

References.

1. L. Henkin, "A problem on inverse mapping systems", *Proc. Amer. Math. Soc.*, 1 (1950), 224-225.
2. D. Zelinsky, "Rings with ideal nuclei", *Duke Math. J.*, 18 (1951), 431-442.

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GROUPS COVERED BY PERMUTABLE SUBSETS

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§1. *Introduction.*

In this paper we shall be mainly concerned with groups which can be covered by (in other words, are unions of) permutable boundedly finite subsets. Obvious examples of such groups are the finite groups, where a single finite set suffices, namely the group itself; and abelian groups, where all one-element subsets will serve. The main result (Theorem 7.1) characterizes these groups completely as those groups which possess a subgroup of finite index with finite derived group.

Another, closely related, class of groups, which also includes all finite groups and all abelian groups, is that of the groups with only finite classes of conjugate elements. Such groups are called *FC*-groups†; they have been studied in an earlier paper‡, and their investigation is carried a little further in the present paper. It will be shown (Theorem 3.1) that if the classes of conjugate elements in a group H are *boundedly* finite, then the derived group of H is finite. The converse is also true, and nearly trivial.

The present investigation arose from the question whether a simple and direct proof of the following theorem of Mautner§ could be devised:

Let the group G possess a finite subgroup K whose double cosets in G are permutable. If H denotes the union of all finite classes of conjugate elements of G , then H (easily seen to be a subgroup of G) has finite index in G .

This is in fact a corollary of our more general results; we give, however, an independent proof in a sharpened form, namely (Theorem 5.1)

$$G = KH.$$

* Received 11 November, 1953; read 19 November, 1953.

† Following Baer (2).

‡ Neumann (5).

§ Unpublished; it is a corollary of deeper results on unitary representations in (4).

I am indebted to F. I. Mautner for a derivation of this theorem from his, *op. cit.*, p. 438, and also to Kurt A. Hirsch for having drawn my attention to the above question.

The proof is completely elementary in the sense that it uses only fundamental notions and facts of group theory.

We also investigate (in §8) the problem how far the boundedness or even the finiteness of the permutable subsets covering the group can be relaxed. We show, for instance, that every finitely generated free group but no non-denumerable free group can be covered by permutable finite subsets; but our results are very far from a complete solution.

§2. Notation and preliminaries.

We use the following notation. Groups are written multiplicatively, the unit element is 1, and the trivial subgroup is $E = \{1\}$. If g is an element of a group G , then g^G is the class of its conjugates, $C(g)$ its centralizer. If S is a set, $|S|$ denotes its cardinal. If S is a subgroup of G then $|G:S|$ is its index; thus in particular if S is a normal subgroup of G , then

$$|G:S| = |G/S|.$$

The centralizer of a set S of elements of G is

$$C(S) = \bigcap_{s \in S} C(s).$$

We note the (well-known) relation

$$|g^G| = |G:C(g)|.$$

If we denote by H the union of all finite classes of conjugates in G , then an element $g \in G$ belongs to H if, and only if, $|g^G|$ is finite. It is not difficult to see that H is a subgroup of G , and in fact a characteristic subgroup. Every element of H has only a finite number of conjugates in H , for g^H is a subset of g^G ; a group with this property is called an *FC*-group.

The following facts* about *FC*-groups will be used.

(2.1) **THEOREM.** *If H is an FC-group, then the periodic elements of H form a (periodic) subgroup P of H ; this contains the derived group H' of H . If H is finitely generated, then P is finite.*

I am indebted to Philip Hall for the following argument which is much shorter than my proof *op. cit.* Let a, b be two elements of the *FC*-group H ; denote by A the group they generate. Then the centre of A is the intersection of the centralizers of a and b in A , and hence of finite index in A . It follows from a result implicit in Schur (6)† that the derived group A' of A is finite. If a, b are periodic, then A/A' is also finite, thus A itself is then

* Neumann (5, Theorem 5.1).

† This is (5, Theorem 5.3); my proof used Schur's basic idea. Cf. also Baer (1, §6, Theorem 4).

finite. It follows at once that the periodic elements of H form a subgroup, and the rest of Theorem 2.1 is an easy consequence*.

