(S) = \{\sigma(n_1), \dots, \sigma(n_k)\}$ . It follows that

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{\sigma(n)} \in [0, \infty].$$

**Absolute Convergence:** We have the following satisfying result.

**Theorem 2.5.2.** *(Weierstrass) Let  $\sum_{n=1}^{\infty} a_n$  be an absolutely convergent series with sum  $A$ . Then for every permutation  $\sigma$  of  $\mathbb{Z}^+$ , the rearranged series  $\sum_{n=1}^{\infty} a_{\sigma(n)}$  converges to  $A$ .*

*Proof.* Fix  $\epsilon > 0$ , and let  $N_1 \in \mathbb{Z}^+$  be such that  $\sum_{n=N_1+1}^{\infty} |a_n| < \epsilon$ . Let  $N_2 \geq N_1$  be such that the terms  $a_{\sigma(1)}, \dots, a_{\sigma(N_2)}$  include all the terms  $a_1, \dots, a_{N_1}$  and possibly others. Then for all  $n \geq N_2$  we have

$$\left| \sum_{k=1}^n a_{\sigma(k)} - A \right| = \left| \sum_{k=1}^n a_{\sigma(k)} - \sum_{k=1}^{\infty} a_k \right| \leq \sum_{k=N_1+1}^{\infty} |a_k| < \epsilon.$$

Indeed: by our choice of  $N_2$  we know that all the terms  $a_1, \dots, a_{N_1}$  appear in both  $\sum_{k=1}^n a_{\sigma(k)}$  and  $\sum_{k=1}^{\infty} a_k$  and thus they get cancelled. Every term in  $\sum_{k=1}^n a_{\sigma(k)}$  is cancelled by some term of  $\sum_{k=1}^{\infty} a_k$ , so what is left is a sum over some subset of  $\{k \in \mathbb{Z}^+ \mid k \geq N_1 + 1\}$ ; applying the triangle inequality, that is bounded above by  $\sum_{k \geq N_1+1} |a_k|$ . It follows that  $\sum_{n=1}^{\infty} a_{\sigma(k)} = A$ .  $\square$

**Nonabsolute Convergence:** We need some preliminary considerations before treating this case. For a real number  $r$ , we define its **positive part**

$$r^+ = \max(r, 0)$$

and its **negative part**

$$r^- = -\min(r, 0).$$

**Lemma 2.5.3.** *For all  $r \in \mathbb{R}$ , we have:*

$$r = r^+ - r^-, \quad |r| = r^+ + r^-.$$

You are asked to prove this result in Exercise 1.

For any real series  $\sum_n a_n$ , we get a decomposition

$$\sum_n a_n = \sum_n a_n^+ - \sum_n a_n^-,$$

at least if all three series converge. Let us call  $\sum_n a_n^+$  the **positive part** of  $\sum_n a_n$  and  $\sum_n a_n^-$  the **negative part** of  $\sum_n a_n$ .

Suppose that  $\sum_n a_n$  converges. There are two possibilities:

Case 1: Both  $\sum_n a_n^+$  and  $\sum_n a_n^-$  converge. Hence  $\sum_n |a_n| = \sum_n a_n^+ + \sum_n a_n^-$  converges: that is,  $\sum_n a_n$  is absolutely convergent.



Case 2: Both  $\sum_n a_n^+$  and  $\sum_n a_n^-$  diverge. Hence  $\sum_n |a_n| = \sum_n a_n^+ + a_n^-$  diverges: indeed, if it converged, then adding and subtracting  $\sum_n a_n$  gives that  $2\sum_n a_n^+$  and  $2\sum_n a_n^-$  converge, contradiction.

What cannot happen is that exactly one of  $\sum_n a_n^+$  and  $\sum_n a_n^-$  converges. Since  $a_n = a_n^+ - a_n^-$  and  $\sum_n a_n$  converges, that would violate the Three Series Principle. So:

**Proposition 2.5.4.** *a) If a series is absolutely convergent, both its positive and negative parts converge.*

*b) If a series is nonabsolutely convergent, then both its positive and negative parts diverge.*

**Theorem 2.5.5.** *(Riemann Rearrangement Theorem) Let  $\sum_{n=1}^{\infty} a_n$  be a non-absolutely convergent series. For any  $B \in [-\infty, \infty]$ , there exists a permutation  $\sigma$  of  $\mathbb{Z}^+$  such that  $\sum_{n=1}^{\infty} a_{\sigma(n)} = B$ .*

*Proof.*

Step 1: Since  $\sum_n a_n$  is convergent, we have  $a_n \rightarrow 0$  and thus that  $\{a_n\}$  is bounded, so we may choose  $M$  such that  $|a_n| \leq M$  for all  $n$ .

We are not going to give an explicit “formula” for  $\sigma$ ; rather, we are going to describe  $\sigma$  by a certain process. For this it is convenient to imagine that the sequence  $\{a_n\}$  has been sifted into a disjoint union of two subsequences, one consisting of the positive terms and one consisting of the negative terms (we may assume without loss of generality that  $a_n \neq 0$  for all  $n$ ). If we like, we may even imagine both of these subsequence ordered so that they are decreasing in absolute value. Thus we have two sequences

$$p_1 \geq p_2 \geq \dots \geq p_n \geq \dots > 0,$$

$$n_1 \leq n_2 \leq \dots \leq n_n \leq \dots < 0$$

so that together  $\{p_n, n_n\}$  comprise the terms of the series. The key point here is Proposition 2.5.4 which tells us that since the convergence is nonabsolute,  $\sum_n p_n = \infty$ ,  $\sum_n n_n = -\infty$ . So we may specify a rearrangement as follows: we specify a choice of a certain number of positive terms – taken in decreasing order – and then a choice of a certain number of negative terms – taken in order of decreasing absolute value – and then a certain number of positive terms, and so on. As long as we include a finite, positive number of terms at each step, then in the end we will have included every term  $p_n$  and  $n_n$

eventually, hence we will get a rearrangement.

Step 2 (diverging to  $\infty$ ): to get a rearrangement diverging to  $\infty$ , we proceed as follows: we take positive terms  $p_1, p_2, \dots$  in order until we arrive at a partial sum which is at least  $M + 1$ ; then we take the first negative term  $n_1$ . Since  $|n_1| \leq M$ , the partial sum  $p_1 + \dots + p_{N_1} + n_1$  is still at least 1. Then we take at least one more positive term  $p_{N_1+1}$  and possibly further terms until we arrive at a partial sum which is at least  $M + 2$ . Then we take one more negative term  $n_2$ , and note that the partial sum is still at least 2. And we continue in this manner: after the  $k$ th step we have used at least  $k$  positive terms, at least  $k$  negative terms, and *all* the partial sums from that point on will be at least  $k$ . Therefore every term gets included eventually and the sequence of partial sums diverges to  $+\infty$ .

Step 3 (diverging to  $-\infty$ ): An easy adaptation of the argument of Step 2 leads to a permutation  $\sigma$  such that  $\sum_{n=0}^{\infty} a_{\sigma(n)} = -\infty$ . Left to you!

Step 4 (converging to  $B \in \mathbb{R}$ ): We first take positive terms  $p_1, \dots, p_{N_1}$ , stopping when the partial sum  $p_1 + \dots + p_{N_1}$  is greater than  $B$ . (We take at least one positive term, even if  $0 > B$ .) Then we take negative terms  $n_1, \dots, n_{N_2}$ , stopping when the partial sum  $p_1 + \dots + p_{N_1} + n_1 + \dots + n_{N_2}$  is less than  $B$ . Then we repeat the process, taking enough positive terms to get a sum strictly larger than  $B$  then enough negative terms to get a sum strictly less than  $B$ , and so forth. Because both the positive and negative parts diverge, this construction can be completed. Because the general term  $a_n \rightarrow 0$ , a little thought shows that the absolute value of the difference between the partial sums of the series and  $B$  approaches zero.  $\square$

**Debriefing:** Theorem 2.5.5 exposes the dark side of nonabsolutely convergent series: just by changing the order of the terms, we can make the series diverge to  $\pm\infty$  or converge to any given real number! Thus nonabsolute convergence is necessarily of a more delicate and less satisfactory nature than absolute convergence. With these issues in mind, we define a series  $\sum_n a_n$  to be **unconditionally convergent** if it is convergent and every rearrangement converges to the same sum, and a series to be **conditionally convergent** if it is convergent but not unconditionally convergent. Then we can (mostly) summarize the results of Weierstrass and Riemann as follows.

**Theorem 2.5.6.** (*Main Rearrangement Theorem*) *A convergent real series is unconditionally convergent if and only if it is absolutely convergent.*

Many texts do not use the term “nonabsolutely convergent” and instead

*define* a series to be conditionally convergent if it is convergent but not absolutely convergent. Aside from the fact that this terminology can be confusing to students to whom this rather intricate story of rearrangements has not been told, it seems correct to make a distinction between the following two *a priori* different phenomena:

- $\sum_n a_n$  converges but  $\sum_n |a_n|$  does not, versus
- $\sum_n a_n$  converges to  $A$  but some rearrangement  $\sum_n a_{\sigma(n)}$  does not.

These two phenomena are equivalent for real series. However the notion of an infinite series  $\sum_n a_n$ , absolute and unconditional convergence makes sense in other contexts, for instance<sup>9</sup> for series with values in an **infinite-dimensional Banach space** or with values in a **p-adic field**. In the former case it is a celebrated theorem of Dvoretzky-Rogers [DR50] that there exists a series which is unconditionally convergent but not absolutely convergent, whereas in the latter case one can show that *every* convergent series is unconditionally convergent whereas there exist nonabsolutely convergent series.

### EXERCISES 2.5

1. Prove Lemma 2.5.3.
2. Let  $\sum_n a_n$  be a real series such that  $a_n \rightarrow 0$ ,  $\sum_n a_n^+ = \infty$  and  $\sum_n a_n^- = -\infty$ . Show that the conclusion of Theorem 2.5.5 holds: for any  $A \in [-\infty, \infty]$ , there exists a permutation  $\sigma$  of  $\mathbb{N}$  such that  $\sum_{n=0}^{\infty} a_{\sigma(n)} = A$ .
3. Let  $\sum_n a_n$  be a real series such that  $\sum_n a_n^+ = \infty$ .
  - a) Suppose that the sequence  $\{a_n\}$  is bounded. Show that there exists a permutation  $\sigma$  of  $\mathbb{Z}^+$  such that  $\sum_n a_{\sigma(n)} = \infty$ .
  - b) Does the conclusion of part a) hold without the assumption that the sequence of terms is bounded?
4. Consider the series  $\sum_{n=1}^{\infty} a_n$ , where for  $n \in \mathbb{Z}^+$  we put

$$a_{2n} = \frac{-1}{(2n)^2}$$

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<sup>9</sup>Both of these are well beyond the scope of these notes, i.e., you are certainly not expected to know what I am talking about here.

$$a_{2n-1} = \frac{1}{2n-1}.$$

Thus

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \dots$$

a) Show:  $\sum_{n=1}^{\infty} a_n = -\infty$ .

b) Let  $b_n = \frac{2(-1)^{n+1}}{n}$  for all  $n$ . Observe that  $\sum_{n=1}^{\infty} b_n$  converges and for all  $n \in \mathbb{Z}^+$  we have  $|a_n| < |b_n|$ . Moral: it is possible to take a nonabsolutely convergent series, replace each term with a term which is smaller in absolute value, and get a divergent series.

c) Show that the above “moral” holds for *every* nonabsolutely convergent series in which each term is nonzero.

## 2.6 Power Series

A *power series* is an expression of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

where  $x$  is a real variable. We can think of this as defining a function whose natural domain of definition is the set of real numbers,  $x$ , such that the sum converges. This set is called the *domain of convergence*.

**Definition 2.6.1.** *The domain of convergence of the power series  $\sum_{n=0}^{\infty} a_n x^n$  is  $\{x \in \mathbb{R} \mid \sum a_n x^n \text{ converges}\}$ .*

**Example 2.6.2.** *a) The domain of convergence of the power series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  is all of  $\mathbb{R}$ . Indeed, for any  $x \in \mathbb{R}$ , the Ratio Test limit is*

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = |x| \cdot 0 = 0.$$

*b) The domain of convergence of the geometric series  $\sum_{n=0}^{\infty} x^n$  is  $(-1, 1)$ . This is nothing but a restatement of Proposition 2.1.12 with  $r$  changed to  $x$ .*

c) The domain of convergence of the power series  $\sum_{n=0}^{\infty} \frac{x^n}{n}$  is  $[-1, 1)$ . To see this, we first apply the Ratio Test:

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n}{n+1} = |x|.$$

So the series converges if  $|x| < 1$  and diverges if  $|x| > 1$ . Of course the test tells us nothing if  $|x| = 1$  so we check those two cases separately. If  $x = 1$ , the series becomes the harmonic series which diverges, and if  $x = -1$  the series becomes the alternating harmonic series, which converges.

d) The domain of convergence of the power series  $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n}$  is  $(-1, 1]$ . To see this we could go through an analysis just like the one in part c), and the reader who does not find such calculations routine should probably do so. However, we can also observe that the series is obtained from the series in part c) by replacing  $x$  with  $-x$ , and therefore the domain of convergence must be  $\{-x \mid x \in [-1, 1)\} = (-1, 1]$ .

e) The domain of convergence of the power series  $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$  is  $[-1, 1]$ . To see this, we first apply the Ratio Test:

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}/(n+1)^2}{x^n/n^2} \right| = |x| \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = |x|.$$

So the series converges when  $|x| < 1$  and diverges when  $|x| > 1$ . When  $x = 1$  we get the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , which is a convergent  $p$ -series. When  $x = -1$  we get the alternating  $p$ -series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  whose absolute series is the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  that we just said is convergent.

f) The domain of convergence of the power series  $\sum_{n=0}^{\infty} n^n x^n$  is  $\{0\}$ . Certainly we have convergence at  $x = 0$ : every power series converges at 0"

$$\sum_{n=0}^{\infty} a_n 0^n = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + \dots = a_0 + 0 + 0 + \dots = a_0.$$

For  $x \neq 0$ , the Root Test gives

$$\theta = \lim_{n \rightarrow \infty} |n^n x^n|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} |n|x = \infty,$$

so the series diverges.

We notice that in every case above the domain of convergence was an interval. This holds in general, as we now show.

**Proposition 2.6.3.** *Suppose that the power series  $\sum_{n=0}^{\infty} a_n x^n$  converges for  $x = c$ . Then it converges absolutely for all  $x$  such that  $|x| < |c|$ .*

*Proof.* Since  $\sum_{n=0}^{\infty} a_n c^n$  converges, the sequences of terms converges to 0 and thus is bounded: there is  $K \geq 0$  such that  $|a_n c^n| \leq K$  for all  $n \in \mathbb{N}$ . Let  $x \in \mathbb{R}$  be such that  $|x| < |c|$ , and put  $d = \frac{|x|}{|c|}$ , so  $d \in (0, 1)$ . Then

$$|a_n x^n| = |a_n| |c|^n d^n \leq K d^n,$$

so

$$\sum_{n=0}^{\infty} |a_n x^n| \leq K \sum_{n=0}^{\infty} d^n < \infty. \quad \square$$

**Corollary 2.6.4.** *Let  $\sum_{n=0}^{\infty} a_n x^n$  be a power series, and let*

$$D = \left\{ x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}.$$

- a) *If  $D$  is unbounded, then  $D = \mathbb{R}$ .*  
 b) *If  $D$  is bounded, let  $R = \sup D$ . Then*

$$(-R, R) \subseteq D \subseteq [R, R].$$

*Proof.* a) Suppose  $D$  is unbounded, so for all  $n \in \mathbb{Z}^+$  there is  $c \in D$  such that  $|c| > n$ . By Proposition 2.6.3 it follows that  $[-n, n] \subseteq D$ . Since this holds for all  $n \in \mathbb{Z}^+$ , we have  $D = \mathbb{R}$ .

b) Suppose  $D$  is bounded and  $R = \sup D$ . Since  $0 \in D$ , we have  $R \geq 0$ . First suppose  $x \in \mathbb{R}$  is such that  $|x| > R$ . Then we claim that  $x \notin D$ ; indeed, if  $x \in D$ , then by Proposition 2.6.3 we have  $\frac{|x|+R}{2} \in D$ , and since  $\frac{|x|+R}{2} > R$ , this is a contradiction. Now suppose  $|x| < R$ . Since  $R = \sup D$ , there is  $y \in D$  such that  $|x| < y \leq R$ , and by Proposition 2.6.3 we have  $x \in D$ .  $\square$

When the domain  $D$  of convergence of  $\sum_{n=0}^{\infty} a_n x^n$  is bounded, we call  $R = \sup D$  the **radius of convergence** of the power series. When  $D = \mathbb{R}$  we define the radius of convergence to be  $R = \infty$ .

