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THE CONGRUENCE $(p-1/2)! \equiv \pm 1 \pmod p$

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Let p be an odd prime. Then Wilson's classical result states that $(p-1)!+1 \equiv 0 \pmod p$. On noting that $p-r \equiv -r \pmod p$, this gives, when $p \equiv 1 \pmod 4$, as is well known,

$$\left\{\left(\frac{p-1}{2}\right)!\right\}^2 + 1 \equiv 0 \pmod p.$$

However, when $p \equiv 3 \pmod 4$, we have

$$\left\{\left(\frac{p-1}{2}\right)!\right\}^2 - 1 \equiv 0 \pmod p.$$

Hence

$$\left(\frac{p-1}{2}\right)! \equiv (-1)^a \pmod p, \tag{1}$$

where $a=0$ or 1. In view of the history of the question, it may perhaps be worth while to state and prove the

Theorem.* *If p is a prime* $\equiv 3 \pmod 4$ *and* $p>3$, *then in* (1)

$$a \equiv \tfrac{1}{2}[1 + h(-p)] \pmod 2 \tag{2}$$

where $h(-p)$ is the class number of the quadratic field $k\{\sqrt{(-p)}\}$.

This result does not appear to have been explicitly stated or at any rate does not seem well known. It is, however, implicit in the literature, and it is now a trivial deduction from results long known, *e.g.*, an old one of Dirichlet's (1828) given here as (3). In fact, Jacobi (1832) conjectured a result equivalent to (2) at a time when the class-number formula was not known. For the history of the subject, see Dickson's *History of the Theory of Numbers*, Vol. 1, page 275.

Write $E=[\frac{1}{2}(p-1)]!$ Denote by $r_1, r_2, \cdots$ the R quadratic residues of p less than $\frac{1}{2}p$, and by $n_1, n_2, \cdots$ the N quadratic nonresidues less than $\frac{1}{2}p$. Then the quadratic residues $r_1', r_2', \cdots$ greater than $\frac{1}{2}p$ are given by $p-n_1, p-n_2, \cdots$ since $p\equiv 3 \pmod 4$. Then

$$E = r_1r_2 \cdots n_1n_2 \cdots \equiv (-1)^N r_1r_2 \cdots r_1'r_2' \cdots \equiv (-1)^N \bmod p \text{ if } p > 3, \tag{3}$$

* Professor Chowla informs me that he found the result about the same time that I did.

since $(-1)^N E \equiv g^{2+4+\cdots+(p-1)} = g^{\frac{1}{4}(p^2-1)} = (g^{\frac{1}{2}(p+1)})^{\frac{1}{2}(p+1)} \equiv 1 \pmod p$, where g is a primitive root of p.

Now $R+N=\frac{1}{2}(p-1)$, and it is known* from the class-number formula that

$$R - N = \delta h(-p), \qquad \begin{cases} \delta = 1 \text{ if } p \equiv 7 \pmod 8, \\ \delta = 3 \text{ if } p \equiv 3 \pmod 8), p > 3. \end{cases}$$

Hence $2N=\frac{1}{2}(p-1)-\delta h(-p)$. Then if $p\equiv 7 \pmod 8$, $2N\equiv 3-h(-p) \pmod 4$, and if $p\equiv 3 \pmod 8$, $2N\equiv 1-3h(-p) \pmod 4$. The first is $N\equiv\frac{1}{2}[-1-h(-p)] \pmod 2$ and the second is $N\equiv\frac{1}{2}[1+h(-p)] \pmod 2$. These are both included in $N\equiv\frac{1}{2}[1+h(-p)] \pmod 2$.

* L. Holzer, Zahlentheorie II, 1959, Leipzig, pp. 91–93, and H. Hasse, Vorlesungen über Zahlentheorie, 1950, Berlin, pp. 386–390.

A GENERALIZED TURÁN EXPRESSION FOR THE BESSEL FUNCTIONS

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1. In a recent paper Toscano [3] has proved the formula

$$(1.1)\qquad \sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} H_{n+r}(x)H_{n-r}(x) = \frac{(2m)!(n-m)!}{m!} \sum_{j=m}^{n} \binom{j-1}{m-1} \frac{H_{n-j}^2(x)}{(n-j)!} \qquad (m \leqq n)$$

where $H_n(x)$ is the Hermite polynomial of order n. The expression in the left hand side may be regarded as a generalization of the Turán expression $H_n^2(x) - H_{n+1}(x)H_{n-1}(x)$. Indeed (1.1) reduces to the Demir-Hsü formula [2] when $m=1$. Other proofs of (1.1) as well as extensions to the Laguerre and ultraspherical polynomials and other hypergeometric functions are given in [1].

In the present note we obtain a similar formula involving the Bessel functions. We prove

$$(1.2)\qquad \Omega_n^{(m)}(x) = \sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} J_{n-r}(x)J_{n+r}(x) = \frac{4^m(2m)!}{x^{2m}m!(m-1)!} \sum_{k=0}^{\infty} (n+m+2k)(k+1)_{m-1}(n+k-1)_{m-1}J_{n+m+2k}^2(x)$$

where $(a)_m = a(a+1)(a+2)\cdots(a+m-1)$, $(a)_0=1$. For definition of the Bessel function $J_n(x)$ see [4]. This formula reduces, for $m=1$, to Lommel's formula [4, p. 152]

$$(1.3)\qquad \tfrac{1}{4}x^2\{J_n^2(x) - J_{n+1}(x)J_{n-1}(x)\} = \tfrac{1}{4}x^2\Delta_n(x) = \sum_{k=0}^{\infty} (n+1+2k)J_{n+1+2k}^2(x).$$

It is also a positive representation as sum of squares.