§3. *Groups with boundedly finite classes of conjugate elements.*

We begin by considering FC -groups in which the classes of conjugate elements are not only finite, but boundedly finite. These include all groups whose centre has finite index; and thus in particular they include all finitely generated FC -groups.

The example of the (restricted) direct product of infinitely many quaternion groups with amalgamated centre shows, however, that in such a group the centre can also have infinite index, and can indeed be finite though the group be infinite. In this example the centre and the derived group coincide with the only minimal normal subgroup, of order 2. Every element outside this subgroup has precisely two conjugates, including itself†.

These groups are capable of a very simple characterization:

(3.1) THEOREM. *The classes of conjugate elements of a group H are boundedly finite if, and only if, the derived group H' of H is finite.*

Proof. If H' is finite then $|H'|$ is a finite bound for the cardinal of every class of conjugate elements; for if h_1, h_2, \dots, h_n are different conjugates, then $1, h_1^{-1}h_2, h_1^{-1}h_3, \dots, h_1^{-1}h_n$ are different commutators‡.

Assume now conversely that the classes of conjugates in H are boundedly finite, and let n be the least upper bound of their cardinals. Let a be an element of H with exactly n conjugates, and let $1 = b_1, b_2, \dots, b_n$ form a set of right coset representatives of H modulo $C(a)$. Thus

$$a_1 = a, \quad a_2 = b_2^{-1}ab_2, \quad \dots, \quad a_n = b_n^{-1}ab_n$$

are the n distinct conjugates of a . Next let

$$U = C(\{b_1, b_2, \dots, b_n\})$$

be the centralizer of b_1, b_2, \dots, b_n in H . This is the intersection of a finite number of groups each of finite index in H :

$$U = C(b_1) \cap C(b_2) \cap \dots \cap C(b_n).$$

* Another elegant simplification of my proof has been found by J. Erdős (3). I owe this information to Tibor Szele.

† Other such examples, made independently by J. Erdős (3), have been kindly communicated to me by Tibor Szele.

‡ This reasoning has already been applied to prove (5, Theorem 5.4).

Hence its index $|H:U| = m$, say, is again finite. Let c_1, c_2, \dots, c_m be a set of right coset representatives of H modulo U , and let V be the least normal subgroup of H containing a, c_1, c_2, \dots, c_m . Then V is finitely generated, namely by the finitely many conjugates of a, c_1, c_2, \dots, c_m . Also

$$H = UV.$$

Let $h = uv$ and $h' = u'v'$ be two arbitrary elements of H . We form their commutator

$$[h, h'] = [uv, u'v'] \equiv [u, u'] \pmod{V},$$

and show that this lies in V . In fact we show that $[u, u']$ lies in $\{a\}^H$, the normal closure of a in H . Consider the element $w = ua$. As u permutes with b_1, b_2, \dots, b_n , the conjugates

$$w \doteq ua, \quad b_2^{-1}wb_2 = ua_2, \quad \dots, \quad b_n^{-1}wb_n = ua_n$$

of w are all different. They must be all the conjugates of w , as n was the greatest number of conjugates any element of H could possess. Thus

$$u'^{-1}wu' = ua_i$$

for some i ; and as also

$$u'^{-1}au' = a,$$

for some j , then

$$[u, u'] = a_i a_j^{-1} \in \{a\}^H \subseteq V.$$

Thus H/V is abelian, and $H' \subseteq V$. But H' is a periodic group by Theorem 2.1, and the subgroup of periodic elements of V is finite, also by Theorem 2.1, because V is finitely generated. Thus H' is finite, and the theorem follows.

I do not know whether one can refine this argument to give a bound for the order $|H'|$ of the derived group in terms of the bound n for the cardinals of classes of conjugate elements.

§4. *Groups covered by finitely many cosets.*

In this section we derive a lemma which will be required later. It is quite possibly known, but I know of no reference in the literature.