Let us spell out Corollary 2.6.4 a bit more explicitly: it says that if  $R = \infty$  then the domain of convergence is  $\mathbb{R}$ ; if  $R = 0$  then the domain of convergence is  $\{0\}$ , and if  $R \in (0, \infty)$  then the radius of convergence is one of the following:  $(-R, R)$ ,  $[-R, R)$ ,  $(-R, R]$  or  $[-R, R]$ . Example 2.6.2 shows that  $R = 0$

and  $R = \infty$  can both occur, and that when  $R = 1$  all four kinds of intervals can occur. In Exercise 10 you are asked to find, for each  $R \in (0, \infty)$ , power series with domains of convergence  $(-R, R)$ ,  $[-R, R)$ ,  $(-R, R]$  and  $[-R, R]$ .

The radius of convergence of a power series  $\sum_{n=0}^{\infty} a_n x^n$  has a simple relation to the Ratio and Root Test limits of the numerical series  $\sum_{n=0}^{\infty} a_n$ . In the following result we use the conventions

$$1/0 = \infty, \quad 1/\infty = 0.$$

(Note that  $1/0 = \infty$  is only a sensible convention when we restrict to non-negative numbers. That is the case here.)

**Proposition 2.6.5.** *Let  $\sum_{n=0}^{\infty} a_n x^n$  be a power series.*

a) *Suppose that the Ratio Test limit*

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

*exists. Then the radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n x^n$  is  $\frac{1}{\rho}$ .*

b) *Suppose that the Root Test limit*

$$\theta = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$

*exists. Then the radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n x^n$  is  $\frac{1}{\theta}$ .*

You are asked to prove this in Exercise 9. (If you don't see how to proceed, try first tracking the relationship between the Ratio Test limit and the radius of convergence in any particular example.)

We now pursue a slightly more general notion of a power series: instead of powers of  $x$ , we look at powers of  $x - a$  for some fixed  $a \in \mathbb{R}$ . Of course whenever we replace a variable  $x$  by  $x - a$  this has the effect of translating everything  $a$  units to the right. There really is no more to it than that.

A power series *centered at  $a$*  is a series of the form

$$\sum_{n=0}^{\infty} a_n (x - a)^n$$

where  $a$  is a fixed real number.

If the domain of convergence of  $\sum_{n=0}^{\infty} a_n x^n$  is  $D$ , then the domain of convergence of  $\sum_{n=0}^{\infty} a_n (x - a)^n$  is

$$D + a = \{x + a \mid x \in D\}.$$

So the radius of convergence of  $\sum_{n=0}^{\infty} a_n (x - a)^n$  is the same as the radius of convergence of  $\sum_{n=0}^{\infty} a_n x^n$ , but the center moves from 0 to  $a$ . For instance, if the domain of convergence of the original power series is  $(-R, R)$ , then the domain of convergence of the shifted power series is  $(a - R, a + R)$ .

**Example 2.6.6.** *Since we know that the power series  $\sum_{n=0}^{\infty} \frac{x^n}{n}$  has domain of convergence  $[-1, 1)$  we can conclude that the power series  $\sum_{n=0}^{\infty} \frac{(x+2)^n}{n}$  has domain of convergence  $[-3, -1)$ .*

The above shifting technique can be thought of as the *composition* of the function defined by the power series  $f(x) = \sum a_n x^n$  with the function  $g(x) = x - a$ . Composition with other simple functions can lead to similar generalizations.

**Example 2.6.7.** *The power series  $h(x) = \sum_{n=0}^{\infty} x^{2n}/2^n$  can be thought of as the composition of the power series  $f(x) = \sum_{n=0}^{\infty} x^n$  with the function  $g(x) = x^2/2$ . Since we know that the domain of convergence for  $f(x)$  is  $(-1, 1)$  we can conclude that  $h(x)$  converges exactly when  $-1 < x^2/2 < 1$ . Thus, the domain of convergence for  $h(x)$  must be  $(-\sqrt{2}, \sqrt{2})$ .*

Although power series define functions on their domains of convergence, we generally cannot find closed form expressions for these functions. In the next chapter we will extend the list of power series for which we can determine closed form expressions, but for now we can do so only for geometric series.

## EXERCISES 2.6

1. Compute the domains of convergence for the following power series.

a)  $\sum_{n=0}^{\infty} n x^n$

b)  $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$

c)  $\sum_{n=0}^{\infty} \frac{n^n x^n}{n!}$

d)  $\sum_{n=0}^{\infty} 3^n (x - 1)^n$



e.) 
$$\sum_{n=0}^{\infty} \frac{n!x^n}{2^n}$$

f.) 
$$\sum_{n=0}^{\infty} \frac{3x^n}{n^2}$$

g.) 
$$\sum_{n=0}^{\infty} \frac{(3x)^n}{n^2}$$

h.) 
$$\sum_{n=0}^{\infty} \frac{(x-5)^n}{n^4}$$

i.) 
$$\sum_{n=0}^{\infty} \frac{x^{3n}}{2^n}$$

j.) 
$$\sum_{n=0}^{\infty} \frac{(x-2)^{2n}}{3^n}$$

k.) 
$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

l.) 
$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

2. Find closed forms for the following power series.

a) 
$$\sum_{n=3}^{\infty} x^n$$

b) 
$$\sum_{n=0}^{\infty} 2x^n$$

c) 
$$\sum_{n=0}^{\infty} (2x)^n$$

d) 
$$\sum_{n=1}^{\infty} (2x-1)^n$$

e.) 
$$\sum_{n=1}^{\infty} x^{2n}$$

3. Find a
- power series*
- representation for the following functions. Be sure to indicate the
- domain of convergence*
- and
- radius of convergence*
- in each case:

(a) 
$$f(x) = \frac{1}{2+x^2}$$

(b) 
$$g(x) = \frac{4}{x+4}$$

4. a) Find a power series which converges to
- $f(x) = \frac{1}{5-x}$
- on the interval
- $(-5, 5)$
- . (Hint:
- $5-x = 5(1-x/5)$
- .)
- 
- b) Find a power series which converges to
- $g(x) = \frac{1}{5+x}$
- on the interval
- $(-5, 5)$
- .

5. Find a power series which converges to  $f(x) = \frac{1}{5-x}$  on the interval  $(3, 5)$ . (Hint:  $5 - x = 1 - (x - 4)$ .)
6. Find a power series which converges to  $f(x) = \frac{1}{(x-1)(x-2)}$  on the interval  $(-1, 1)$ . (Hint: partial fractions.)
7. Let  $\{F_n\}$  be the Fibonacci sequence, so  $F_0 = 1, F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 2$ . Let  $F(x)$  be the function given by

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} F_n x^n \\ &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots \end{aligned}$$

$F(x)$  is called the *generating function* for  $\{F_n\}$ .

a) By writing out  $F(x) = 1 + x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2})x^n$  show that  $(1 - x - x^2)F(x) = 1$ .

b) Expand  $\frac{1}{1-x-x^2}$  by partial fractions to write  $F(x)$  in the form

$$F(x) = \frac{A}{\alpha - x} + \frac{B}{\beta - x}$$

for some numbers  $A$  and  $B$ , where  $\alpha = (-1 + \sqrt{5})/2$  and  $\beta = (-1 - \sqrt{5})/2$  are the roots of  $1 - x - x^2 = 0$ .

c) By combining the power series for  $\frac{A}{\alpha - x}$  and  $\frac{B}{\beta - x}$ , (and using that  $1/\alpha = -\beta$ ) show that

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].$$

- d) Check that this formula really works for  $n = 0, 1$ , and  $2$ .
8. (From the William Lowell Putnam Mathematical Competition, 1999)  
Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer  $n \geq 0$ , there is an integer  $m$  such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

9. Prove Proposition 2.6.5.
10. Let  $R \in (0, \infty)$ .
- Exhibit a power series with domain of convergence  $(-R, R)$ .
  - Exhibit a power series with domain of convergence  $[-R, R)$ .
  - Exhibit a power series with domain of convergence  $(-R, R]$ .
  - Exhibit a power series with domain of convergence  $[-R, R]$ .
11. Let  $\sum_{n=0}^{\infty} a_n x^n$  and  $\sum_{n=0}^{\infty} b_n x^n$  be two power series with radii of convergence  $R_a > 0$  and  $R_b > 0$ . Let  $R = \min(R_a, R_b)$ . Put  $c_n = \sum_{k=0}^n a_k b_{n-k}$ . Show that the “formal identity”

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) \left( \sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n$$

is valid for all  $x \in (-R, R)$ .

12. This exercise explains why it often turns out that the radius of convergence of a power series is 1. Let  $\{a_n\}$  be a real sequence.
- Suppose  $\{a_n\}$  is bounded. Show that the radius of convergence of  $\sum_{n=0}^{\infty} a_n x^n$  is at least 1.
  - Suppose  $a_n \not\rightarrow 0$ . Show that the radius of convergence of  $\sum_{n=0}^{\infty} a_n x^n$  is at most 1.
  - Deduce: if  $\{a_n\}$  is bounded and not convergent to 0, then the radius of convergence of  $\sum_{n=0}^{\infty} a_n x^n$  is 1.



# Chapter 3

## Sequences and Series of Functions

Consider a power series  $\sum_{n=0}^{\infty} a_n x^n$ . The perspective of Chapter 2 was that this is a “parameterized family of series,” i.e., for each  $x \in \mathbb{R}$  we get a different series, and we investigated the set of parameter values for which the series converges.

In this chapter we will discuss such “parameterized families of sequences and series” in general terms. We will establish results of the following character: given a sequence  $\{f_n\}$  of functions such that each  $f_n$  has a certain good property, then the limit function  $f$  also has that good property. Then we will apply these results to establish good properties of a function defined by a convergent power series. Finally we will develop the theory of Taylor series and thereby represent large classes of functions as convergent power series.

### 3.1 Pointwise Convergence

We fix a subset  $D \subset \mathbb{R}$ . In practice  $D$  is usually an interval – keep in mind especially the cases  $D = \mathbb{R}$  and  $D = [a, b]$  – but many of the results hold without any assumptions on  $D$ . Then a sequence of functions is a sequence  $\{f_n : D \rightarrow \mathbb{R}\}_{n=1}^{\infty}$ , i.e., for each  $n \in \mathbb{Z}^+$  we have a function  $f_n : D \rightarrow \mathbb{R}$ .<sup>1</sup> The key point of this setup is the following (familiar) one: for each  $x \in D$ , by “plugging in  $x$ ” we get a sequence of real numbers  $\{f_n(x)\}_{n=1}^{\infty}$ , and we may

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<sup>1</sup>The sequence may also start at 0 or in fact at any integer.

ask whether the sequence converges.

Our first notion of convergence of a sequence of functions is no more and no less than requiring an affirmative answer for every  $x \in D$ . Namely:

Let  $D \subset \mathbb{R}$ , and for each  $n \in \mathbb{Z}^+$  let  $f_n : D \rightarrow \mathbb{R}$  be a function. Let  $f : D \rightarrow \mathbb{R}$  be a function. We say that  $f_n$  **converges pointwise to  $f$**  if for all  $x \in D$  we have

$$\lim_{n \rightarrow \infty} f_n(x) = f(x).$$

To unpack this definition completely, this says: for all  $x \in D$  and for all  $\epsilon > 0$ , there is  $N \in \mathbb{N}$  such that for all  $n > N$  we have

$$|f_n(x) - f(x)| < \epsilon.$$

A key point (and as we will soon see, the fundamental shortcoming!) of this definition is that we are allowed to choose  $N$  *in terms of both  $\epsilon$  and  $x$* .

If  $\{f_n : D \rightarrow \mathbb{R}\}_{n=1}^{\infty}$  is a sequence of functions, we may form the corresponding series of functions  $\sum_{n=1}^{\infty} f_n$ . Again, for each  $x \in D$  after plugging in  $x$  we get a real series  $\sum_{n=1}^{\infty} f_n(x)$ . If  $S : D \rightarrow \mathbb{R}$  is a function, we say that the series  $\sum_{n=1}^{\infty} f_n$  **converges pointwise to  $S$**  if for all  $x \in R$ , the series  $\sum_{n=1}^{\infty} f_n(x)$  converges and has value  $S(x)$ . Again we unpack this completely: for all  $x \in D$  and all  $\epsilon > 0$ , there is  $N \in \mathbb{N}$  such that for all  $n > N$  we have

$$\left| \sum_{k=1}^n f_k(x) - S(x) \right| < \epsilon.$$

We may refer to the function  $S$  as the **sum** of the series  $\sum_{n=1}^{\infty} f_n$  or, extending our earlier slight abuse of notation, we may simply write  $S = \sum_{n=1}^{\infty} f_n$ .

**Example 3.1.1.** a) Let  $\sum_{n=0}^{\infty} a_n x^n$  be a power series, and let

$$D = \left\{ x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}.$$

By definition, the series converges pointwise on  $D$ : that is, if  $S_N(x) = \sum_{n=0}^N a_n x^n$ , then  $S_N$  is pointwise convergent on  $D$ , and the pointwise limit is the function  $\sum_{n=0}^{\infty} a_n x^n$ .

b) Let us be more specific. If  $a_n = 1$  for all  $n$ , then we get the geometric series

$\sum_{n=0}^{\infty} x^n$ . We know that the domain of convergence is  $(-1, 1)$ . Moreover, in this case we know what the sum of the series is, namely  $\frac{1}{1-x}$ . Thus the series  $\sum_{n=0}^{\infty} x^n$  converges pointwise to  $\frac{1}{1-x}$  on  $(-1, 1)$ .

c) We saw in Chapter 2 that the power series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges for all  $x \in \mathbb{R}$ . I claim that in fact for all  $x \in \mathbb{R}$ , the series converges to  $e^x$ . This is the type of result that we have been gunning for in the entire course; we will prove it later in this chapter. Assuming this, the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges pointwise to  $e^x$  on  $\mathbb{R}$ .

Hey: why are we studying sequences and series of functions? The answer is that from the earliest days of calculus (in the 17th century), mathematicians sought to do calculus on complicated functions by writing them as limits of sequences and series of simpler functions. Power series give one of the simplest and most important examples of this: on the domain of convergence  $D$ , the power series  $\sum_{n=0}^{\infty} a_n x^n$  is the pointwise limit of the sequence  $S_N = \sum_{n=0}^N a_n x^n$ . Each  $S_N$  is a polynomial, and calculus on polynomials is easy: the derivative of  $S_N$  is

$$S'_N = \sum_{n=1}^N n a_n x^{n-1}$$

and an antiderivative of  $S_n$  is

$$\int S_N = \sum_{n=0}^N \frac{a_n x^{n+1}}{n+1}.$$

If we dare to admit it, we are certainly hoping that “the same thing” holds for power series on their domain of convergence. In particular, we hope that the derivative of a power series is

$$\left( \sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

The mathematicians of the 17th, 18th and early 19th centuries were often happy to do calculus on the limit of a sequence or series of functions by doing calculus on the  $n$ th term and then taking the limit. They would sometimes make assertions that, in our modern notation, would read as follows:

•: Suppose  $f_n \rightarrow f$  pointwise on  $D$ . Then

$$\forall c \in D, \lim_{x \rightarrow c} \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow c} f_n(x). \quad (3.1)$$

- Suppose  $f_n \rightarrow f$  pointwise on  $[a, b]$ . Then

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx. \quad (3.2)$$

- Suppose  $f_n \rightarrow f$  pointwise on  $D$ . Then

$$f' = \lim_{n \rightarrow \infty} f'_n. \quad (3.3)$$

Such relations between  $f_n$  and  $f$  are both natural and useful. Unfortunately, none of them are true! More precisely, they are *often true* but assuming that  $f_n \rightarrow f$  pointwise is not enough to *ensure* that they are true.

**Example 3.1.2.** *Let*

$$f_n(x) = \begin{cases} -1, & -1 \leq x < \frac{-1}{n} \\ nx, & \frac{-1}{n} \leq x < \frac{1}{n} \\ 1, & \frac{1}{n} \leq x \leq 1 \end{cases}$$

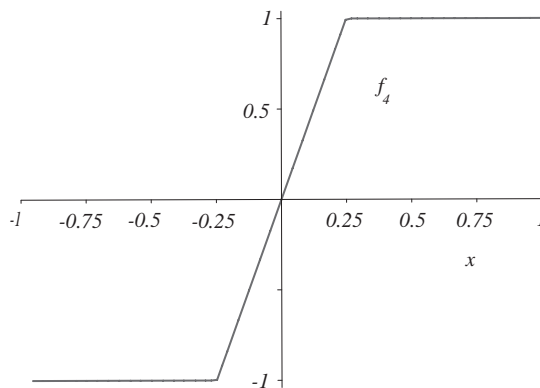


figure 3.1.1a

and let

$$f(x) = \begin{cases} -1, & -1 \leq x < 0 \\ 0, & x = 0 \\ 1, & 0 < x \leq 1 \end{cases}.$$

Then  $f_n \rightarrow f$  pointwise on  $\mathbb{R}$ . Moreover, each  $f_n$  is continuous, but the pointwise limit  $f$  is not.



