CLASSROOM NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

FUNCTIONS WITH ARBITRARILY SMALL PERIODS

J. M. HENLE

Department of Mathematics, Smith College, Northampton, MA 01060

Recently R. Cignoli and J. Hounie [2] gave a new proof, together with applications, of Burstin's Theorem: *A Lebesgue-measurable function $f:R\to R$ having arbitrarily small periods is constant a.e.* The following is a more direct, self-contained proof.

Let I be any closed interval, and let $D=f^{-1}(I)$. Then the measure of D intersected with any interval depends linearly on the length of the interval. To see this, let $\alpha=m(D\cap[0,1])$ and suppose we are given $\varepsilon>0$ and $a<b$. Choose a period p of f so that $p<\varepsilon$ and $|m/n-(b-a)|<\varepsilon$, where $n=[1/p]$ and $m=[(b-a)/p]$. Since p is a period of f, the measure of D intersected with any interval of length p is the same. Thus if $d=m(D\cap[0,p])$, then $\alpha=m(D\cap[0,1])=nd+\varepsilon_1$ and $m(D\cap(a,b))=md+\varepsilon_2$, with ε_1, $\varepsilon_2<\varepsilon$. We then have:

$$\begin{aligned}|m(D\cap(a,b))-\alpha(b-a)|&=\left|nd\left(\frac{m}{n}\right)+\varepsilon_2-(nd+\varepsilon_1)(b-a)\right|\\&=\left|nd\left(\frac{m}{n}-(b-a)\right)+\varepsilon_2-\varepsilon_1(b-a)\right|<\alpha\varepsilon+\varepsilon_2+\varepsilon_1(b-a);\end{aligned}$$

hence $m(D\cap(a,b))=\alpha(b-a)$.

The theorem results from the following lemma.

LEMMA. *If the measure of a set D intersected with any interval depends linearly on its length, then either $m(D)=0$ or $m(D^c)=0$.*

Using this, for each $n>0$, let k_n, $I_n=[k_n/n, (k_n+1)/n]$ be such that $f^{-1}(I_n)$ is not of measure 0. By the lemma, $m(f^{-1}(I_n)^c)=0$. Also, $\cap_{n<\omega}I_n$ is not empty, since

$$m\left(f^{-1}\left(\left(\bigcap_{n<\omega}I_n\right)^c\right)\right)=m\left(f^{-1}\left(\bigcup_{n<\omega}I_n^c\right)\right)=m\left(\bigcup_{n<\omega}f^{-1}(I_n)^c\right)=0.$$

Since there can be no more than one point q in $\cap_{n<\omega}I_n$, $0=m(f^{-1}(\{q\})^c)$ implies $f(x)=q$ a.e.

The lemma is proved as follows: let $\alpha=m(D\cap[0,1])$. Given $\varepsilon>0$, cover $D\cap[0,1]$ with open intervals O_n so that $\Sigma_n m(O_n)<\alpha+\varepsilon$. Since $m(O_n\cap D)=\alpha m(O_n)$, we have

$$\alpha=m(D\cap[0,1])\leqslant\sum_n m(D\cap O_n)\leqslant\alpha\sum_n m(O_n)<\alpha^2+\alpha\varepsilon.$$

As ε is arbitrary, this shows $\alpha\leqslant\alpha^2$, and so $\alpha=0$ or 1. If $\alpha=0$, $m(D)=0$; and if $\alpha=1$, $m(D^c)=0$.

Another proof depends on the well-known principle that a set that covers at most a fixed fraction of every interval covers almost none of every interval.

I am informed that A. B. Novikoff has found that Burstin's original proof is incorrect.

References

1. C. Burstin, Uber eine spezielle Klasse reeller periodischer Funktionen, Monatsh. Math. Phys., 26 (1915) 229–262.

2. R. Cignoli and J. Hounie, Functions with arbitrarily small periods, this MONTHLY, 85 (1978) 582–584.