(4.1) LEMMA. *Let the group G be the union of finitely many, let us say n , cosets of subgroups C_1, C_2, \dots, C_n :*

$$G = \bigcup_{i=1}^n C_i g_i. \tag{4.11}$$

Then the index of (at least) one of these subgroups in G does not exceed n .

It should be noted that we have sacrificed no generality in writing the n cosets as right cosets; for a left coset of a subgroup C is also a right coset

of a conjugate of C :

$$gC = gCg^{-1} \cdot g.$$

It may also be remarked that the lemma is obvious for finite groups G , and that it becomes false if an infinite cardinal is substituted for n . Its proof for arbitrary G is carried out in several steps.

(4.2) *Under the assumptions of the lemma at least one subgroup C_i has finite index in G .*

Proof. We proceed by induction over the number of distinct groups among C_1, C_2, \dots, C_n . If all the groups C_i coincide, that is if G is the union of n right cosets of a single group, then this clearly has finite index. Assume now that the proposition is true when there are $r-1$ or fewer distinct groups C_i ; and let there be $r > 1$ distinct groups among C_1, C_2, \dots, C_n . Consider one of the groups, C_n , say; and assume the groups in (4.11) so arranged that C_1, \dots, C_m are different from C_n , and $C_{m+1} = C_{m+2} = \dots = C_n$. Now either

$$G = \bigcup_{i=m+1}^n C_n g_i, \quad (4.21)$$

in which case C_n clearly has finite index in G , or else there is an element

$$h \notin \bigcup_{i=m+1}^n C_n g_i. \quad (4.22)$$

In this case then

$$C_n h \cap \bigcup_{i=m+1}^n C_n g_i = \phi,$$

and therefore

$$C_n h \subseteq \bigcup_{i=1}^m C_i g_i.$$

Thus

$$C_n g \subseteq \bigcup_{i=1}^m C_i g_i h^{-1} g,$$

that is to say, every right coset of C_n is contained in a finite union of right cosets of the other $r-1$ groups C_i . But then G can also be covered by a union of finitely many right cosets of these $r-1$ groups, and by the induction hypothesis one of them has finite index in G . Thus (4.2) follows.

(4.3) *Let C_1, \dots, C_m have infinite index, and let $C_{m+1} = C_{m+2} = \dots = C_n$. Then*

$$G = \bigcup_{i=1}^n C_i g_i = \bigcup_{i=m+1}^n C_n g_i;$$

that is to say, if only one of the groups has finite index in G , then the cosets of groups of infinite index can be omitted from the covering of G .

This an immediate corollary of the proof of (4.2); for the alternative (4.22) leads to one of C_1, \dots, C_m having finite index, and hence it cannot arise here. We now drop the assumption that there is only one group (possibly repeated) of finite index.

(4.4) Let C_1, \dots, C_m have infinite index. Then

$$G = \bigcup_{i=m+1}^n C_i g_i;$$

that is to say, the cosets of groups of infinite index can in any case be omitted from the covering of G .

Proof. We know from (4.2) that some of C_{m+1}, \dots, C_n have finite index, and we lose no generality if we assume that they all have finite index; for any groups of infinite index can be lumped together with those we want to show superfluous. We put

$$D = \bigcap_{i=m+1}^n C_i.$$

Then D , as the intersection of finitely many groups of finite index, also has finite index in G . Now C_{m+1}, \dots, C_n each contains D , and thus can be written as a union of right cosets of D , finitely many in every case. Thus G is a union of finitely many cosets of C_1, \dots, C_m and D . By (4.3) the cosets of C_1, \dots, C_m can be omitted; then the original cosets of C_{m+1}, \dots, C_n are restored from the cosets of D into which they had been decomposed, and (4.4) follows.