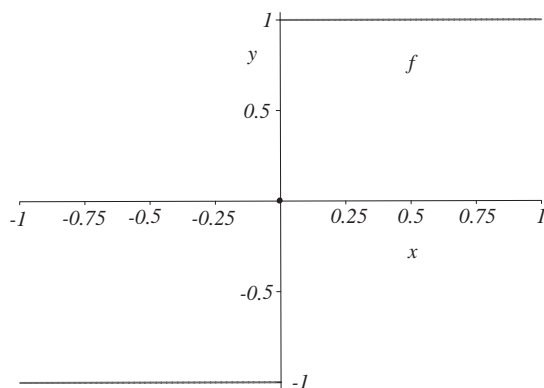


figure 3.1.1b

**Example 3.1.3.** For  $n \in \mathbb{Z}^+$ , let  $f_n : [0, 1] \rightarrow \mathbb{R}$  by  $f_n(x) = x^n$ . If we plug in any  $x \in [0, 1]$ , then the sequence  $\{f_n(x)\} = \{x^n\}$  is geometric with geometric ratio  $x$ . Thus:

- If  $x \in [0, 1)$ , then  $f_n(x) = x^n \rightarrow 0$ .
- If  $x = 1$ , then  $f_n(x) = 1^n = 1 \rightarrow 1$ .

Thus if

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

then  $f_n \rightarrow f$  pointwise on  $[0, 1]$ . As in part a) we have a sequence of continuous functions converging pointwise to a discontinuous function. Unlike part a), each  $f_n$  is much better than continuous: it is smooth (or infinitely differentiable, or  $C^\infty$ : derivatives of all orders exist). One could not ask for a better behaved function on the interval  $[0, 1]$  than  $f_n = x^n$ . However, all this good behavior does not ensure that the pointwise limit is continuous.

To see how these examples fit in with the above “lies”: pointwise convergence tells us precisely that for all  $x \in D$ , we have  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ . If each  $f_n : D \rightarrow \mathbb{R}$  is continuous, then for all  $c \in D$  we have then  $\lim_{x \rightarrow c} f_n(x) = f_n(c)$ , and thus (3.1) reduces to

$$\forall c \in D, \lim_{x \rightarrow c} f(x) = \lim_{n \rightarrow \infty} f_n(c). \quad (3.4)$$

Since, once again, pointwise convergence gives that for all  $c \in D$  we have

$\lim_{n \rightarrow \infty} f_n(c) = f(c)$ , we see that in this case (3.4) is equivalent to

$$\forall c \in D, \lim_{x \rightarrow c} f(x) = f(c),$$

i.e., that  $f$  is continuous.

**Example 3.1.4.** *Let*

$$f_n(x) = \begin{cases} 2n^2x, & 0 \leq x < \frac{1}{2n} \\ 2n - 2n^2x, & \frac{1}{2n} \leq x < \frac{1}{n} \\ 0, & \frac{1}{n} \leq x \leq 1 \end{cases}$$

and let

$$f(x) = 0, \quad 0 \leq x \leq 1.$$

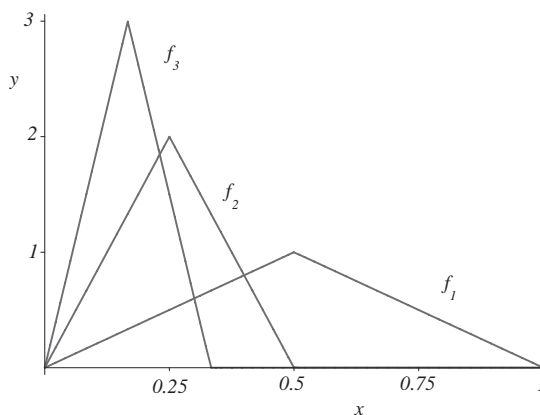


figure 3.1.2

Since the graph of  $f_n$  is a triangle with height  $n$  and base  $\frac{1}{n}$  we have

$$\forall n \in \mathbb{Z}^+, \int_0^1 f_n(x) dx = \frac{1}{2},$$

while

$$\int_0^1 f(x) dx = \int_0^1 0 = 0.$$

Thus

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2} \neq 0 = \int_0^1 f(x) dx.$$

**Example 3.1.5.** Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded, differentiable function such that  $\lim_{n \rightarrow \infty} g'(n)$  does not exist. (For instance, we may take  $g(x) = \sin\left(\frac{\pi x}{2}\right)$ .) For  $n \in \mathbb{Z}^+$ , let

$$f_n : \mathbb{R} \rightarrow \mathbb{R}, \quad f_n(x) = \frac{g(nx)}{n}.$$

Let  $M$  be such that  $|g(x)| \leq M$  for all  $x \in \mathbb{R}$ . (With our specific choice of  $g$  above, we may take  $M = 1$ .) Then for all  $x \in \mathbb{R}$  we have  $|f_n(x)| \leq \frac{M}{n}$ , so  $\lim_{n \rightarrow \infty} f_n(x) = 0$ . Thus if  $f$  is the identically zero function, then  $f_n \rightarrow f$  pointwise on  $\mathbb{R}$ . The derivative of the zero function is again the zero function, and in particular  $f'(1) = 0$ . On the other hand, the Chain Rule gives

$$f'_n(x) = \frac{ng'(nx)}{n} = g'(nx),$$

and thus, taking  $x = 1$ , we get

$$\lim_{n \rightarrow \infty} f'_n(1) = \lim_{n \rightarrow \infty} g'(n) \text{ does not exist,}$$

and thus

$$\lim_{n \rightarrow \infty} f'_n(1) \neq \left( \lim_{n \rightarrow \infty} f_n \right)'(1).$$

## 3.2 Uniform Convergence

All we have to do now is take these lies and make them true somehow. – G. Michael<sup>2</sup>

So unfortunately all three “identities” (3.1), (3.2) and (3.3) used rather blithely by many mathematicians for about 200 years turn out to be false. Well, don’t give up. In fact the best of these mathematicians did not assert these identities in general but only in certain situations. What was actually missing for two hundred years was a clear understanding of exactly what is needed in order to make the lies true. (The situation is complicated by the fact that the precise  $(\epsilon, N)$ -definition of convergence of a sequence of real numbers was not introduced until well into the 19th century. So for instance our great hero A.L. Cauchy, in his 1821 text *Cours d’analyse*, included a result that another (brilliant, leading) mathematician, N.H. Abel, regarded as “admitting exceptions.” However, when 21st century mathematicians look

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<sup>2</sup>George Michael, 1963–

back at Cauchy's work and interpret it using the lens of *20th century innovations in analysis*, they can interpret his statements in a foundational context in which they become correct. A good way to appreciate the power and simplicity of the epsilonic approach to analysis is to read a little of the controversies and confusions that preceded it!

The key idea that makes the lies (most of them: please read on!) true was first given by Gudermann<sup>3</sup> in an 1838 paper on elliptic functions. In 1839-1840 he gave a course on elliptic functions. Weierstrass was a student in Gudermann's course. He took Gudermann's idea, ran with it, and the rest, as they say, is history.

### 3.2.1 Introducing Uniform Convergence

What Gudermann and Weierstrass realized is that most of our problems disappear if we replace pointwise convergence by a subtly stronger definition, uniform convergence.

We say that a sequence  $\{f_n : D \rightarrow \mathbb{R}\}_{n=1}^{\infty}$  of functions **converges uniformly** to a function  $f : D \rightarrow \mathbb{R}$  if for all  $\epsilon > 0$ , there is  $N \in \mathbb{N}$  such that for all  $x \in D$  and all  $n \geq N$  we have

$$|f_n(x) - f(x)| < \epsilon.$$

We will abbreviate “ $f_n$  converges uniformly to  $f$ ” by  $f_n \xrightarrow{u} f$ .<sup>4</sup>

The definitions of pointwise and uniform convergence look very similar at first. Let us unpack them both completely, side by side:

$f_n$  converges to  $f$  pointwise on  $D$  means:

$$\forall x \in D, \forall \epsilon > 0, \exists N \in \mathbb{N} \mid \forall n \geq N, |f_n(x) - f(x)| < \epsilon,$$

while  $f_n$  converges to  $f$  uniformly on  $D$  means:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \mid \forall x \in D, \forall n \geq N, |f_n(x) - f(x)| < \epsilon.$$

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<sup>3</sup>Christoph Gudermann, 1798–1852

<sup>4</sup>This is not standard notation, but we think that making a notational distinction is pedagogically useful.

In fact all the words are the same; the only difference lies in the *order of quantifiers*. For pointwise convergence, because  $\forall x \in D$  appears before  $\exists N \in \mathbb{N}$ , the  $N$  is allowed to depend on  $x$  as well as  $\epsilon$ . For uniform convergence, because  $\exists N \in \mathbb{N}$  appears before  $\forall x \in D$ , the  $N$  is allowed to depend on  $\epsilon$  alone: it must work for all  $x \in D$  at the same time (or “uniformly”; thus the name). Thus the second definition is stronger than the first:  $f_n \xrightarrow{u} f$  on  $D$  certainly implies that  $f_n \rightarrow f$  pointwise on  $D$ , but (as we shall see), the converse is not true.

A student early in her study of theoretical mathematics may have difficulty appreciating the significance of the order of quantifiers. But in fact switching a  $\forall$  and an  $\exists$  often makes an **enormous difference**. It may help to hone your intuition on a simpler example: consider the two statements:

$$\forall x \in \mathbb{R} \exists y \in \mathbb{R} y > x.$$

$$\exists y \in \mathbb{R} \forall x \in \mathbb{R}, y > x.$$

They differ only in swapping the order of an  $\exists$  and a  $\forall$ . The first statement asserts that for every real number  $x$ , there is some larger real number  $y$ . True: given  $x$ , take  $y = x + 1$ . The second statement asserts that there is a real number larger than every real number. False, and even false if there were a largest real number: given  $y$ , take  $x = y$ : it is not the case that  $y > y$ .

As usual, we apply this definition to a series of functions via the sequence of partial sums: thus  $\sum_{n=1}^{\infty} f_n \xrightarrow{u} S$  on  $D$  if for all  $\epsilon > 0$ , there is  $N \in \mathbb{N}$  such that for all  $x \in D$  and all  $n > N$ , we have

$$\left| \sum_{k=0}^n f_k(x) - S(x) \right| < \epsilon.$$

**Lemma 3.2.1.** (*Cauchy Criterion for Uniform Convergence*)

Let  $\{f_n : D \rightarrow \mathbb{R}\}$  be a sequence of functions, and let  $f : D \rightarrow \mathbb{R}$ .

a) The following are equivalent:

(i) We have  $f_n \xrightarrow{u} f$ .

(ii) For all  $\epsilon > 0$ , there is  $N \in \mathbb{Z}^+$  such that for all  $x \in D$  and all  $m, n \geq N$  we have

$$|f_m(x) - f_n(x)| < \epsilon.$$

b) The following are equivalent:

(i) The series  $\sum_{n=1}^{\infty} f_n$  converges uniformly to  $f$  on  $D$ .

(ii) For all  $\epsilon > 0$ , there is  $N \in \mathbb{Z}^+$  such that for all  $x \in D$  and all  $n \geq m \geq N$  we have

$$\left| \sum_{k=m}^n f_k(x) - f(x) \right| < \epsilon.$$

*Proof.* a) (i)  $\implies$  (ii) The proof that convergent sequences are Cauchy carries over without essential change. For completeness, we give the argument. Let  $\epsilon > 0$ , and choose  $N \in \mathbb{Z}^+$  such that for all  $n \geq N$  and all  $x \in D$  we have  $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ . Then for all  $m, n \geq N$  we have

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(ii)  $\implies$  (i): For each  $x \in D$ , the sequence  $\{f_n(x)\}$  is Cauchy, so convergent by Cauchy's Property of Completeness. Calling the limit  $f(x)$ , we get  $f_n \rightarrow f$  pointwise on  $D$ . It remains to show the convergence is uniform. For this fix  $\epsilon > 0$ , and let  $x_0 \in D$ . Choose  $N \in \mathbb{Z}^+$  such that  $|f_m(x) - f_n(x)| < \frac{\epsilon}{2}$  for all  $x \in D$ , choose  $M \in \mathbb{Z}^+$  such that  $|f_m(x_0) - f(x_0)| < \frac{\epsilon}{2}$  for all  $m \geq M$ , and put  $m_0 = \max(M, N)$ . Then for all  $n \geq N$ , we have

$$|f_n(x_0) - f(x_0)| \leq |f_n(x_0) - f_{m_0}(x_0)| + |f_{m_0}(x_0) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

This shows that  $f_n \xrightarrow{u} f$  on  $D$ .

b) We apply part a) to the sequence of partial sums  $S_n = \sum_{k=1}^n f_k$ .  $\square$

We introduce a notation which is convenient for dealing with uniform convergence and also provides a nice way of thinking about the concept. For  $f : D \rightarrow \mathbb{R}$ , we define the **norm** of  $f$

$$\|f\| = \sup_{x \in D} |f(x)| \in [0, \infty].$$

Note here that we are using the convention that the supremum of a nonempty subset of real numbers which is unbounded above is  $\infty$ . Thus,  $\|f\| = \infty$  precisely when  $f$  is unbounded on  $D$ .

Now consider the following key inequality:

$$\|f\| < \epsilon.$$

What it means is that the graph of  $f : D \rightarrow \mathbb{R}$  lies strictly between the horizontal lines  $y = -\epsilon$  and  $y = \epsilon$ . And now observe that if  $\{f_n : D \rightarrow \mathbb{R}\}$  is a sequence of functions and  $f : D \rightarrow \mathbb{R}$  is a function, then the assertion  $f_n \xrightarrow{u} f$  on  $D$  holds iff

$$\|f_n - f\| \rightarrow 0.$$

(Here we have a sequence  $\{a_n\}$  in the extended real numbers  $[0, \infty]$ . Our convention is that for such a sequence to converge to a real number, we must have  $a_n < \infty$  for all sufficiently large  $n$ .) Thus we have reduced the question of uniform convergence of a sequence of functions to the convergence of a single numerical sequence.

### 3.2.2 Consequences of Uniform Convergence

And now we will see that the lies become true if  $f_n \rightarrow f$  is replaced by  $f_n \xrightarrow{u} f$ . (Well, most of them...)

**Theorem 3.2.2.** *Let  $\{f_n\}$  be a sequence of functions with common domain  $D$ , and let  $c$  be a point of  $I$ . Suppose that for all  $n \in \mathbb{Z}^+$  we have*

$$\lim_{x \rightarrow c} f_n(x) = L_n.$$

*Suppose moreover that  $f_n \xrightarrow{u} f$ . Then the sequence  $\{L_n\}$  is convergent,  $\lim_{x \rightarrow c} f(x)$  exists and we have equality:*

$$\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} \lim_{x \rightarrow c} f_n(x) = \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \lim_{n \rightarrow \infty} f_n(x).$$

*Proof.* Step 1: We show that the sequence  $\{L_n\}$  is convergent. Since we don't yet have a real number to show that it converges to, it is natural to try to use the Cauchy criterion, hence to try to bound  $|L_m - L_n|$ . Now comes the trick: for all  $x \in I$  we have

$$|L_m - L_n| \leq |L_m - f_m(x)| + |f_m(x) - f_n(x)| + |f_n(x) - L_n|.$$

By the Cauchy criterion for uniform convergence, for any  $\epsilon > 0$  there exists  $N \in \mathbb{Z}^+$  such that for all  $m, n \geq N$  and all  $x \in I$  we have  $|f_m(x) - f_n(x)| < \frac{\epsilon}{3}$ . Moreover, the fact that  $f_m(x) \rightarrow L_m$  and  $f_n(x) \rightarrow L_n$  give us bounds on the first and last terms: there exists  $\delta > 0$  such that if  $0 < |x - c| < \delta$  then

$|L_n - f_n(x)| < \frac{\epsilon}{3}$  and  $|L_m - f_m(x)| < \frac{\epsilon}{3}$ . Combining these three estimates, we find that by taking  $x \in (c - \delta, c + \delta)$ ,  $x \neq c$  and  $m, n \geq N$ , we have

$$|L_m - L_n| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

So the sequence  $\{L_n\}$  is Cauchy and hence convergent, say to  $L \in \mathbb{R}$ .

