SOLUTION TO MATH 8410, EXERCISE 2.4

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Prerequisites: This exercise uses Exercise 2.2 and Exercise 2.3.

Statement: Let $(K, | \, |)$ be a non-Archimedean normed field, and let L/K be a purely transcendental extension. Show that $| \, |$ extends to a norm on L.

Background: We will review the definition of a purely transcendental extension. More significantly, we will prove the result by **transfinite induction**, so let's begin with a refresher[1] on that.

Recall that a **well-ordered set** is a set S endowed with a total ordering relation $\leq$ (a reflexive, anti-symmetric, transitive relation such that for all $x, y \in S$, at least one of $x \leq y$ and $y \leq x$ holds) which has the additional property that every nonempty subset T of S has a least element. For instance, the natural numbers $\mathbb{N}$ with their usual ordering are a well-ordered set, as are $\mathbb{N} \cup \{\infty\}$, the natural numbers with an additional element ∞ such that $x < \infty$ for all $x \in \mathbb{N}$. In fact, for any well-ordered set S, it turns out to be useful to define the well-ordered set $S^+ = S \cup \{\infty\}$ in the same way. (In particular, S^+ has a maximal element, whereas S need not.)

The empty set *is* well-ordered. Any nonempty well-ordered set has a least element, which we might as well call 0.[2] If S is well-ordered, and x is any element of S except possibly the maximal element (if any), then the set $\{y \in S \mid x < y\}$ is nonempty hence has a least element, the **successor** of x. In the natural numbers, the successor of n is nothing else than $n+1$, so we may as well denote the successor of x by $x + 1$ in the general case. An element x is called a **successor element** if $x = y+1$ for some y, i.e., if x is the successor of some other element. An element x which is not a successor element is called a **limit element**. Note that technically 0 is a limit element, but not such a great example of one. A better example is the element ∞ in $\mathbb{N}^+$: it has the property that for any $x < \infty$, the interval (x, ∞) contains infinitely many elements, which is characteristic of nonzero limit elements.

Principle of Transfinite Induction: Let $(S, \leq)$ be a nonempty well-ordered set. Let $\{P(s)\}_{s \in S}$ be a family of statements indexed by the elements of S. Suppose that both of the following hold:

(PTI1) $P(0)$ is true.
(PTI2) For all $s \in S$, (if $P(s')$ holds for all $s' < s$, then $P(s)$ holds) is true.
Then $P(s)$ is true for all $s \in S$.

[1]Suitable for those who have never seen it before!
[2]This is an allusion to ordinal arithmetic, which is in no way necessary for us here.

Note that this is a direct generalization of the principle of mathematical induction, which we recover by taking $S = \mathbb{N}$. The proof is the same in the general case, the **method of minimal counterexample**: let T be the set of all elements s of S for which $P(s)$ is false. We wish to show that $T = \emptyset$, so assume not. Since T is a nonempty subset of a well-ordered set, it has a least element, say t. By minimality of t, $P(t')$ is true for all $t' < t$. So either $t = 0$, which contradicts (PTI1), or $t > 0$, which contradicts (PTI2).

Remark 1: Note that there is absolutely no set-theoretic funny business here: in particular, we are not (yet) assuming the Axiom of Choice or anything like that.

Remark 2: If in fact we apply the condition (PTI2) with $s = 0$, we see that we must assume nothing and deduce $P(0)$, i.e., we recover (PTI1). However it seems less confusing to state PTI in this slightly redundant way.

Remark 3: In practice to apply PTI one considers not two but three separate cases of (PTI2): $s = 0$, s a successor element, and s a limit element. This is indeed what we are about to do.

Purely transcendental extensions: A purely transcendental extension of a field k is a field extension obtained by adjoining an arbitrary indexed set $\{t_i\}_{i\in I}$ of indeterminates. Precisely, one first defines the polynomial ring $k[\{t_i\}_{i\in I}]$ (see e.g. X.X for a rigorous definition of a polynomial ring in an arbitrary set of indeterminates) and then forms the field of fractions, a rational function field in the indetermiates $\{t_i\}_{i\in I}$. It may be worth emphasizing that any given rational function may involve only finitely many indeterminates, so that a purely transcendental extension is simply the direct limit of its subextensions $k(\{t_i\}_{i\in J})$ as J ranges over the finite subsets of I.

Solution: Let $T = \{t_i\}_{i\in S}$ be a set of indeterminates indexed by any nonempty set I. Choose a well-ordering $\leq$ on S.[3] For $i \in S^+$, define $K_i = k(\{t_{i'}\}_{i'<i})$. Thus if the elements of S^+ begin $0, 1, \ldots, n, \ldots$, then $K_0 = k$, $K_1 = k(t_0), \ldots K_n = k(t_0, \ldots, t_{n-1})$, and so forth; finally $K_\infty = k(T)$. Now, for $i \in S^+$, let $P(i)$ be the following statement: for every $i' \leq i$, there exists a norm $|\ |_{i'}$ on $K_{i'}$ extending $|\ |$ on k, and this family of norms is compatible in the sense that for all $i' < i'' \leq i$, the natural inclusion $K_{i'} \hookrightarrow K_{i''}$ is a homomorphism of normed fields. We will prove by transfinite induction that $P(i)$ holds for all $i \in S^+$; applying this with $i = \infty$ gives the result we want.

Case 1: $i = 0$. $P(0)$ says that there exists a norm on k which is compatible with the given norm on k. True!

Case 2: Suppose $i = i' + 1$ is a successor element. Then $K_i = K_{i'}(t_{i'})$. By our induction hypothesis, we are assuming that we have a norm $|\ |_{i'}$ on $K_{i'}$ which is compatible with all the previous norms. So we can just endow K_i with the Gauss

[3]That each set can be well-ordered follows from, and also implies, the Axiom of Choice. This is where the set-theoretic funny business comes in!

norm of Exercise 2.3.

Case 3: Suppose that i is a limit element. Then $K_i = \bigcup_{i'<i} K_{i'}$ (this is proabably the step you want to think about the most if you have not seen a proof by transfinite induction before; every PTI I have seen uses this kind of argument for the case of limit elements) and we already have a compatible family of norms on each $K_{i'}$. It follows from Exercise 2.2 that there is a unique norm on K_i compatible with the norms on all the subfields $K_{i'}$.

Applying the principle of transfinite induction, we're finished.

Note that the norm we end up constructing on $k(T)$ is nothing close to unique. Well-ordering S corresponds to making a bewilderingly complicated set of choices, and moreover in Case 2 we *chose* to extend a norm from a field L to $L(t)$ by the Gauss norm, rather than, for instance, the image of the Gauss norm under a linear fractional transformation $M \in \mathrm{PGL}_2(L) \subset \mathrm{Aut}(L(t))$.