(4.5) Define the "density" $\delta(C)$ of a subgroup C of G to be the reciprocal of its index $|G:C|$ if this is finite, and zero if $|G:C|$ is infinite. Then under the assumptions of the lemma

$$\sum_{i=1}^n \delta(C_i) \geq 1. \tag{4.51}$$

Proof. We begin by omitting the cosets of groups of infinite index; and we again denote by D the intersection of the remaining groups C_i . Next we decompose the cosets $C_i g_i$ that remain into cosets of D . Each such coset $C_i g_i$ is the union of $|C_i:D|$ cosets of D . Thus G will be contained in the union of $\Sigma |C_i:D|$ cosets of D , the summation extending over the C_i of finite index. It follows that

$$\Sigma |C_i:D| \geq |G:D|,$$

and this, together with the identity

$$|C_i:D| = |G:D| \cdot \delta(C_i),$$

proves (4.5).

Lemma 4.1 is an immediate corollary of (4.5).

§5. Mautner's Theorem.

We now consider Mautner's Theorem, and prove it in the following more precise form:

(5.1) THEOREM. Let the group G possess a finite subgroup K with the

property that the double cosets of K in G permute. Denote by H the union of all finite classes of conjugates in G . Then

$$G = KH.$$

Proof. By assumption

$$KyKxK = KxKyK$$

for all $x, y \in G$. Thus in particular there is, to each pair x, y of elements of G , a triplet k, k', k'' of elements of K such that

$$yx = kxk'yk''. \quad (5.2)$$

(There may be more than one such triplet for some pairs x, y .)

We now keep y fixed for the moment, and denote by $X(k, k', k'')$ the set of all those $x \in G$ which satisfy (5.2) with a given triplet k, k', k'' . Some of these sets may be empty, and others may overlap; but every element x of G occurs in at least one of them. Moreover there are finitely many such sets only, namely $|K|^3$ —one to each ordered triplet of elements of K .

Next we observe that these sets $X(k, k', k'')$ are certain cosets of subgroups of G . Specifically, if $g \in X(k, k', k'')$, then

$$X(k, k', k'') = C(k^{-1}y)g.$$

For if x is any other element of the same set $X(k, k', k'')$, then

$$yg = kgk'yk''$$

and

$$yx = kxk'yk''.$$

Hence

$$g^{-1}k^{-1}yg = k'yk'' = x^{-1}k^{-1}yx,$$

and

$$xg^{-1} \in C(k^{-1}y).$$

Conversely, if $x \in C(k^{-1}y)g$, then

$$x^{-1}k^{-1}yx = g^{-1}k^{-1}yg = k'yk'',$$

and $x \in X(k, k', k'')$.

Thus G is seen to be the union of finitely many cosets of the form $C(k^{-1}y)g$. It follows from Lemma 4.1—or even the weaker (4.2)—that at least one $C(k^{-1}y)$ has finite index in G . But this means that at least one $k^{-1}y$ has only finitely many conjugates, and lies in H . Thus

$$y \in KH,$$

and as y was an arbitrary element of G , the theorem follows.

(5.3) COROLLARY. $|G:H| \leq |K|.$

It is not difficult to refine this proof so as to obtain the further result that the finite classes of conjugates of G —that is the classes contained in H —are boundedly finite. However, this will also be true in the more general situation studied in the next section, and we therefore defer the proof.

§6. *Groups covered by permutable boundedly finite subsets.*

We now come to study groups which can be covered by permutable boundedly finite subsets. Throughout this section, let G be such a group, and let \mathfrak{F} be a family of subsets such that

$$G = \bigcup_{F \in \mathfrak{F}} F, \tag{6.11}$$

$$FF' = F'F \text{ for all } F, F' \in \mathfrak{F}, \tag{6.12}$$

$$|F| \leq n \text{ for all } F \in \mathfrak{F}. \tag{6.13}$$

We again denote by H the union of all finite classes of conjugates of G ; and we denote by $H^{(m)}$ the union of all classes of conjugates with at most m elements. Thus

$$H^{(m)} = \bigcup_{|g^G| \leq m} g^G,$$

and

$$H = \bigcup_{m=1}^{\infty} H^{(m)}.$$

$H^{(1)}$ is the centre of G , but $H^{(m)}$ for $m > 1$ need not be a group. We now prove that $H^{(m)}$, for suitably large m , has "finite index" in G , that is to say, G is the union of a finite number of translates $H^{(m)}g$ of $H^{(m)}$.