Step 2: We show that  $\lim_{x \rightarrow c} f(x) = L$  (so in particular the limit exists!).

Actually the argument for this is very similar to that of Step 1:

$$|f(x) - L| \leq |f(x) - f_n(x)| + |f_n(x) - L_n| + |L_n - L|.$$

Since  $L_n \rightarrow L$  and  $f_n(x) \rightarrow f(x)$ , the first and last term will each be less than  $\frac{\epsilon}{3}$  for sufficiently large  $n$ . Since  $f_n(x) \rightarrow L_n$ , the middle term will be less than  $\frac{\epsilon}{3}$  for  $x$  sufficiently close to  $c$ . Overall we find that by taking  $x$  sufficiently close to (but not equal to)  $c$ , we get  $|f(x) - L| < \epsilon$  and thus  $\lim_{x \rightarrow c} f(x) = L$ .  $\square$

**Corollary 3.2.3.** *Let  $f_n$  be a sequence of continuous functions with common domain  $D$  and suppose that  $f_n \xrightarrow{u} f$  on  $D$ . Then  $f$  is continuous on  $I$ .*

For the convenience of readers who would rather focus on Corollary 3.2.3 than Theorem 3.2.2, we give a separate (easier) proof.

*Proof.* Let  $x \in I$ . We need to show that  $\lim_{x \rightarrow c} f(x) = f(c)$ , thus we need to show that for any  $\epsilon > 0$  there exists  $\delta > 0$  such that for all  $x$  with  $|x - c| < \delta$  we have  $|f(x) - f(c)| < \epsilon$ . The idea – again! – is to trade this one quantity for three quantities that we have an immediate handle on by writing

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|.$$

By uniform convergence, there exists  $n \in \mathbb{Z}^+$  such that  $|f(x) - f_n(x)| < \frac{\epsilon}{3}$  for all  $x \in I$ : in particular  $|f_n(c) - f(c)| = |f(c) - f_n(c)| < \frac{\epsilon}{3}$ . Further, since  $f_n(x)$  is continuous, there exists  $\delta > 0$  such that for all  $x$  with  $|x - c| < \delta$  we have  $|f_n(x) - f_n(c)| < \frac{\epsilon}{3}$ . Consolidating these estimates, we get

$$|f(x) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \quad \square$$

**Theorem 3.2.4.** *Let  $\{f_n\}$  be a sequence of Riemann integrable functions with common domain  $[a, b]$ . Suppose that  $f_n \xrightarrow{u} f$ . Then  $f$  is Riemann integrable and*

$$\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b \lim_{n \rightarrow \infty} f_n = \int_a^b f. \quad (3.5)$$



*Proof.* We find ourselves at an awkward position here: we have not given any discussion of Riemann integrability of functions in this course – not even the definition. Moreover, it is not a matter of inserting a few paragraphs: the theory of the Riemann integral is one of the more technically complicated parts of undergraduate analysis. So we are not in a position to formally prove either part of the theorem here. A proof of this result can be found e.g. in [Cl-HC, §13.2] (though you will also have to read parts of Chapter 8, on integration, in order to understand it).

We have already proved that a uniform limit of continuous functions is continuous, and every continuous function is Riemann integrable [Cl-HC, Thm. 8.8], so the *integrability* of the limit function is not a key point for us. However, we do want to give some account of (3.5). In fact the basic idea here is so simple that one does not need a careful definition of the integral to appreciate it. All we need to know is the following facts about the integral:

(F1) If  $f, g : [a, b] \rightarrow \mathbb{R}$  are Riemann integrable, then  $\int_a^b f - g = \int_a^b f - \int_a^b g$ .

(F2) If  $f, g : [a, b] \rightarrow \mathbb{R}$  are Riemann integrable and  $f(x) \leq g(x)$  for all  $x \in [a, b]$ , then  $\int_a^b f \leq \int_a^b g$ .

(F3) For all  $C \in \mathbb{R}$  we have  $\int_a^b C = C(b - a)$ .

Notice that (F2) and (F3) are especially easy to swallow: they have simple interpretations in terms of  $\int_a^b f$  as the signed area under the curve  $y = f(x)$  from  $x = a$  to  $x = b$ . Now, since  $f_n \xrightarrow{u} f$ , for all  $\epsilon > 0$ , there is  $N \in \mathbb{Z}^+$  such that for all  $n \geq N$  we have  $\|f_n - f\| < \frac{\epsilon}{b-a}$ . Unpacking this, we have that for all  $x \in [a, b]$ ,

$$\frac{-\epsilon}{b-a} \leq f_n(x) - f(x) \leq \frac{\epsilon}{b-a}.$$

Using (F2) and (F3) we get

$$-\epsilon = \int_a^b \frac{-\epsilon}{b-a} \leq \int_a^b f_n - f \leq \int_a^b \frac{\epsilon}{b-a} = \epsilon.$$

So for all  $n \geq N$  we have

$$\left| \int_a^b f_n - f \right| \leq \epsilon$$

and thus (using (F1))

$$\int_a^b f_n - \int_a^b f = \int_a^b f_n - f \rightarrow 0,$$

so finally

$$\int_a^b f_n \rightarrow \int_a^b f \quad \square.$$

**Corollary 3.2.5.** *Let  $\{f_n\}$  be a sequence of continuous functions defined on the interval  $[a, b]$  such that  $\sum_{n=0}^{\infty} f_n \xrightarrow{u} f$ . For each  $n$ , let  $F_n : [a, b] \rightarrow \mathbb{R}$  be the unique function with  $F_n' = f_n$  and  $F_n(a) = 0$ , and similarly let  $F : [a, b] \rightarrow \mathbb{R}$  be the unique function with  $F' = f$  and  $F(a) = 0$ . Then  $\sum_{n=0}^{\infty} F_n \xrightarrow{u} F$ .*

*Proof.* By the Fundamental Theorem of Calculus, for all  $x \in [a, b]$  we have

$$F_n(x) = \int_a^x f_n,$$

$$F(x) = \int_a^x f.$$

Thus if  $S_N = \sum_{n=0}^N f_n$ , we also have

$$\int_a^x S_N = \int_a^x \sum_{n=0}^N f_n = \sum_{n=0}^N F_n.$$

Applying Theorem 3.2.4 to the sequence of partial sums  $\{S_N\}$  on the interval  $[a, x]$  gives the result.  $\square$

Our next order of business is to discuss differentiation of sequences of functions. For this we should reconsider Example 3.1.5: let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded differentiable function such that  $\lim_{n \rightarrow \infty} g(n)$  does not exist, and let  $f_n(x) = \frac{g(nx)}{n}$ . Then for all  $x \in \mathbb{R}$  we have

$$|f_n(x)| = \left| \frac{g(nx)}{n} \right| \leq \frac{\|g\|}{n},$$

so  $f_n \xrightarrow{u} 0$  on  $\mathbb{R}$ . But as we saw above,  $\lim_{n \rightarrow \infty} f_n'(1)$  does not exist. We have shown the following somewhat distressing fact: *uniform convergence of  $f_n$  to  $f$  does not imply that  $f_n'$  converges*. Our last lie is still a lie!

Well, don't panic. As before, the result *can be made true* by inserting suitable hypotheses. The following simple result is sufficient for our later applications.

**Theorem 3.2.6.** Let  $\{f_n : [a, b] \rightarrow \mathbb{R}\}_{n=1}^{\infty}$  be a sequence. We suppose:

- (i) Each  $f_n$  is continuously differentiable on  $[a, b]$ ,
- (ii) The functions  $f_n$  converge pointwise on  $[a, b]$  to some function  $f$ , and
- (iii) The functions  $f'_n$  converge uniformly on  $[a, b]$  to some function  $g$ .

Then  $f$  is differentiable and  $f' = g$ , or in other words

$$\left(\lim_{n \rightarrow \infty} f_n\right)' = \lim_{n \rightarrow \infty} f'_n.$$

*Proof.* Let  $x \in [a, b]$ . Since  $f'_n \xrightarrow{u} g$  on  $[a, b]$ , certainly  $f'_n \xrightarrow{u} g$  on  $[a, x]$ . Since each  $f'_n$  is continuous, by Corollary 3.2.3  $g$  is continuous. Now applying Theorem 3.2.4 and the Fundamental Theorem of Calculus we have

$$\int_a^x g = \int_a^x \lim_{n \rightarrow \infty} f'_n = \lim_{n \rightarrow \infty} \int_a^x f'_n = \lim_{n \rightarrow \infty} f_n(x) - f_n(a) = f(x) - f(a).$$

Differentiating and applying the Fundamental Theorem of Calculus, we get

$$g = (f(x) - f(a))' = f'. \quad \square$$

For future use we record the analogous result for infinite series. (It is, of course, proved by applying Theorem 3.2.6 to the sequence of partial sums.)

**Corollary 3.2.7.** Let  $\sum_{n=0}^{\infty} f_n(x)$  be a series of functions converging pointwise to  $f(x)$ . Suppose that each  $f'_n$  is continuously differentiable and  $\sum_{n=0}^{\infty} f'_n(x) \xrightarrow{u} g$ . Then  $f$  is differentiable and  $f' = g$ :

$$\left(\sum_{n=0}^{\infty} f_n\right)' = \sum_{n=0}^{\infty} f'_n. \quad (3.6)$$

When it holds that  $(\sum_n f_n)' = \sum_n f'_n$ , we say that the series can be differentiated **termwise** or **term-by-term**. Thus Corollary 3.2.7 gives a condition under which a series of functions can be differentiated termwise.

Although Theorem 3.2.6 (or more precisely, Corollary 3.2.7) will be sufficient for our needs, we cannot help but record the following stronger version.

**Theorem 3.2.8.** Let  $\{f_n\}$  be differentiable functions on the interval  $[a, b]$  such that  $\{f_n(x_0)\}$  is convergent for some  $x_0 \in [a, b]$ . If there is  $g : [a, b] \rightarrow \mathbb{R}$  such that  $f'_n \xrightarrow{u} g$  on  $[a, b]$ , then there is  $f : [a, b] \rightarrow \mathbb{R}$  such that  $f_n \xrightarrow{u} f$  on  $[a, b]$  and  $f' = g$ .

*Proof.* [R, pp.152-153]

Step 1: Fix  $\epsilon > 0$ , and choose  $N \in \mathbb{Z}^+$  such that  $m, n \geq N$  implies  $|f_m(x_0) - f_n(x_0)| \leq \frac{\epsilon}{2}$  and  $|f'_m(t) - f'_n(t)| < \frac{\epsilon}{2(b-a)}$  for all  $t \in [a, b]$ . The latter inequality is telling us that the derivative of  $g := f_m - f_n$  is small on the entire interval  $[a, b]$ . Applying the Mean Value Theorem to  $g$ , we get a  $c \in (a, b)$  such that for all  $x, t \in [a, b]$  and all  $m, n \geq N$ ,

$$|g(x) - g(t)| = |x - t||g'(c)| \leq |x - t| \left( \frac{\epsilon}{2(b-a)} \right) \leq \frac{\epsilon}{2}. \quad (3.7)$$

It follows that for all  $x \in [a, b]$ ,

$$|f_m(x) - f_n(x)| = |g(x)| \leq |g(x) - g(x_0)| + |g(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

By the Cauchy Criterion,  $f_n$  is uniformly convergent on  $[a, b]$  to some  $f$ .

Step 2: Now **fix**  $x \in [a, b]$  and define

$$\varphi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}$$

and

$$\varphi(t) = \frac{f(t) - f(x)}{t - x},$$

so that for all  $n \in \mathbb{Z}^+$ ,  $\lim_{t \rightarrow x} \varphi_n(t) = f'_n(x)$ . Now by (3.7) we have

$$|\varphi_m(t) - \varphi_n(t)| \leq \frac{\epsilon}{2(b-a)}$$

for all  $m, n \geq N$ , so once again by the Cauchy criterion  $\varphi_n$  converges uniformly for all  $t \neq x$ . Since  $f_n \xrightarrow{u} f$ , we get  $\varphi_n \xrightarrow{u} \varphi$  for all  $t \neq x$ . Finally we apply Theorem 3.2.2 on the interchange of limit operations:

$$f'(x) = \lim_{t \rightarrow x} \varphi(t) = \lim_{t \rightarrow x} \lim_{n \rightarrow \infty} \varphi_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} \varphi_n(t) = \lim_{n \rightarrow \infty} f'_n(x). \quad \square$$

### 3.2.3 Showing Uniform Convergence

Now that we believe that uniform convergence of a sequence or series is what we really want, the next question becomes how often it occurs in nature.

In some sense it may be *easier* to show  $f_n \xrightarrow{u} f$  on  $D$  than to show  $f_n \rightarrow f$  on

$D$ : whereas for the latter we need to show that  $f_n(x) \rightarrow f(x)$  for all  $x \in D$ , as we saw above,  $f_n \xrightarrow{u} f$  on  $D$  is equivalent to

$$\|f_n - f\| \rightarrow 0,$$

i.e., we just have to show that one sequence converges to 0. When we unpack the notation, we see that we must show that

$$\sup_{x \in D} |f_n(x) - f(x)| \rightarrow 0.$$

Notice that if  $D = [a, b]$  and  $f_n$  and  $f$  are continuous then by the Extreme Value Theorem the supremum can be replaced with a maximum. How do we compute the maximum of the function  $|f_n - f|$  on  $[a, b]$ ? Well, there is something called calculus that can help with this! In fact, even when the domain is not a closed bounded interval, if the calculus works out right we can still use it to find the maximum value. Here is an example.

**Example 3.2.9.** For each  $n \in \mathbb{Z}^+$ , let  $f_n : [0, \infty) \rightarrow \mathbb{R}$  by  $f_n(x) = xe^{-nx}$ . The graphs of the first three functions in this sequence are depicted in figure 3.1.3. We claim that again  $\|f_n\| \rightarrow 0$  so that  $f_n \xrightarrow{u} 0$  on  $[0, \infty)$ . To show

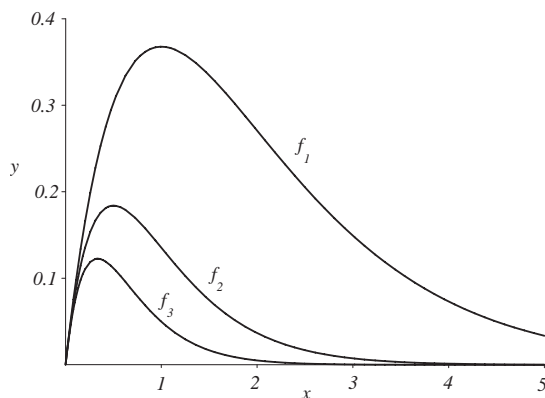


figure 3.1.3

this we use calculus: for all  $n \in \mathbb{Z}^+$  and  $x \in [0, \infty)$  we have

$$f'_n = e^{-nx} - nxe^{-nx} = (1 - nx)e^{-nx},$$

so  $f'_n$  is positive on  $[0, \frac{1}{n})$ , zero at  $\frac{1}{n}$  and negative on  $(\frac{1}{n}, \infty)$ . It follows that  $f_n$  has a maximum at  $x = \frac{1}{n}$ . Since  $f_n(x) \geq 0$  for all  $x$ , we find

$$\|f_n\| = f_n\left(\frac{1}{n}\right) = \frac{1}{ne} \rightarrow 0.$$

Thus indeed  $f_n \xrightarrow{u} 0$  on  $[0, \infty)$ .

This kind of technique can often be used to show directly that  $f_n \xrightarrow{u} f$ .

We will give one further criterion for showing uniform convergence.

**Theorem 3.2.10.** (Weierstrass M-test) Let  $D \subset \mathbb{R}$ , and let  $\{f_n : D \rightarrow \mathbb{R}\}$  be a sequence of functions. If

$$\sum_n \|f_n\| < \infty,$$

then the infinite series  $\sum_n f_n$  is uniformly and absolutely convergent on  $D$ .

*Proof.* This follows from the Cauchy criteria for convergence and the triangle inequality. Namely, since  $\sum_n \|f_n\| < \infty$ , for all  $\epsilon > 0$  there is  $N \in \mathbb{Z}^+$  such that for all  $n \geq m \geq N$  we have  $\sum_{k=m}^n \|f_k\| < \epsilon$ . Then for all  $n \geq m \geq N$ ,

$$\begin{aligned} \left\| \sum_{k=m}^n f_k \right\| &= \sup_{x \in D} \left| \sum_{k=m}^n f_k(x) \right| \\ &\leq \sup_{x \in D} \sum_{k=m}^n |f_k(x)| \leq \sum_{k=m}^n \sup_{x \in D} |f_k(x)| = \sum_{k=m}^n \|f_k\| < \epsilon. \quad \square \end{aligned}$$

From this we deduce the following important result on convergence of power series.

