$$f''(0) = -(1/\sqrt{\pi})\int_{-\infty}^{+\infty} x^2 \exp(-x^2)dx = -\tfrac{1}{2}. \tag{8}$$

By (7), (8) we have $\alpha = -\frac{1}{4}$. Hence, by (7) we have $f(a) = \exp(-a^2/4)$.

References

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ON THE MATRIX EQUATION $AB=I$

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In a course in linear algebra, and elsewhere, it is a great computational convenience to know that if A and B are n-square matrices such that $AB=I$, where I is the identity matrix of order n, then necessarily $BA=I$ and B is the inverse of A. A standard proof of this result is the identification of a matrix with its corresponding linear transformation and the proof that a linear transformation of a finite dimensional vector space is onto if and only if it is one-to-one. There is, however, an elementary direct proof of an even stronger result which is almost an immediate consequence of an interesting theorem about matrices that is not so well known, although its proof is quite simple.

Suppose that R is a ring and that n is a positive integer. Let R_n denote the ring of all n-square matrices over R. We emphasize that R need not be commutative.

THEOREM. *If R satisfies the ascending chain condition for right ideals, then so does R_n.*

Proof. Denote by $f_{i,j}$ the function from R_n to R that sends a matrix onto its (i, j)-component. Observe that if S is a nonempty subset of R_n which is closed with respect to addition, subtraction, and scalar multiplication on the right, then $f_{i,j}(S)$ is a right ideal of R.

Assume that $\{E_k\}$ is a sequence of right ideals of R_n such that E_k is properly contained in E_{k+1} for each positive integer k. For each positive integer t not exceeding n^2 let $E_{k,t}$ denote the subset of E_k consisting of those matrices in E_k that have zero in each of the first t components—for definiteness, count the components by rows. Now suppose that t is less than n^2 and that we have already shown that $E_{k,t}$ is properly contained in $E_{k+1,t}$ for all but a finite number of k. We wish to show that the latter result must also hold for $t+1$. Since $f_{i,j}(E_{k,t})$ is a right ideal of R and since R satisfies the ascending chain condition, we know that $f_{i,j}(E_{k,t}) = f_{i,j}(E_{k+1,t})$ for all but a finite number of k. If this equality holds, however, and if $E_{k,t}$ is proper in $E_{k+1,t}$, then $E_{k,t+1}$ is proper in $E_{k+1,t+1}$; for if A is in $E_{k+1,t}$ but not in $E_{k,t}$, then there exists B in $E_{k,t}$ having the same $(t+1)$-

component as A and $B-A$ is in $E_{k+1,t+1}$ but not in $E_{k,t+1}$. We are led to the conclusion that the set consisting of only the zero matrix in R_n is a proper subset of itself. Thus our assumption must be denied, and R_n satisfies the ascending chain condition for right ideals.

The next theorem is a weak version of Theorem 4 in [1].

THEOREM. *If R is a ring with unit which satisfies the ascending chain condition for right ideals, then the equation $ab=1$ in R implies that $ba=1$.*

Proof. Define a mapping π from R into itself by: $x \rightarrow ax$. Since $\pi(bx)=x$ for each x in R, π is onto. Let $K_n = \{x \in R \mid \pi^n(x)=0\}$ denote the kernel of π^n for each positive integer n. Since K_n is a right ideal of R, there must be a positive integer n such that $K_n = K_{n+1}$. For any such n, we see that $\pi(\pi^n(x))=0$ implies that $\pi^n(x)=0$. Since π^n is onto and since π preserves addition, π must be one-to-one. Since $\pi(1)=\pi(ba)$, $ba=1$ and the theorem is proved.

COROLLARY. *If R is a field or division ring and if A and B are n-square matrices over R such that $AB=I$, then $BA=I$.*

Proof. R has only the two trivial right ideals.

Both of the above theorems can be proved, in much the same way, for the descending chain condition and also for left ideals, so we actually have

THEOREM. *If R is a ring with unit which satisfies the ascending (descending) chain condition for right (left) ideals and if A and B are n-square matrices over R such that $AB=I$, then $BA=I$.*

Reference

1. R. Baer, Inverses and zero divisors, Bull. Amer. Math. Soc., 48 (1942).

TITCHMARSH'S THEOREM FOR ANALYTIC FUNCTIONS

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The convolution of two functions f and g written $f * g$ is a function defined on $[0, \infty)$ by $f * g(t) = \int_0^t f(u)g(t-u)du$. An important property of convolutions, which was first proved (in [1]) by E. Titchmarsh, is: If f and g are continuous on $[0, \infty)$ and if $f * g = 0$, then $f=0$ or $g=0$.

Jan Mikusinski makes use of this property of convolution products to construct convolution quotients, the theory of which provides a rigorous basis for Heaviside's operational calculus. He presents a different proof of Titchmarsh's theorem in [2].

An elementary proof can be given for the following special case of Titchmarsh's theorem:

THEOREM. *If f and g are analytic functions on an interval I which includes the origin and if $f * g = 0$ on I, then either $f=0$ or $g=0$ on I.*