(6.2) LEMMA. *Let $m = \frac{1}{2}n^3(n+1)$, and let g_1, g_2, \dots be a sequence of elements of G such that*

$$g_i \notin H^{(m)}g_j \text{ when } i < j. \tag{6.21}$$

Then the sequence breaks off after at most n terms. In other words, there is a number $p \leq n$ such that

$$G = \bigcup_{i=1}^p H^{(m)}g_i.$$

Proof. Assume the contrary. Then it is possible to find $n+1$ elements g_1, g_2, \dots, g_{n+1} satisfying (6.21). To each g_i we select a set $F_i \in \mathfrak{F}$ which contains g_i . Let x be an arbitrary element of G ; we also select a set $F \in \mathfrak{F}$ containing x . Using the permutability of the F_i and F we can find elements $f_i \in F_i$ and $x_i \in F$ such that

$$g_i x = x_i f_i \quad (i = 1, 2, \dots, n+1).$$

As $|F| \leq n$, there must be two different suffixes s, t such that $x_s = x_t$; we may take $s < t$. We obtain

$$x^{-1}g_s^{-1}g_t x = f_s^{-1}f_t. \tag{6.3}$$

Let us denote by $X(s, t, f_s, f_t)$ the set of those elements $x \in G$ which satisfy (6.3), that is to say, which transform $g_s^{-1}g_t$ into $f_s^{-1}f_t$. Some of these sets may be empty, others may overlap; but in any case every element of G belongs to at least one of them. Moreover there are only a finite number of such sets, in fact at most $\frac{1}{2}n^3(n+1) = m$; for $1 \leq s < t \leq n+1$, whence there are $\frac{1}{2}n(n+1)$ possible choices of s, t ; and f_s and f_t belong to F_s and F_t respectively, and are each capable of at most n values.

Next we observe that these sets $X(s, t, f_s, f_t)$ are certain cosets of subgroups of G . Specifically, if $g \in X(s, t, f_s, f_t)$, then

$$X(s, t, f_s, f_t) = C(g_s^{-1}g_t)g;$$

for any other element $x \in X(s, t, f_s, f_t)$ transforms $g_s^{-1}g_t$ into $f_s^{-1}f_t$, like g .

Thus G is seen to be the union of at most m cosets of the form $C(g_s^{-1}g_t)g$. It follows from Lemma 4.1 that one of the $C(g_s^{-1}g_t)$ has index at most m in G . This means that $g_s^{-1}g_t$, for some s, t with

$$1 \leq s < t \leq n+1,$$

has at most m conjugates. The same is then true of $g_t g_s^{-1}$ —which is in fact one of these conjugates. Thus

$$g_t g_s^{-1} \in H^{(m)}$$

and
$$g_t \in H^{(m)} g_s,$$

contrary to (6.21); and the lemma follows.

(6.3) COROLLARY. $|G:H| \leq n.$

This is now obvious. It should be remarked that Corollary 5.3 is in general sharper than this; for some double cosets of K may contain $n = |K|^2$ elements.

(6.4) LEMMA. *The finite classes of conjugate elements of G are boundedly finite; equivalently, there is a number q such that $H = H^{(q)}$.*

Proof. We choose a sequence of elements h_1, h_2, \dots , of H such that

$$h_j \notin H^{(m)} h_i \text{ for } i < j.$$

This sequence breaks off after at most n terms. Then

$$H = \bigcup H^{(m)} h_i.$$

An arbitrary element $h \in H$ is of the form $h = h^{(m)} h_i$, with $h^{(m)} \in H^{(m)}$ and h_i out of our finite sequence. A conjugate of h is a product of a conjugate of $h^{(m)}$ and a conjugate of h_i ; thus the number of conjugates of h is at most the product of m and the number of conjugates of h_i :

$$|h^G| \leq m \cdot |h_i^G| \leq m \cdot \max_i |h_i^G|.$$

The maximum is finite because each $|h_i^G|$ is finite and there are only a finite number of h_i . Thus $|h^G|$ is bounded, and the lemma follows.