**Theorem 3.2.11.** Let  $\sum_{n=0}^{\infty} a_n x^n$  be a power series with radius of convergence  $R > 0$ . Then for all  $c \in (0, R)$ , the series converges absolutely and uniformly on the interval  $[-c, c]$ .

*Proof.* First we observe that the series  $\sum_{n=0}^{\infty} a_n c^n$  is absolutely convergent. To see this, take  $r = \frac{|c|+R}{2}$ . Since  $r < R$ , by definition of the interval of

convergence,  $\sum_{n=0}^{\infty} a_n r^n$  converges. Since  $|c| < r$ , it follows from Proposition 2.6.3 that  $\sum_{n=0}^{\infty} |a_n c^n| < \infty$ . Now observe that for all  $x \in [-c, c]$ , we have

$$|a_n x^n| \leq |a_n c^n|,$$

so taking  $D = [-c, c]$  as our domain, we have

$$\sum_{n=0}^{\infty} \|a_n x^n\| \leq \sum_{n=0}^{\infty} |a_n c^n| < \infty.$$

The absolute and uniform convergence of  $\sum_{n=0}^{\infty} a_n x^n$  on  $[-c, c]$  now follows from the Weierstrass M-Test.  $\square$

### 3.2.4 Applications to Power Series

**Lemma 3.2.12.** *Let  $\sum_n a_n x^n$  be a power series with radius of convergence  $R \in [0, \infty]$ . Then the radius of convergence of  $\sum_n n a_n x^n$  is also  $R$ .*

*Proof.* The radius of convergence  $R$  of  $\sum_n a_n x^n$  is characterized by the fact that for all  $x$  with  $|x| < R$  we have that  $\sum_n a_n x^n$  converges, while for all  $x$  with  $|x| > R$  we have that  $\sum_n a_n x^n$  diverges. In fact, if  $|x| > R$  then the sequence  $\{a_n x^n\}$  is unbounded: indeed, in the proof of Proposition 2.6.3 we used only the assumption that  $\{a_n x^n\}$  is bounded to show absolute convergence of  $\sum_n a_n c^n$  for all  $c$  with  $|c| < |x|$ , so if  $\{a_n x^n\}$  were bounded, the power series  $\sum_n a_n (\frac{|x|+R}{2})^n$  would converge, which is a contradiction. Since the sequence  $a_n x^n$  is unbounded, so is the sequence  $n a_n x^n$ , and thus the series  $\sum_n n a_n x^n$  also diverges for all  $x$  with  $|x| > R$ .

Suppose now that  $|x| < R$ . Let  $y = \frac{|x|+R}{2}$ . Since  $|y| < R$ , we have  $\sum_n |a_n y^n| < \infty$ . But for all sufficiently large  $n$  we have  $n|a_n||x|^n \leq |a_n||y|^n$ . Indeed, this clearly holds when  $a_n = 0$ ; if not, the inequality is equivalent to

$$n \leq \left|\frac{y}{x}\right|^n.$$

Since  $|\frac{y}{x}| > 1$ , that this holds for all sufficiently large  $n$  follows from the fact that for all  $C > 1$  we have  $\lim_{x \rightarrow \infty} \frac{x}{C^x} = 0$  (e.g. by L'Hôpital). Therefore  $\sum_n n a_n x^n$  is absolutely convergent by comparison to  $\sum_n |a_n y^n|$ .  $\square$

We are now ready to state and prove a big result on functions defined by convergent power series.

**Theorem 3.2.13.** (*Wonderful Properties of Power Series*) Let  $\sum_{n=0}^{\infty} a_n x^n$  be a power series with radius of convergence  $R > 0$ . Consider the function

$$f : (-R, R) \rightarrow \mathbb{R}, \quad f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- a) The function  $f$  is continuous.  
 b) The function  $f$  is differentiable. Moreover its derivative may be computed termwise: for all  $x \in (-R, R)$  we have

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}. \quad (3.8)$$

c) Since the power series in the right hand side of (3.8) has the same radius of convergence  $R$  as  $f$ , the function  $f$  is in fact infinitely differentiable.

d) For all  $n \in \mathbb{N}$ , we have  $f^{(n)}(0) = n! a_n$ .

e) The power series  $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$  also has radius of convergence  $R$ , and for all  $x \in (-R, R)$  we have  $F'(x) = f(x)$ .

*Proof.* a) Let  $0 < A < R$  and consider  $f$  as a function from  $[-A, A]$  to  $\mathbb{R}$ . We claim that the series  $\sum_n a_n x^n$  converges to  $f$  uniformly on  $[-A, A]$ . Indeed, as a function on  $[-A, A]$  we have

$$\|a_n x^n\| = |a_n| A^n$$

and thus

$$\sum_n \|a_n x^n\| = \sum_n |a_n| A^n < \infty,$$

since by Proposition 2.6.3 power series converge absolutely on the interior of their interval of convergence. So the Weierstrass  $M$ -Test gives that  $f$  is the uniform limit of the sequence  $S_n(x) = \sum_{n=0}^N a_n x^n$ . Each  $S_n$  is a polynomial function, hence continuous and indeed infinitely differentiable. By Theorem 3.2.3 we get that  $f$  is continuous on  $[-A, A]$ . Since any  $x \in (-R, R)$  lies in  $[-A, A]$  for some  $0 < A < R$ , in fact  $f$  is continuous on  $(-R, R)$ .

b) According to Corollary 3.2.7, in order to show that  $f = \sum_n a_n x^n = \sum_n f_n$  is differentiable and the derivative may be computed termwise, it is enough to check first that each  $f_n$  is continuous differentiable and second that  $\sum_n f'_n$  is uniformly convergent. The first point is clear:  $f_n = a_n x^n$  is a polynomial. As for the second point, we find that

$$\sum_n f'_n = \sum_n (a_n x^n)' = \sum_n n a_n x^{n-1}.$$



For  $x \neq 0$ , certainly  $\sum_n na_n x^{n-1}$  converges iff  $x \sum_n na_n x^{n-1} = \sum_n na_n x^n$  converges, which by Lemma 3.2.12 occurs for all  $|x| < R$ . We may now apply the result of part a) to the power series  $\sum_n na_n x^{n-1}$  to get that it is uniformly convergent on  $[-A, A]$  for all  $0 < A < R$ . Thus Corollary 3.2.7 applies to show that  $f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$ .

c) We have just seen that for a power series  $f$  convergent on  $(-R, R)$ , its derivative  $f'$  is also given by a power series convergent on  $(-R, R)$ . So we may continue in this way: by induction, derivatives of all orders exist.

d) The formula  $f^{(n)}(0) = n!a_n$  is what one obtains by repeated termwise differentiation and evaluation at 0. We leave this as an exercise.

e) We leave this as an exercise.  $\square$

**Example 3.2.14.** a) Since we know that  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$  when  $-1 < x < 1$ , we can use Theorem 3.2.13 to conclude that  $\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$  when  $-1 < x < 1$ .

b) Applying Theorem 3.2.13 to the same series tells us that  $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$  converges to  $\int_0^x \frac{ds}{1-s} = -\log(1-x)$  when  $-1 < x < 1$ . By re-labeling the index, we see that

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$$

for  $-1 < x < 1$ . Notice that even though the series converges at  $x = -1$ , the theorem does not tell us anything about what it converges to. On the other hand, we know from exercise 2.3.7 that the alternating harmonic series converges to  $\log(1/2)$  so the above equality is indeed true on the interval  $[-1, 1)$ .

### EXERCISES 3.2

- Consider the sequence of functions  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  defined in Example 3.1.2, which converges pointwise to  $f : \mathbb{R} \rightarrow \mathbb{R}$ .
  - Use one of the results of §3.2.2 to deduce that the convergence of  $f_n$  to  $f$  cannot be uniform.
  - Show directly from the definition that the convergence of  $f_n$  to  $f$  is not uniform.
- Consider the sequence of functions  $f_n : [0, 1] \rightarrow \mathbb{R}$  defined in Example 3.1.3, which converges pointwise to  $f : [0, 1] \rightarrow \mathbb{R}$ .

- a) Use one of the results of §3.2.2 to deduce that the convergence of  $f_n$  to  $f$  cannot be uniform.
- b) Show directly from the definition that the convergence of  $f_n$  to  $f$  is not uniform.
3. Consider the sequence of functions  $f_n : [0, 1] \rightarrow \mathbb{R}$  defined in Example 3.1.4, which converges pointwise to  $f : [0, 1] \rightarrow \mathbb{R}$ .
- a) Use one of the results of §3.2.2 to deduce that the convergence of  $f_n$  to  $f$  cannot be uniform.
- b) Show directly from the definition that the convergence of  $f_n$  to  $f$  is not uniform.
4. For each of the following sequences of functions  $\{f_n\}$  defined on the given interval  $J$ , determine if the sequence converges pointwise to a limit function  $f$ . If the sequence does converge pointwise, determine whether the convergence is uniform.
- a)  $f_n(x) = \frac{x}{x+n}$ ,  $J = [0, \infty)$ .
- b)  $f_n(x) = \frac{nx}{1+n^2x^2}$ ,  $J = (-\infty, \infty)$ .
- c)  $f_n(x) = \frac{x^n}{1+x^n}$ ,  $J = [0, \infty)$ .
- d)  $f_n(x) = \arctan(nx)$ ,  $J = (-\infty, \infty)$ .
- e.)  $f_n(x) = e^{-nx}$ ,  $J = [0, \infty)$ .
- f.)  $f_n(x) = e^{-nx}$ ,  $J = [1, \infty)$ .
- g.)  $f_n(x) = xe^{-nx}$ ,  $J = (-\infty, \infty)$ .
- h.)  $f_n(x) = x^2e^{-nx}$ ,  $J = [0, \infty)$ .

5. a) Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  be a sequence of functions converging uniformly to  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Show: if each  $f_n$  is bounded, then so is  $f$ .
- b) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be any continuous function. Define  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  as follows:

$$f_n(x) = \begin{cases} f(-n) & x \leq -n \\ f(x) & x \in [-n, n] \\ f(n) & x \geq n. \end{cases}$$

Show: each  $f_n$  is bounded and continuous, and  $f_n \rightarrow f$  pointwise on  $\mathbb{R}$ .

6. Let  $D_1 \subset D_2 \subset \mathbb{R}$ , and suppose that

$$D_2 \setminus D_1 = \{x \in D_2 \mid x \notin D_1\}$$

is finite. Let  $\{f_n : D_2 \rightarrow \mathbb{R}\}$  be a sequence of functions and let  $f : D_2 \rightarrow \mathbb{R}$  be a function. Suppose that  $f_n \xrightarrow{u} f$  on  $D_1$  and  $f_n \rightarrow f$  pointwise on  $D_2$ . Show that  $f_n \xrightarrow{u} f$  on  $D_2$ .

7. Find “closed form” expressions for the following power series and determine their domains of convergence:

a)  $\sum_{n=0}^{\infty} nx^n$

b)  $\sum_{n=0}^{\infty} n^2x^n$

c)  $\sum_{n=0}^{\infty} nx^{2n}$

d)  $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n}$

e.)  $\sum_{n=2}^{\infty} \frac{x^n}{n^2-n}$

8. Evaluate:

a)  $\sum_{n=1}^{\infty} \frac{n}{3^n}$

b)  $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

c)  $\sum_{n=1}^{\infty} \frac{1}{n3^n}$

9. (From the William Lowell Putnam Mathematical Competition, 1999)

Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2n}{3^m(n3^m + m3^n)}.$$

10. a) Show that  $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$  for  $x \in (-1, 1)$ . (Hint: differentiate!)

b) Using the trigonometric identity  $\frac{\pi}{4} = 4 \arctan(1/5) - \arctan(1/239)$  show that  $\pi = 3.14159\dots$

11. Prove that the series  $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$  converges uniformly on  $A = \mathbb{R}$ .
12. Prove Theorem 3.2.13d).
13. Prove Theorem 3.2.13e).
14. a) Prove: for all  $\alpha > 1$ , the series  $\sum_{n=1}^{\infty} \frac{1}{n^x}$  converges uniformly on  $[\alpha, \infty)$ .  
The function defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

is called the *Riemann zeta function*.

b) Deduce:  $\zeta : (1, \infty) \rightarrow \mathbb{R}$  is a continuous function.

15. a) Suppose the power series  $\sum_{n=0}^{\infty} a_n x^n$  converges absolutely at some  $x \in \mathbb{R}$ . Show that the series converges absolutely and uniformly on  $[-|x|, |x|]$ .  
b) Without the hypothesis of *absolute* convergence at  $x$ , the power series need not converge at  $-x$ . Give an example of this.  
c) One may still ask: suppose a power series converges on  $[-R, R]$  but that the convergence is not assumed to be absolute at  $x = \pm R$ . Must the convergence then be uniform on  $[-R, R]$ ? The answer is *yes*, but this is a theorem, not an exercise. (See [Cl-HC, Thm. 14.15].)
16. Let  $\sum_{n=0}^{\infty} a_n x^n$  be a power series with radius of convergence  $R = \infty$ : i.e., it converges for all real numbers.  
a) Show that if  $a_n = 0$  for all sufficiently large  $n$ , the convergence is uniform on  $\mathbb{R}$ .  
(Hint: this is almost trivial.)  
b) Show that if the convergence is uniform on  $\mathbb{R}$ , then  $a_n = 0$  for all sufficiently large  $n$ .  
(Hint: if  $\sum_n f_n$  converges uniformly on  $D$ , then by the Cauchy Criterion we have  $\|f_n\| \leq 1$  for all sufficiently large  $n$ .)

## 3.3 Taylor Series

### 3.3.1 Introducing Taylor Series

In Theorem 3.2.13d) we saw that if  $f(x) = \sum_{n=1}^{\infty} a_n x^n$  then the coefficient,  $a_n$ , is intimately related to the value of the  $n^{\text{th}}$  derivative of  $f$  at zero, namely

$$f^{(n)}(0) = n!a_n.$$

(Here we should mention the convention that  $0! = 1$ .) The above observation leads to a method of constructing a power series which might converge to a given function  $f$ . In particular, if the series  $\sum_{n=1}^{\infty} a_n x^n$  has any chance of converging to  $f(x)$ , then the coefficients must be given by

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

This leads to the following:

**Definition 3.3.1.** *Let  $a \in \mathbb{R}$ , and let  $f$  be an infinitely differentiable function defined on an interval containing  $a$ , then the Taylor series for  $f$ , centered at  $a$  is given by*

$$T_{f,a}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n.$$

For  $N \in \mathbb{N}$ , we also define the  $N$ th order Taylor polynomial centered at  $a$  as

$$T_{N,f,a}(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x - a)^n.$$

Thus  $T_{N,f,a}$  is the  $N$ th partial sum of the infinite series  $T_{f,a}$ .

**Remark 3.3.2.** *When  $a = 0$  one often calls the Taylor series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

the “Maclaurin series.” We will for the most part not use this terminology.

**Example 3.3.3.** *Consider  $f(x) = e^x$ . Since  $f^{(n)}(x) = e^x$  for all  $n$ , we see that  $f^{(n)}(0) = 1$  for all  $n$  and so the Taylor series for  $f$  at 0 is given by*

$$T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

**Example 3.3.4.** a) Let  $f(x) = \cos(x)$ . Then

$$\begin{aligned}f'(x) &= -\sin(x) \\f''(x) &= -\cos(x) \\f^{(3)}(x) &= \sin(x) \\f^{(4)}(x) &= \cos(x)\end{aligned}$$

This pattern repeats with period 4. Thus  $f(0) = 1, f'(0) = 0, f''(0) = -1, f^{(3)}(0) = 0, f^{(4)}(0) = 1$  and this pattern repeats. A good way to write down the general form of the  $n^{\text{th}}$  derivative at 0 is  $f^{(2n+1)}(0) = 0$  and  $f^{(2n)}(0) = (-1)^n$ . Thus the Taylor series expansion of  $f(x) = \cos x$  centered at 0 is

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

b) A very similar computation, which we leave to the reader as an exercise, shows that if  $f(x) = \sin x$ , the Taylor series expansion centered at 0 is

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

### 3.3.2 The Local Taylor Theorem

The question of whether or not the Taylor series for  $f$  converges to  $f$  is the same as the question of whether or not the sequence of Taylor polynomials converges to  $f$ . It seems wise to start by considering why we might believe that this is true.

For a smooth function  $f$  defined on an open interval around  $c \in \mathbb{R}$ , we have defined the  $N$ th order Taylor polynomial

$$T_{N,f,a}(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

The first key idea is that we are performing a certain kind of *polynomial interpolation*, as captured by the following result.

**Theorem 3.3.5.** *The Taylor polynomial  $T_{N,f,a}(x)$  is the unique polynomial  $P(x)$  of degree at most  $N$  such that*

$$\forall 0 \leq n \leq N, P^{(n)}(a) = f^{(n)}(a).$$

*Proof.*

Step 1: Let  $P$  be a polynomial of degree at most  $N$ . Then we may write

$$P(x) = \sum_{n=0}^N a_n(x-a)^n$$

for some real numbers  $a_0, \dots, a_N$ . We prove this by induction on  $N$ . The base case is  $N = 0$ , and this is trivial: a degree 0 polynomial is a constant, thus of the form  $a_0$ . Now let  $N \in \mathbb{Z}^+$ , suppose that all polynomials of degree  $N - 1$  can be expressed in the desired form, and let  $P(x) = \sum_{n=0}^N b_n x^n$  be a polynomial of degree at most  $N$ . Let  $a_N = b_N$ , and put

$$Q(x) = P(x) - b_N(x-a)^N = \sum_{n=0}^N b_n x^n - a_N(x-a)^N.$$

The key observation here is that when we expand  $a_N(x-a)^N$  out in powers of  $x$ , the coefficient of  $x^N$  is simply  $a_N = b_N$ , which is the same as the coefficient of  $x^N$  in  $P(x) = \sum_{n=0}^N b_n x^n$ . Thus  $Q(x)$  is a polynomial of degree at most  $N - 1$ . By our induction hypothesis, we may write it as  $\sum_{n=0}^{N-1} a_n(x-a)^n$ , and thus

$$P(x) = Q(x) + a_N(x-a)^N = \sum_{n=0}^N a_n(x-a)^n,$$

completing the induction step.

Step 2: Now consider the polynomial

$$P(x) = \sum_{n=0}^N a_n(x-a)^n.$$

Evaluating at  $x = a$ , we get

$$P(a) = a_0 + a_1(a-a) + a_2(a-a)^2 + \dots + a_N(a-a)^N = a_0.$$

Differentiating, we get

$$P'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \dots + Na_N(x-a)^{N-1}.$$

Evaluating the derivative at  $x = a$ , we get

$$P'(a) = a_1 + 2a_2(a-a) + 3a_3(a-a)^2 + \dots + Na_N(a-a)^{N-1} = a_1.$$

Continuing in this manner – i.e., alternately differentiating and evaluating at  $x = a$  – we will successively obtain formulas for all the coefficients  $a_n$  in terms of the higher derivatives at  $a$ . If we differentiate once more we get

$$P''(x) = 2a_2 + 3 \cdot 2(x - a) + 4 \cdot 3(x - a)^2 + \dots + N(N - 1)a_N(x - a)^{N-2}$$

and evaluating at  $x = a$  gives

$$P''(a) = 2a_2,$$

or

$$a_2 = \frac{P''(a)}{2}.$$

In general, if we differentiate  $n$  times and evaluate at  $x = a$ , then in  $\sum_{k=0}^N a_k(x - a)^k$ , then if  $k < n$  the  $n$ th derivative of  $a_k(x - a)^k$  will be zero, whereas if  $k > n$  then the  $n$ th derivative of  $a_k(x - a)^k$  will still involve a positive power of  $(x - a)$  so will be zero *when evaluated at  $x = a$* . So the  $k$ th derivative of  $P(x)$  evaluated at  $a$  is equal to the  $k$ th derivative of  $a_k x^k$  evaluated at  $a$ , which a little thought shows to be  $k!a_k$ . Thus

$$P^{(k)}(a) = k!a_k$$

or

$$a_k = \frac{P^{(k)}(a)}{k!}.$$

Step 3: We notice that the formula just obtained immediately implies not just the existence but the *uniqueness* of the expression of a polynomial  $P(x)$  of degree at most  $N$  in the form  $\sum_{n=0}^N a_n(x - a)^n$ : indeed we have

$$P(x) = \sum_{n=0}^N \frac{P^{(n)}(a)}{n!} (x - a)^n.$$

Applying this to the Taylor polynomial

$$T_N(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x - a)^n,$$

we get that for all  $0 \leq n \leq N$ ,

$$T_N^{(n)}(a) = n! \frac{f^{(n)}(a)}{n!} = f^{(n)}(a).$$



So indeed the Taylor polynomial interpolates the value of  $f$  and the derivatives up through the  $N$ th derivative of  $f$  at  $x = a$ , and it is the only polynomial of degree at most  $N$  with this property.  $\square$

The 0th order Taylor polynomial is

$$T_0(x) = f(a),$$

a constant function. This is the crudest possible approximation to  $f$  near  $a$ . The 1st order Taylor polynomial is

$$T_1(x) = f(a) + f'(a)(x - a).$$

Thus  $T_1(x)$  is a linear function which passes through the point  $(a, f(a))$  and has slope  $f'(a)$ . By definition then  $T_1(x)$  is the *tangent line* to  $y = f(x)$  at  $x = a$ . In freshman calculus one hears that the tangent line is the “best linear approximation” to  $y = f(x)$  at  $x = a$ ...but other than the idea that approximating  $f$  by its tangent line is a better idea the closer  $x$  is to  $a$ , one does not really get a clear quantitative explanation of what that means.

The 2nd order Taylor polynomial is

$$T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

This is a parabola (unless  $f''(a) = 0$ , in which case it is the tangent line again). I encourage the reader to use software to actually graph the second order Taylor polynomial along with  $y = f(x)$  for various smooth functions  $f$  and various choices of  $a$ . After a while you will start to suspect that  $T_2(x)$  is somehow the “best parabolic approximation” to  $y = f(x)$  near  $x = a$ . But what does that mean?

In the rest of the section we work towards an answer to the general claim that  $T_N(x)$  is the “best order  $N$  approximation to  $y = f(x)$  near  $a$ .”

For  $f : D \rightarrow \mathbb{R}$  and  $a \in D$ , we will use the shorthand  $a \in D^\circ$  to mean that there is some  $\delta > 0$  such that  $(a - \delta, a + \delta) \subset D$ .

For  $n \in \mathbb{N}$  and  $a \in D^\circ$ , we say two functions  $f, g : D \rightarrow \mathbb{R}$  **agree to order  $n$  at  $a$**  if

$$\lim_{x \rightarrow a} \frac{f(x) - g(x)}{(x - a)^n} = 0.$$

**Lemma 3.3.6.** *Suppose  $0 \leq m \leq n$  and  $f$  and  $g$  agree to order  $n$  at  $a$ . Then also  $f$  and  $g$  agree to order  $m$  at  $a$ .*

You are asked to prove Lemma 3.3.6 in Exercise ??.

**Example 3.3.7.** *We claim that two continuous functions  $f$  and  $g$  agree to order 0 at  $a$  if and only if  $f(a) = g(a)$ . Indeed, suppose that  $f$  and  $g$  agree to order 0 at  $a$ . Since  $f$  and  $g$  are continuous, we have*

$$0 = \lim_{x \rightarrow a} \frac{f(x) - g(x)}{(x - a)^0} = \lim_{x \rightarrow a} f(x) - g(x) = f(a) - g(a).$$

*The converse – if  $f(a) = g(a)$  then  $\lim_{x \rightarrow a} f(x) - g(x) = 0$  – is equally clear.*

**Example 3.3.8.** *We claim that two differentiable functions  $f$  and  $g$  agree to order 1 at  $a$  if and only if  $f(a) = g(a)$  and  $f'(a) = g'(a)$ . Both hypotheses imply  $f(a) = g(a)$  so we may assume that, and then we find*

$$\lim_{x \rightarrow a} \frac{f(x) - g(x)}{x - a} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} - \frac{g(x) - g(a)}{x - a} = f'(a) - g'(a).$$

*Thus assuming  $f(a) = g(a)$ ,  $f$  and  $g$  agree to order 1 at  $a$  if and only if  $f'(a) = g'(a)$ .*

The following result gives the expected generalization of these two examples. It is generally attributed to Taylor,<sup>5</sup> probably correctly, although special cases were known to earlier mathematicians.

**Theorem 3.3.9.** (*Local Taylor Theorem*) *Let  $N \in \mathbb{N}$  and  $f, g : I \rightarrow \mathbb{R}$  be two  $N$  times differentiable functions, and let  $a$  be an interior point of  $I$ . The following are equivalent:*

- (i) *We have  $f(a) = g(a), f'(a) = g'(a), \dots, f^{(N)}(a) = g^{(N)}(a)$ .*
- (ii)  *$f$  and  $g$  agree to order  $N$  at  $a$ .*

*Proof.* Let  $h = f - g$ . Then (i) holds iff  $h(a) = h'(a) = \dots = h^{(n)}(a) = 0$  and (ii) holds iff  $\lim_{x \rightarrow a} \frac{h(x)}{(x-a)^n} = 0$ . So we may work with  $h$  instead of  $f$  and  $g$ . Since we dealt with  $N = 0$  and  $N = 1$  above, we may assume  $N \geq 2$ .

(i)  $\implies$  (ii):  $L = \lim_{x \rightarrow a} \frac{h(x)}{(x-a)^N}$  is of the form  $\frac{0}{0}$ , so L'Hôpital's Rule gives

$$L = \lim_{x \rightarrow a} \frac{h'(x)}{N(x-a)^{N-1}},$$

---

<sup>5</sup>Brook Taylor, 1685 - 1731

provided the latter limit exists. By our assumptions, this latter limit is still of the form  $\frac{0}{0}$ , so we may apply L'Hôpital's Rule again. We do so iff  $N > 2$ . In general, we apply L'Hôpital's Rule  $N - 1$  times, getting

$$L = \lim_{x \rightarrow a} \frac{h^{(N-1)}(x)}{N!(x-a)} = \frac{1}{N!} \left( \lim_{x \rightarrow a} \frac{h^{(N-1)}(x) - h^{(N-1)}(a)}{x-a} \right),$$

provided the latter limit exists. But the expression in parentheses is nothing else than the derivative of the function  $h^{(N-1)}(x)$  at  $x = a$  — i.e., it is  $h^{(N)}(a) = 0$  (and, in particular the limit exists; only now have the  $N - 1$  applications of L'Hôpital's Rule been unconditionally justified), so  $L = 0$ . Thus (ii) holds. (ii)  $\implies$  (i): Let  $T_N(x)$  be the degree  $N$  Taylor polynomial to  $h$  at  $a$ . By Theorem 3.3.5 we have

$$\forall 0 \leq n \leq N, f^{(n)}(a) = T_N^{(n)}(a),$$

so by the just proved implication (i)  $\implies$  (ii),  $h(x)$  and  $T_N(x)$  agree to order  $N$  at  $x = a$ :

$$\lim_{x \rightarrow a} \frac{h(x) - T_N(x)}{(x-a)^N} = 0.$$

Moreover, by assumption  $h(x)$  agrees to order  $N$  with the zero function:

$$\lim_{x \rightarrow a} \frac{h(x)}{(x-a)^N} = 0.$$

Subtracting these limits gives

$$\lim_{x \rightarrow a} \frac{T_N(x)}{(x-a)^N} = \lim_{x \rightarrow a} \frac{h(a) + h'(a)(x-a) + \frac{h''(a)}{2}(x-a)^2 + \dots + \frac{h^{(N)}(a)}{N!}(x-a)^N}{(x-a)^N} = 0. \quad (3.9)$$

Clearly  $\lim_{x \rightarrow a} T_N(x) = T_N(a)$ , so if  $T_N(a) \neq 0$ , then  $\lim_{x \rightarrow a} \frac{T_N(x)}{(x-a)^N}$  would not exist, so we must have  $T_N(a) = h(a) = 0$ . Therefore

$$\lim_{x \rightarrow a} \frac{T_N(x)}{(x-a)^N} = \lim_{x \rightarrow a} \frac{h'(a) + \frac{h''(a)}{2}(x-a) + \dots + \frac{h^{(N)}(a)}{N!}(x-a)^{N-1}}{(x-a)^{N-1}} = 0.$$

As above, we have the limit of a quotient of continuous functions which we know exists such that the denominator approaches 0, so the numerator must also approach zero (otherwise the limit would be infinite): evaluating the

numerator at  $a$  gives  $h'(a) = 0$ . And so forth: continuing in this way we find that the existence of the limit in (3.9) implies that

$$h(a) = h'(a) = \dots = h^{(N-1)}(a) = 0,$$

so (3.9) simplifies to

$$0 = \lim_{x \rightarrow a} \frac{\frac{h^{(N)}(a)}{N!}(x-a)^N}{(x-a)^N} = \frac{h^{(N)}(a)}{N!},$$

so  $h^{(N)}(a) = 0$ . □

**Remark 3.3.10.** *Professor Krystal Taylor (!) of Ohio State University has a nice handout on the use and misuse of l'Hôpital's Rule [Ta]. It ends with the following remark (quoted verbatim, except for the orthography of "l'Hôpital's Rule"): It is not much of an exaggeration to say that the version of Taylor's formula just discussed is the only application of l'Hôpital's Rule that is not silly. Anytime you find yourself using l'Hôpital's Rule to evaluate a concrete limit, you should stop and ask yourself whether there is any good reason for you to appeal to l'Hôpital's Rule.*

### 3.3.3 Taylor's Theorem With Remainder

Let  $I$  be an interval, let  $f : I \rightarrow \mathbb{R}$  be a smooth function, and let  $a$  be an interior point of  $I$ . The Local Taylor Theorem gives a sense in which the  $N$ th Taylor polynomial  $T_N(x)$  is a good approximation to  $f$  as  $x$  approaches  $a$ . But since this sense involves taking a limit as  $x$  approaches  $a$ , it does not in fact tell us anything about the difference between  $f(x)$  and  $T_N(x)$  for a fixed  $x \neq a$ . To analyze this, it is natural to analyze the *remainder function*

$$R_N(x) = f(x) - T_N(x).$$

And now a simple but crucial observation: for  $x \in I$ , we have  $\lim_{N \rightarrow \infty} R_N(x) = 0$  if and only if  $f(x) = \lim_{N \rightarrow \infty} T_N(x) = T(x)$ . Thus, in order to show that the Taylor series is both convergent at  $x$  and convergent to  $f(x)$ , it is necessary and sufficient to show that  $R_N(x)$  approaches 0 as  $N \rightarrow \infty$ .

Unfortunately even when  $T(x)$  converges for all  $x \in \mathbb{R}$ , it need not converge to  $f(x)$  for any  $x \neq a$ , as the following example shows.

**Example 3.3.11.** Consider the following function  $f : \mathbb{R} \rightarrow \mathbb{R}$ :

$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x^2}, & x > 0 \end{cases}.$$

Then one can show – with some work; this is Exercise 13 – that  $f$  is infinitely differentiable at 0 and for all  $n \geq 0$  we have  $f^{(n)}(0) = 0$ . Therefore the Taylor series expansion of  $f$  at 0 is

$$T(x) = 0 + 0x + \frac{0}{2!}x^2 + \dots + \frac{0}{n!}x^n + \dots = 0,$$

i.e., the zero function. Evidently then  $f(x) \neq T(x)$  for all  $x \neq 0$ .

So we cannot give a general theorem saying that  $f$  is equal to its Taylor series. Instead what we can do is to give *information about the remainder function* which can be used in favorable cases to show that  $R_N(x) \rightarrow 0$  for a certain range of  $x$ . The following result has a complicated statement, so let us lead with the basic idea: if one wants to analyze the difference between  $f$  and its  $N$ th order approximation at  $a$ , the key fact is how large the  $N+1$ st derivative is on the interval between  $a$  and  $x$ . Indeed, if the  $(N+1)$ st derivative were identically zero then  $f$  would be a polynomial of degree at most  $N$  and thus  $f = T_N$ . Certainly we can produce examples in which  $f$  differs by  $T_N$  by a lot: e.g. take  $f = T_N + a_{N+1}(x-a)^{N+1}$ : then if we fix  $x$  and  $a$  and make  $|a_{N+1}|$  very large, then  $f - T_N$  becomes very large at  $x$ .

One piece of notation: for real numbers  $a, b$ , we denote by  $[[a, b]]$  the closed, bounded interval with endpoints  $a$  and  $b$ . Specifically, if  $a \leq b$  then  $[[a, b]] = [a, b]$ , while if  $b \leq a$  then  $[[a, b]] = [b, a]$ .