§7. *The main result.*

We are now in a position to characterize the groups of the preceding section completely.

(7.1) THEOREM. *A necessary and sufficient condition for the group G to be the union of permutable boundedly finite subsets is that G possesses a subgroup of finite index whose derived group is finite; or, equivalently, that G has a normal series*

$$G \supseteq H \supseteq H' \supseteq E$$

with G/H and H' finite and H/H' abelian.

Proof. We first remark that the two stated forms of the condition are indeed equivalent: for if G has a subgroup S , say, of finite index and with finite derived group S' , then the intersection of the conjugates of S in G is a normal subgroup H of G , still of finite index in G , and with derived group H' contained in S' and thus also finite.

Next we see that if G is the union of permutable boundedly finite subsets, then the union of its finite classes H is a subgroup of finite index by Corollary 6.3. Moreover the classes of conjugates h^H in H are boundedly finite because—by Lemma 6.4—even the finite classes h^G in G are boundedly finite. By Theorem 3.1 then the derived group H' is finite, and the necessity of the condition follows.

Conversely let G have a subgroup H , which we may assume normal, of finite index n , and with derived group H' finite. Let g_1, g_2, \dots, g_n be a set of representatives of G modulo H . If $h \in H$ then

$$h^G = \bigcup_{i=1}^n g_i^{-1} h^H g_i.$$

This is a boundedly finite set because it is the union of n sets each of bounded cardinal; in fact

$$|h^G| \leq n \cdot |h^H| \leq n \cdot |H'|.$$

We now define to each element $g \in G$ a set F_g containing g as follows. If $g = h g_p$ with $h \in H$ and $1 \leq p \leq n$, we put

$$F_g = \bigcup_{i=1}^n h^G g_i.$$

Then $|F'_g| \leq n \cdot |h^G| \leq n^2 |H'|$, that is F'_g is boundedly finite. Also if $g' = h' g_q$ with $h' \in H$, $1 \leq q \leq n$, then

$$F_g F'_g = \bigcup_{i,j=1}^n h^G h'^G g_i g_j = \bigcup_{i,j=1}^n h'^G h^G g_i g_j = F'_g F_g.$$

Thus G is the union of permutable boundedly finite subsets, the condition is seen to be sufficient, and the theorem is proved.

§8. *Further results.*

What rôle does the boundedness of the permutable finite subsets play? Is it possible to characterize the groups which can be covered by permutable finite subsets, if these subsets may be arbitrarily large?

I do not know the answer; the following partial results may indicate some of the difficulties involved.

(8.1) LEMMA. *Every finitely generated group is the union of permutable finite subsets.*

Proof. Let g_1, g_2, \dots, g_n form a finite set of generators of the group G . We denote by F_λ the set of elements of G which can be expressed as words of length not exceeding λ in g_1, g_2, \dots, g_n . Then F_λ is clearly finite, and

$$F_\lambda F_\mu = F_{\lambda+\mu} = F_\mu F_\lambda.$$

Also G is evidently the union of the F_λ , and the lemma follows.

It may be remarked that a finitely generated *free* group can even be covered by *disjoint* permutable finite subsets: one uses the sets of elements of length exactly, instead of at most, λ .

(8.2) LEMMA. *Every countable locally finite group is the union of permutable finite subsets.*

Proof. If G is countable and locally finite, then it is the union of an ascending sequence

$$G_1 \subseteq G_2 \subseteq G_3 \subseteq \dots$$

of finite groups: $G = \bigcup_{i=1}^\infty G_i.$

These G_i will themselves serve as the required subsets. If instead we put

$$F_1 = G_1, \quad F_2 = G_2 - G_1, \quad F_3 = G_3 - G_2, \quad \dots$$

then the F_i are also permutable and finite and they cover G ; and they are, moreover, disjoint.

These results indicate that the class of groups which can be covered by permutable finite subsets is wide; it may possibly include all countable

groups. It does not, however, include *all* groups; to show this we extend the argument of §6.