**Theorem 3.3.12.** (*Taylor's Theorem With Remainder*) Let  $n \in \mathbb{N}$ , let  $I$  be an interval, and let  $f : I \rightarrow \mathbb{R}$  be defined and infinitely differentiable on  $I$ . Let  $a$  be an interior point of  $I$ , and let  $x \in I$ . Let

$$T_N(x) = \sum_{n=0}^N \frac{f^{(n)}(a)(x-a)^n}{n!}$$

be the degree  $N$  Taylor polynomial of  $f$  at  $c$ , and let

$$R_N(x) = f(x) - T_n(x)$$

be the remainder function. We give three expressions for  $R_N(x)$ .

a) (Cauchy Form of the Remainder) There is  $z \in ]a, x[$  such that

$$R_N(x) = \frac{f^{N+1}(z)}{(N+1)!} (x-z)^N (x-a).$$

b) (Lagrange Form of the Remainder) There is  $z \in ]a, x[$  such that

$$R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!} (x-a)^{N+1}.$$

c) (Integral Form of the Remainder) We have

$$R_N(x) = \int_a^x \frac{f^{(N+1)}(t)(x-t)^N dt}{N!}.$$

*Proof.* The proof of this theorem is somewhat technical, and parts of it use results from differential calculus that are not covered in this text. So we refer the interested reader to, e.g., [Cl-HC, Thm. 12.4].  $\square$

Theorem 3.3.12 gives several different descriptions of the remainder  $R_N(x)$ . Indeed it is the case that in certain circumstances one is preferable to the other, and in truth the study of  $R_N(x)$  for all but the simplest functions  $f$  becomes rather intricate. One example – already treated by Newton (who did many intricate, difficult things!) – is the Taylor series expansion of  $f(x) = (1+x)^\alpha$  at  $x = 0$ .<sup>6</sup> (The function  $f$  is a polynomial if and only if  $\alpha \in \mathbb{N}$ , so the interesting case is when  $\alpha \in \mathbb{R} \setminus \mathbb{N}$ .) Clearly  $f(0) = 1$ , and it is not difficult to compute that for all  $n \in \mathbb{Z}^+$  we have

$$\frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!}.$$

Because of this calculation, we put

$$\forall n \in \mathbb{Z}^+, \binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!}, \binom{\alpha}{0} := 1.$$

---

<sup>6</sup>As usual, the actual history is much more complicated than side remarks in contemporary textbooks lead you to believe: see e.g. [Co49] for a more extensive discussion of early work on the binomial theorem.

Thus the Taylor series of  $f(x) = (1+x)^\alpha$  is the **binomial series**

$$T(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

It is not hard to see that the radius of convergence of the binomial series is  $R = 1$ . (Recall that we have assumed  $\alpha \notin \mathbb{N}$ . If  $\alpha \in \mathbb{N}$  then  $f(x)$  is a polynomial and only finitely many terms of the binomial series are nonzero.) The convergence at the boundary points  $x = \pm 1$  is a bit delicate and depends on  $\alpha$ . One can use Theorem 3.3.12c) to show that

$$\forall x \in (-1, 1), (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

and you are asked to do so in Exercise 14...but the argument for this is really not a routine one. See e.g. [Cl-HC, §12.3] for complete details.

In the present text we will be less ambitious: most of all we want to show that  $e^x$ ,  $\cos x$  and  $\sin x$  are equal to their Taylor series. For this we can use a corollary of Theorem 3.3.12c): rather than giving an exact expression for  $R_N(x)$ , we will content ourselves with a relatively crude upper bound. This upper bound is much easier to prove and nevertheless carries the essential idea behind the theorem: the  $N$ th Taylor polynomial of  $f$  centered at  $a$  is a good approximation to the function  $f$  at a point  $x$  provided the  $(N+1)$ st derivative  $f^{(N+1)}$  does not get too large on the interval  $[[a, x]]$ .

Every version of Taylor's Theorem With Remainder makes use of some form of the Mean Value Theorem from differential calculus. Whereas the proof of Theorem 3.3.12 requires an extension of the Mean Value Theorem due to Cauchy, the less ambitious corollary requires only the following result.

**Proposition 3.3.13.** (*Racetrack Principle*) *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be differentiable functions. We suppose:*

- $g(a) \geq f(a)$  and
- for all  $x \in [a, b]$ , we have  $g'(x) \geq f'(x)$ .

*Then  $g(b) \geq f(b)$ .<sup>7</sup>*

---

<sup>7</sup>In other words, if at the start of a race you are not behind your competitor and at no point is your instantaneous velocity less than your competitor's, you cannot lose the race!

*Proof.* We use a standard trick: let  $h = g - f$ . Then it is enough to assume that  $h(a) \geq 0$  and  $h'(x) \geq 0$  for all  $x \in [a, b]$  and prove that  $h(b) \geq 0$ . Assume not: if  $h(b) < 0$ , then applying the Mean Value Theorem to  $h$  on  $[a, b]$  we get that there is  $c \in (a, b)$  such that  $h'(c) = \frac{h(b)-h(a)}{b-a} < 0$ , contradiction.  $\square$

And now for the result on the remainder that we will actually prove and use.

**Corollary 3.3.14.** (*Taylor's Corollary With Remainder*) Let  $a \in \mathbb{R}$ , and let  $R > 0$ . Let  $f : [a - R, a + R] \rightarrow \mathbb{R}$  be  $N + 1$  times differentiable. Let  $x \in [a - R, a + R]$ . Then for all  $N \in \mathbb{Z}^+$ , we have

$$|R_N(x)| \leq \frac{\|f^{(N+1)}\|}{(N+1)!} |x - a|^{N+1},$$

where

$$\|f^{(N+1)}\| = \sup_{y \in [a, x]} |f^{(N+1)}(y)|.$$

*Proof.* Our setup allows the possibility that  $f^{(N+1)}$  is unbounded on the interval between  $x$  and  $a$ . In this case (and only in this case) we have  $\|f^{(N+1)}\| = \infty$ , so the result asserts that  $|R_N(x)| \leq \infty$ : true but vacuous. Henceforth we assume that  $\|f^{(N+1)}\| < \infty$ .

We will give the proof under the assumption  $a < x$ . The other case is merely notationally different, and we leave it for the reader.

Let  $R_N = f - T_N$ . Then for all  $0 \leq n \leq N$ , we have

$$R_N^{(n)}(a) = f^{(n)}(a) - T_N^{(n)}(a) = 0. \quad (3.10)$$

We also have – by definition! – that

$$\forall y \in [a, x], |R_N^{(N+1)}(y)| \leq \|R_N^{(N+1)}\|. \quad (3.11)$$

Now we want to use the bounds on the derivatives of (3.11) and the initial conditions (3.10) to get an upper bound on  $|R_N(x)|$ . In fact this is quite straightforward, albeit somewhat notationally complicated. At the first stage, we compare the function  $R_N^{(N)}$  to the function

$$U_N : [a, x] \rightarrow \mathbb{R}, \quad U_N(y) = \|R^{(N+1)}(a)\|(y - a).$$

The point here is that we have

$$-U_N(a) = U_N(a) = 0 = R_N^{(N)}(a)$$



and for all  $y \in [a, x]$ ,

$$-U'_N(y) = -\|R_N^{(N+1)}\| \leq R_N^{(N+1)}(y) \leq \|R_N^{(N+1)}\| = U'_N(y),$$

so applying the Racetrack Principle (twice) we get

$$\forall y \in [a, x], \quad -U_N(y) \leq R_N^{(N)}(y) \leq U_N(y).$$

Observe that  $U_N$  is the function with derivative the constant function  $\|R_N^{(N+1)}\|$  and having value 0 at  $x = a$ : in other words,

$$U_N = \int_a^y \|R_N^{(N+1)}\| dt.$$

We continue in the same manner: for  $0 \leq n \leq N-1$ , having defined  $U_{n+1}$ , let  $U_n : [a, x] \rightarrow \mathbb{R}$  be the function with derivative  $U_{n+1}$  and satisfying  $U_n(a) = 0$ :

$$U_n = \int_a^y U_{n+1}(t) dt.$$

Repeated application of the Racetrack Principle gives: for all  $0 \leq n \leq N$ ,

$$\forall y \in [a, x], \quad |R_N^{(n)}(y)| \leq U_n(y). \quad (3.12)$$

We find that

$$U_{N-1}(y) = \|R_N^{(N+1)}\| \frac{(y-a)^2}{2},$$

$$U_{N-2}(y) = \|R_N^{(N+1)}\| \frac{(y-a)^3}{3!},$$

and so forth: finally we get

$$U_0(y) = \|R_N^{(N+1)}\| \frac{(y-a)^{(N+1)}}{(N+1)!}. \quad (3.13)$$

Taking  $n = 0$  and combining (3.12) and (3.13) we get

$$\forall y \in [a, x], \quad |R_N(y)| = |R_N^{(0)}(y)| \leq U_0(y) = \|R_N^{(N+1)}\| \frac{(y-a)^{N+1}}{(N+1)!}.$$

Finally, take  $y = x$  to get

$$|R_N(x)| \leq \|R_N^{(N+1)}\| \frac{(x-a)^{(N+1)}}{(N+1)!}. \quad \square$$

**Remark 3.3.15.** *The (more difficult and more important) implication (i)  $\implies$  (ii) in the Local Taylor Theorem (Theorem 3.3.9) is almost a consequence of Taylor's Corollary With Remainder (Corollary 3.3.14). Namely, if we assume that  $\|f^{(N+1)}\| < \infty$ , then the fact that*

$$|R_N(x)| \leq \frac{\|f^{(N+1)}\|}{(N+1)!} |x-a|^{N+1},$$

*immediately implies that  $f$  agrees with  $T_N(x)$  to order  $N$  at  $a$ : indeed,*

$$\frac{|f(x) - T_N(x)|}{|x-a|^N} = \frac{|R_N(x)|}{|x-a|^N} \leq \frac{\|f^{(N+1)}\|}{(N+1)!} |x-a|.$$

*Since the right hand side approaches 0 as  $x \rightarrow a$ , so does the left hand side.*

*The only reason that Taylor's Corollary With Remainder does not render the Local Taylor Theorem obsolete is that the latter result holds without the assumption of boundedness of the  $(N+1)$ st derivative. We emphasize that if we are interested in Taylor series then we need our functions to be smooth, i.e., to have infinitely many derivatives. So for a smooth function all derivatives are themselves differentiable, hence continuous, hence bounded.*

*It is however in the nature of mathematics that truly basic and important results (and theorems on Taylor polynomials are about as basic and important as it gets) continually get revisited, refined and improved by various mathematicians over the years. A mathematician is happiest with her theorem when every time a hypothesis is removed or even weakened she knows a counterexample. A mathematician is generally unhappy with a theorem that contains hypotheses that are not really used in the proof. We admit however that sometimes including "extraneous hypotheses" makes for theorems that are easier for a beginning student to understand and appreciate.*

**Example 3.3.16.** *We claim that for all  $x \in \mathbb{R}$ , the function  $f(x) = e^x$  is equal to its Taylor series expansion at  $x = 0$ :*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

*First we compute the Taylor series expansion:  $f^{(0)}(0) = f(0) = e^0 = 1$ , and  $f'(x) = e^x$ , hence every derivative of  $e^x$  is just  $e^x$  again. We conclude that  $f^{(n)}(0) = 1$  for all  $n$  and thus the Taylor series is  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ , as claimed. Next*

note that this power series converges for all real  $x$ , as we have already seen: just apply the Ratio Test. Finally, we apply Corollary 3.3.14: note that no matter what  $N$  is,  $f^{(N+1)}(x) = e^x$ . If we fix  $A > 0$ , then since  $e^x$  is positive and increasing, the maximum value of  $|e^x|$  on  $[-A, A]$  occurs at the right endpoint  $x = A$ , so  $\|f^{N+1}\| = e^A$ . Thus

$$|R_N(x)| \leq e^A \frac{|x|^{N+1}}{(N+1)!}.$$

By the  $N$ th Term Test, we know that

$$\lim_{N \rightarrow \infty} e^A \frac{|x|^{N+1}}{(N+1)!} = e^A \lim_{N \rightarrow \infty} \frac{|x|^{N+1}}{(N+1)!} = e^A \cdot 0 = 0.$$

Therefore  $R_N(x) \rightarrow 0$  for all  $x \in [-A, A]$ . Since  $A$  was arbitrary,  $R_N(x) \rightarrow 0$  for all  $x \in \mathbb{R}$ , i.e.,

$$e^x = T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

In fact, since

$$|R_N(x)| \leq e^A \frac{A^{N+1}}{(N+1)!} \rightarrow 0,$$

this argument shows that  $R_N(x)$  converges uniformly to 0 on  $[-A, A]$ : this is something that we proved earlier for power series with infinite radius of convergence using the Weierstrass  $M$ -Test, but in this case it follows directly from Taylor's Theorem.

The above argument can be generalized, as follows.

**Theorem 3.3.17.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be smooth, and let  $a \in \mathbb{R}$ . We suppose:

- The Taylor series  $T(x)$  of  $x$  centered at  $a$  converges for all  $x \in \mathbb{R}$ .
  - For each  $A > 0$ , the sequence  $\|f^{(N)}\| = \max_{x \in [a-A, a+A]} |f^{(N)}(x)|$  is bounded.
- Then  $R_N(x) \xrightarrow{u} 0$  on  $[a-A, a+A]$ , and thus for all  $x \in \mathbb{R}$  we have

$$f(x) = T(x).$$

*Proof.* This is almost the same as above: since  $\frac{A^{N+1}}{(N+1)!} \rightarrow 0$  and  $\|f^{(N)}\|$  is bounded, we have for all  $x \in [a-A, a+A]$  that

$$|R_N(x)| \leq \|f^{(N)}\| \frac{A^{N+1}}{(N+1)!} \rightarrow 0. \quad \square$$

The sequence  $\|f^{(N)}\|$  being bounded on  $[-A, A]$  for all  $A > 0$  is quite a strong one. We showed that it holds for  $f(x) = e^x$  because  $f = f^{(N)}$  for all  $N$ . Similarly, it holds if as we range over all  $N \in \mathbb{N}$ , we get only finitely many different functions as  $f^{(N)}$ . This is more specialized still, but it applies to two more all-important functions: namely, for  $f(x) = \sin x$  and  $f(x) = \cos x$  – in these cases we have  $f^{(4)} = f$  so the sequence of derivatives is periodic. Let us record the result.

**Theorem 3.3.18.** *For all  $x \in \mathbb{R}$  we have*

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Taylor's Theorem With Remainder can also be applied to give estimates on how well a Taylor polynomial approximates the original function.

**Example 3.3.19.** *As above, the  $n^{\text{th}}$  Taylor polynomial for  $f(x) = e^x$  is given by  $P_n^f(x) = \sum_{j=0}^n x^j/j!$  and Taylor's Theorem gives us an estimate for the error term  $|f(x) - P_n^f(x)|$ . Let's use this to find a decimal approximation for  $f(1) = e$  which is correct to 4 decimal places. To do this, we must figure out which  $n$  will give us a small enough error. By Taylor's Theorem, we have*

$$|e - P_n^f(1)| \leq \left| \frac{f^{(n+1)}(t)}{(n+1)!} (1)^n \right| = \left| \frac{f^{(n+1)}(t)}{(n+1)!} \right|$$

for some  $t$  with  $0 < t < 1$ . Now, as above,  $f^{(n)}(x) = e^x$  so, if  $0 < t < 1$ , then  $f^{(n+1)}(t) < e$ . Thus we can estimate that

$$\left| \frac{f^{(n+1)}(t)}{(n+1)!} \right| \leq \frac{e}{(n+1)!}.$$

We find ourselves in an apparently circular situation: the error in our approximation for  $e$  includes  $e$  itself! But we can resolve this: a very crude upper bound on  $e$  obtained by other means will convert the above into a procedure for giving arbitrarily good approximations to  $e$ . So:

$$e = e^1 = 1 + 1 + \sum_{n=2}^{\infty} \frac{1}{n!} < 1 + 1 + \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} = 2 + \sum_{n=1}^{\infty} \frac{1}{2^n} = 2 + 1 = 3.$$

Using this fact, we see that we can make the error estimate less than  $10^{-5}$  as long as  $3/(n+1)! < 10^{-5}$ , i.e., we need  $(n+1)! > 3 \times 10^5$ . Making a list of values of  $n!$  we have

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \\ 3! &= 6 \\ 4! &= 24 \\ 5! &= 120 \\ 6! &= 720 \\ 7! &= 5,040 \\ 8! &= 40,320 \\ 9! &= 362,880 \\ 10! &= 3,628,800. \end{aligned}$$

Since  $9! > 3 \times 10^5$  we can take  $P_8^f(1)$  to approximate  $e = f(1)$  and be sure that it is correct to four decimal places. i.e.,

$$e - \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320}\right) < 10^{-5}.$$

Evaluating this sum on a calculator yields  $e \approx 2.71827$ .

### EXERCISES 3.3

1. Find the Taylor series at 0 for the following functions.

$$\begin{array}{ll} \text{(a) } f(x) = \frac{1}{1-x} & \text{(b) } g(x) = \frac{1}{1+x^2} \\ \text{(c) } F(x) = \sin x & \text{(d) } G(x) = e^{x^2} \end{array}$$

2. Find the Taylor series centered at  $a$  for  $f(x) = 1 + x + x^2 + x^3$  when

$$\text{(a) } a = 0 \qquad \text{(b) } a = 1 \qquad \text{(c) } a = 2$$

3. Find the sum of these series

$$\text{(a) } \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} \qquad \text{(b) } \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{6^{2n}(2n)!} \qquad \text{(c) } \sum_{n=0}^{\infty} \frac{x^n}{2^n(n+1)!}$$

4. Use the 5th degree Taylor polynomial for  $f(x) = e^x$  and Taylor's theorem to obtain the estimate

$$\frac{1957}{720} \leq e \leq \frac{1956}{719}$$

5. For what values of  $x$  do the following polynomials approximate  $\sin x$  to within 0.01

$$(a) P_1(x) = x \quad (b) P_3(x) = x - x^3/6 \quad (c) P_5(x) = x - x^3/6 + x^5/120$$

6. How accurately does  $1+x+x^2/2$  approximate  $e^x$  for  $x \in [-1, 1]$ ? Can you find a polynomial that approximates  $e^x$  to within 0.001 on this interval?

7. Use Taylor's Theorem to approximate  $\cos(\pi/4)$  to 5 decimal places.

8. Find the Taylor series for  $x^2 \sin(x^2)$ .

9. Find the  $10^{th}$  degree Taylor polynomial for the following functions:

a)  $\cos(x^2)$

b)  $\sin(2x)$

c)  $e^{x+1}$

d)  $e^{x^2} \cos(x^3)$

e.)  $\frac{\sin(x^2)}{1+x^2}$

10. a) Find the Taylor series for  $f(x) = \frac{1-\cos(x^6)}{x^{12}}$ .  
 b) Use part a) to determine  $f^{(n)}(0)$  for  $n = 0, 1, 2, 3, \dots, 12$ .

11. Evaluate

$$\int_0^1 e^{-x^2} dx$$

to within 0.001.

12. Approximate  $\int_0^1 \frac{\sin(x)}{x} dx$  to 3 decimal places.

13. Consider the function

$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x^2}, & x > 0 \end{cases}.$$

- a) Show that  $f$  is continuous at 0.
- b) Show that  $f$  is differentiable at 0 and that  $f'(0) = 0$ . (Hint: you must use the definition of the derivative to prove this.)
- c) Show that  $f'(x)$  is differentiable at 0 and that  $f''(0) = 0$ .
- d) Show by induction that  $f^{(n)}(x)$  is differentiable at 0 and that  $f^{(n+1)}(0) = 0$ , for all  $n \in \mathbb{Z}^+$ .
- e.) To what function does the Taylor series for  $f$  converge?
14. Let  $r$  be a nonzero real number and define the generalized binomial coefficient  $\binom{r}{k}$  by  $\binom{r}{0} = 1$ , and

$$\binom{r}{k} = \frac{r \cdot (r-1) \cdot (r-2) \cdots (r-k+1)}{k!}, \text{ for } k \geq 1.$$

- a) Prove the formula

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$$

for  $-1 < x < 1$ .

- b) Write out the first 5 terms in this series for  $r = -1$ ,  $r = 1/2$ , and  $r = 3$ .
15. Prove Lemma 3.3.6.

## 3.4 Complex Numbers

To introduce *complex numbers* we define a new “number”, usually denoted by  $i$  which satisfies  $i^2 = -1$ . That is,  $i$  is a “square root of  $-1$ ”. Notice that  $-i$  is also a square root of  $-1$ , so be careful not to mistake  $i$  for *the* square root of  $-1$ .

Once we accept the use of this new number, we can define the set of *complex numbers* as

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}.$$

If  $z = a + ib$  is a complex number, we call  $a$  the *real part* of  $z$  and  $b$  the *imaginary part* of  $z$ . These are denoted respectively by  $\operatorname{Re}(z)$  and  $\operatorname{Im}(z)$ . Two complex numbers  $z_1$  and  $z_2$  are equal when their real and imaginary parts are equal, i.e.,  $z_1 = z_2$  if and only if  $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$  and  $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$ .

Complex numbers can be added and multiplied using the following definition.

**Definition 3.4.1.** *Let  $z, w \in \mathbb{C}$  with  $z = a + ib$ , and  $w = c + id$ ,  $a, b, c, d \in \mathbb{R}$ . Then*

- a)  $z + w = (a + c) + i(b + d)$
- b)  $zw = (ac - bd) + i(ad + bc)$ .

One thing to be careful about is that imaginary numbers can't be made into an "ordered field" (see Appendix A), so it makes no sense to use inequalities with complex numbers. On the other hand, there is a real quantity attached to a complex number called the modulus that behaves like the absolute value for real numbers, in particular the triangle inequality holds.

**Definition 3.4.2.** *Let  $z = a + ib$  with  $a, b \in \mathbb{R}$ . The modulus of  $z$  is given by  $\sqrt{a^2 + b^2}$ . The modulus of  $z$  is denoted by  $|z|$ .*

**Proposition 3.4.3.** *Let  $z, w \in \mathbb{C}$ . Then*

- a)  $|\operatorname{Re}(z)| \leq |z|$
- b)  $|\operatorname{Im}(z)| \leq |z|$
- c)  $|z + w| \leq |z| + |w|$ .

It is useful to think of the complex numbers geometrically in terms of the *complex plane*. Any complex number  $z = a + ib$  can be identified to the ordered pair of real numbers  $(a, b)$ , which is understood geometrically as a point in the Cartesian plane. The modulus of  $z$  then has the geometric interpretation as the distance between the point  $(a, b)$  and the origin  $(0, 0)$ . A little thought reveals then that if  $z$  and  $w$  are complex numbers then  $|z - w|$  is the distance between the points in the plane related to  $z$  and  $w$ . In particular, if  $\epsilon > 0$  and  $c \in \mathbb{C}$  the set  $\{z \in \mathbb{C} \mid |z - c| < \epsilon\}$  is the open disc of points centered at  $c$  with radius  $\epsilon$ .

Much of the work on sequences and series of real numbers carries over directly to sequences and series of complex numbers if we just use the complex modulus in place of the absolute values. In particular if we let  $\{c_n\}$  be a sequence of complex numbers, we have the following



**Definition 3.4.4.** The sequence  $\{c_n\}$  of complex numbers converges to the complex number  $c$  if for every  $\epsilon > 0$  there is an  $N \in \mathbb{Z}^+$  such that

$$|c_n - c| < \epsilon$$

for every  $n > N$ . As usual, this is written as  $\lim_{n \rightarrow \infty} c_n = c$ .

In general terms, we have  $\lim_{n \rightarrow \infty} c_n = c$  if “eventually” all of the terms of the sequence are situated in the open disc of radius  $\epsilon$  centered at  $c$ .

**Proposition 3.4.5.** Let  $\{c_n\}$  be a sequence of complex numbers with  $c_n = a_n + ib_n$ , ( $a_n, b_n \in \mathbb{R}$ ). Let  $c = a + ib$  be a complex number. Then we have  $\lim_{n \rightarrow \infty} c_n = c$  if and only if  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ .

**Proof:** This follows from the inequalities

$$|a_n - a| \leq |c_n - c|, \quad |b_n - b| \leq |c_n - c|, \quad \text{and} \quad |c_n - c| \leq |a_n - a| + |b_n - b|,$$

which all follow from Proposition 3.3.3. □

Most of the results of chapter 2 (especially those in section 2.3) can be generalized to complex sequences. However, at the moment we are more interested in complex series. Let  $\{c_n\}$  be a sequence of complex numbers. We can form a *sequence of partial sums*

$$s_n = \sum_{j=0}^n c_j.$$

We say that the series  $\sum c_n$  *converges* if the sequence  $\{s_n\}$  converges. Otherwise, it *diverges*. A direct consequence of Proposition 3.3.5 is the following:

**Proposition 3.4.6.** Let  $\{c_n\}$  be a sequence of complex numbers with  $c_n = a_n + ib_n$  ( $a_n, b_n \in \mathbb{R}$ ). Then  $\sum c_n$  converges if and only if both  $\sum a_n$  and  $\sum b_n$  converge. Furthermore, if  $\sum_{n=0}^{\infty} a_n = a$  and  $\sum_{n=0}^{\infty} b_n = b$  then  $\sum_{n=0}^{\infty} c_n = a + ib$ .

**Proof:** This follows from applying Proposition 3.3.5 to the partial sums of the series. □

The above Proposition is not much help in determining the convergence of geometric series like

$$\sum_{n=0}^{\infty} \left( \frac{1+i}{2} \right)^n$$

since it is not that easy to find the real and imaginary parts of  $\left(\frac{1+i}{2}\right)^n$ . On the other hand, this Proposition does help to prove for complex series many of the results we know about real series, for example, we have

**Proposition 3.4.7.** *Suppose that  $\{c_n\}$  is a sequence of complex numbers and that  $\sum c_n$  converges. Then  $\lim_{n \rightarrow \infty} c_n = 0$ .*

**Proof:** Let  $c_n = a_n + ib_n$  where  $a_n$  and  $b_n$  are real. Then since  $\sum c_n$  converges, we know that both  $\sum a_n$  and  $\sum b_n$  converge. These are real series, so we know from Proposition 2.1.7 that  $\lim a_n = 0$  and  $\lim b_n = 0$ . Finally, from Proposition 3.3.5 we conclude that  $\lim c_n = 0$ . □

To help address the question of convergence of the above series we introduce

**Definition 3.4.8.** *We say that the series  $\sum c_n$  converges absolutely if the series  $\sum |c_n|$  converges.*

Notice that the series  $\sum |c_n|$  has real, nonnegative terms so we can check its convergence using the tests from section 2.2.

**Proposition 3.4.9.** *If a series converges absolutely, then it converges.*

**Proof:** Let  $c_n = a_n + ib_n$  where  $a_n$  and  $b_n$  are real. Then since  $|a_n| \leq |c_n|$ , the convergence of  $\sum |c_n|$  implies the convergence of  $\sum |a_n|$  (by comparison). Likewise, since  $|b_n| \leq |c_n|$ , the convergence of  $\sum |c_n|$  also implies the convergence of  $\sum |b_n|$ . Now, since  $\sum a_n$  and  $\sum b_n$  are real series, we can apply proposition 2.3.1 to conclude that they both converge. Then, by Proposition 3.3.6, we see that  $\sum c_n$  converges. □

**Example 3.4.10.** *The terms of the series  $\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n$  have modulus*

$$\left| \left( \frac{1+i}{2} \right)^n \right| = \left( \frac{\sqrt{2}}{2} \right)^n.$$

*Since that sequence is summable, the series is absolutely convergent.*

Finally, for power series, we have the following:

**Proposition 3.4.11.** *Let  $\sum a_n z^n$  be a complex power series and assume that for some fixed  $c \in \mathbb{C}$  we know that  $\sum a_n c^n$  converges. Then  $\sum a_n z^n$  converges for any  $z$  with  $|z| < |c|$ .*

**Proof:** The proof is similar to that of Proposition 2.4.3. We begin by noting that since  $\sum a_n c^n$  converges, we know that  $\lim a_n c^n = 0$  and thus we can conclude that the sequence  $\{a_n c^n\}$  is bounded in the sense that there is some real number  $M > 0$  so that  $|a_n c^n| < M$  for all  $n \in \mathbb{Z}^+$ . (This follows from 3.3.5 and the analogous fact for real sequences.) Now choose  $z$  with  $|z| < |c|$  and let  $d = |z|/|c|$ . Then

$$\begin{aligned} \sum |a_n z^n| &= \sum |a_n| |z|^n \\ &= \sum |a_n| |c|^n d^n \\ &= \sum |a_n c^n| d^n \\ &\leq \sum M d^n. \end{aligned} \tag{3.14}$$

But the last of these sums is finite since  $d < 1$  so we conclude by the comparison test that the first sum is also finite, i.e.,  $\sum a_n z^n$  converges absolutely.  $\square$

**Corollary 3.4.12.** *Let  $\sum a_n z^n$  be a complex power series. Then either*  
 1.)  $\sum a_n z^n$  converges for all  $z \in \mathbb{C}$ , or  
 2.) there is a nonnegative real number  $R$  so that  $\sum a_n z^n$  converges if  $|z| < R$  and diverges if  $|z| > R$ .

In case 2.),  $R$  is called the *radius of convergence*, in case 1.) the radius of convergence is said to be infinite. In the second case, there is no information about convergence on the circle  $|z| = R$ . Convergence at such points must be dealt with in more detail in order to describe the full domain of convergence.

**Example 3.4.13.** *Thinking of power series in the complex domain gives some intuition about “why” the Taylor series,  $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ , for  $f(x) = 1/(1+x^2)$ , converges only for  $|x| < 1$  even though  $f$  makes perfectly good sense for all real values of  $x$ . The point is that if we think of  $x$  as a complex variable, then  $f$  has a “singularity” at  $x = i$  and  $x = -i$ , so the radius of convergence for  $\sum_{n=0}^{\infty} (-1)^n x^{2n}$  cannot possibly be bigger than 1.*

**Example 3.4.14.** The power series  $\sum_{n=0}^{\infty} x^n/n!$  converges to  $f(x) = e^x$  for all real  $x$ . Hence its radius of convergence is infinite. Therefore the complex power series  $\sum_{n=0}^{\infty} z^n/n!$  converges for all complex  $z$ . It makes sense to think of the function defined by this complex power series to be an extension of  $f(x) = e^x$ . That is, we use the power series  $\sum_{n=0}^{\infty} z^n/n!$  to define  $e^z$  for complex numbers  $z$ . Of course, just naming this  $e^z$  does not mean that it necessarily behaves like an exponential. For instance, it is not at all clear from this definition that the usual exponent rule,  $e^{z+w} = e^z e^w$  is still valid. It turns out that this rule does remain valid for complex exponents, but we won't go into the general proof of that now. Instead, let's just look at the special case where  $z$  and  $w$  are purely imaginary, i.e.,  $z = ix$  and  $w = iy$ ,  $x$  and  $y$  real. Before looking at the rules of exponents, we need to look a little more closely at the power series for  $e^z$  when  $z = ix$ . Noticing that  $z^2 = -x^2$ ,  $z^3 = -ix^3$ ,  $z^4 = x^4$  and this pattern repeats with period four, we get that

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \dots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + \dots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right). \end{aligned}$$

Recalling, however, that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

and that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

we see that

$$e^{ix} = \cos(x) + i \sin(x).$$

Now it is just a matter of using some addition of angle formulas to see that

$e^{ix+iy} = e^{ix}e^{iy}$ . *Indeed*

$$\begin{aligned}e^{ix+iy} &= \cos(x+y) + i \sin(x+y) \\ &= (\cos(x)\cos(y) - \sin(x)\sin(y)) + i(\cos(x)\sin(y) + \cos(y)\sin(x)) \\ &= (\cos(x) + i \sin(x))(\cos(y) + i \sin(y)) \\ &= e^{ix}e^{iy}.\end{aligned}$$

*(Actually, this author admits to having a hard time remembering the addition of angle formulas, but since he knows that  $e^{ix} = \cos(x) + i \sin(x)$ , and the exponent laws, he can always “rederive” the angle addition formulas.) By the way, the formula  $e^{ix} = \cos(x) + i \sin(x)$  also gives some interesting identities by plugging in particular values of  $x$ , for example*

$$e^{i\pi} = -1.$$

## EXERCISES 3.4

1. Find the domains of convergence of the following complex power series
  - a)  $\sum \frac{z^n}{n^2}$
  - b)  $\sum \frac{z^n}{2^n n^2}$
  - c)  $\sum \frac{z^n}{n!}$
  - d)  $\sum (-1)^n \frac{z^n}{n! 2^n}$
  - e.)  $\sum z^n$ .
2. Show that if  $|z| < 1$  then  $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ .
3. Using exercise 2, evaluate  $\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n$ .
4. Evaluate  $\sum_{n=0}^{\infty} n \left(\frac{1+i}{2}\right)^n$ .
5.
  - a) Show that any complex number  $z = a + ib$  can be written in the *polar form*,  $z = r(\cos(\theta) + i \sin(\theta))$ , where  $r = |z|$  and  $\theta$  is the angle between the real axis and the line segment passing from the origin to the point  $(a, b)$ .
  - b) Notice that the polar form of  $z$  can be rewritten using the complex exponential,  $z = r(\cos(\theta) + i \sin(\theta)) = re^{i\theta}$ . Use this to prove *de Moivre's theorem*:
$$z^n = r^n(\cos(n\theta) + i \sin(n\theta)).$$
  - c) Interpret de Moivre's theorem geometrically in the complex plane.

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