(8.3) THEOREM. *Let the infinite group G be the union of a family \mathfrak{F} of permutable subsets whose cardinals are (strictly) less than a cardinal n (strictly) less than that of G :*

$$G = \bigcup_{F \in \mathfrak{F}} F; \tag{8.31}$$

$$FF' = F'F \text{ for all } F, F' \in \mathfrak{F}; \tag{8.32}$$

$$|F| < n < |G| \text{ for all } F \in \mathfrak{F}. \tag{8.33}$$

Then G has a subgroup C whose centre is not trivial and whose order exceeds n :

$$|C| > n.$$

Proof. In G we choose a subset Y of cardinal

$$|Y| = n.$$

To every element $y \in Y$ we select—using Zermelo’s Axiom—a set $F_y \in \mathfrak{F}$ which contains y . Let x be an arbitrary element of G ; we also select a set $F \in \mathfrak{F}$ containing x . Using the permutability of the F_y and F , we can find elements $f_y \in F_y$ and $x_y \in F$ such that

$$yx = x_y f_y \quad (y \in Y).$$

As $|F| < |Y|$, there must be two different elements s, t in Y such that $x_s = x_t$. Then

$$x^{-1} s^{-1} t x = f_s^{-1} f_t. \tag{8.4}$$

Denote by $X(s, t, f_s, f_t)$ the set of those elements $x \in G$ which satisfy (8.4), that is to say, which transform $s^{-1}t$ into $f_s^{-1}f_t$. Then G is the union of these sets:

$$G = \bigcup X(s, t, f_s, f_t). \tag{8.5}$$

Here s and t range over Y and f_s, f_t over F_s, F_t respectively. Thus the union (8.5) has at most n^4 terms. At least one of these terms must have cardinal greater than n ; for otherwise the union would have cardinal $\leq n^5$, but $|G|$ being infinite and (strictly) greater than n implies $|G| > n^5$. Next we observe that these sets $X(s, t, f_s, f_t)$ are again, as in §6, cosets of certain centralizers in G . Specifically, if $g \in X(s, t, f_s, f_t)$, then

$$X(s, t, f_s, f_t) = C(s^{-1}t)g.$$

The cardinal of a coset is, of course, that of the subgroup. Hence there is a centralizer $C(s^{-1}t)$ whose cardinal exceeds n . Such a centralizer

contains the element $s^{-1}t \neq 1$ in its centre, and will, therefore, serve as the subgroup C of the theorem. This completes the proof.

It may be remarked that if n is a finite cardinal, the theorem asserts much less than we know to be true: for we know that G has subgroups of finite index with non-trivial centre.

(8.6) COROLLARY. *Let n be an infinite cardinal and let G be a locally free group of order greater than n . Then G cannot be covered by permutable subsets of cardinal less than n . In particular no non-denumerable locally free group is the union of permutable finite subsets.*

This follows immediately from Theorem 8.3 when it is observed that a locally free group with non-trivial centre is denumerable.

References.

1. Reinhold Baer, "Representations of groups as quotient groups. II. Minimal central chains of a group", *Trans. American Math. Soc.*, 58 (1945), 348-389.
2. ———, "Finiteness properties of groups", *Duke Math. J.*, 15 (1948), 1021-1032.
3. J. Erdős (to be published).
4. F. I. Mautner, "The structure of the regular representation of certain discrete groups", *Duke Math. J.*, 17 (1950), 437-441.
5. B. H. Neumann, "Groups with finite classes of conjugate elements. In memoriam Issai Schur", *Proc. London Math. Soc.* (3), 1 (1951), 178-187.
6. J. Schur, "Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen", *J. reine angew. Math.*, 132 (1907), 85-137.

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A COUNTABLY GENERATED GROUP WHICH CANNOT BE COVERED BY FINITE PERMUTABLE SUBSETS

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Let G be a group and consider an expression of G in the form

$$G = \bigcup_{A \in \mathfrak{A}} A, \tag{1}$$

where \mathfrak{A} is a family of subsets of G such that

$$(i) \quad AB = BA \text{ for } A, B \in \mathfrak{A},$$

and either

$$(ii) \quad |A| \text{ is bounded for } A \in \mathfrak{A},$$

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