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A new approach to the construction of subsystems of complex root systems

by Himmet Can

Department of Mathematics, Faculty of Arts & Sciences, Erciyes University, 38039 Kayseri, Turkey

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ABSTRACT

Dynkin has shown how subsystems of real root systems may be constructed. As the concept of subsystems of complex root systems is not as well developed as in the real case, in this paper we give an algorithm to classify the proper subsystems of complex proper root systems. Furthermore, as an application of this algorithm, we determine the proper subsystems of imprimitive complex proper root systems. These proper subsystems are useful in giving combinatorial constructions of irreducible representations of properly generated finite complex reflection groups.

1. INTRODUCTION

In a now classic paper, Shephard and Todd [24] have given a complete classification of the finite irreducible complex reflection groups. In recent years, these groups have been the subject of considerable study, see for example [5,6], and more recently [19,21] and [22]. The results in the real case are well developed and documented (see, e.g., [2]). In the context of complex reflection groups however, some of the fundamental ideas are not as well developed with no universally accepted analogues for such basic concepts as root systems and their subsystems and positive systems or a length function; for some recent attempts, see [17,9,3] and [4].

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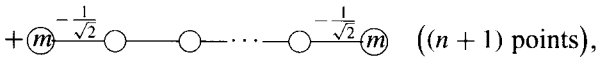
E-mail: can@erciyes.edu.tr (H. Can).

All the subsystems of a real root system Φ relating to a Weyl group may be obtained up to conjugacy by a standard algorithm due independently to Dynkin [15], Borel and de Siebenthal [1]. This is described by Carter [12] as follows: Form the extended Dynkin diagram of Φ by adding one further node to the Dynkin diagram of Φ corresponding to the negative of the highest root. Remove one or more nodes in all possible ways from the extended Dynkin diagram of Φ. Take also the duals of the diagrams obtained in the same way from the dual root system $\widetilde{\Phi}$ of Φ which is obtained from Φ by interchanging long and short roots. Then repeat the process with the diagrams obtained, and continue any number of times. In this algorithm, the concept of the extended Dynkin diagrams is important. Inspired by these, Hughes [17,18] introduced what he called extended Cohen diagrams in order to give an algorithm for obtaining subsystems of complex root systems. Unfortunately, this algorithm has its shortcomings, since for example for type $\pi(m, 1, n) = B_n^m$, he gives the following graph



$((n+1)$ points$)$,

as an extended Cohen diagram, where the adjoined root is marked with the sign "+". However, when m is odd, there does not exist a root in $\Phi(m, 1, n)$ which can be adjoined in this way. Moreover, as there could be more than one highest primary root in the complex case and since a number of equivalent diagrams must be considered, Hughes' algorithm is more difficult to apply in the complex case in comparison with the real case. Furthermore, neither Dynkin's nor Hughes' algorithm leads directly to simple systems for subsystems which are subsets of the primary roots.

Proper subsystems of complex proper root systems are useful in giving combinatorial constructions of representations of properly generated complex reflection groups. For example, they have been used in [8] where the Young tableaux method for generalized symmetric groups [7] has been further generalized.

As the concept of subsystems of root systems for complex reflection groups is not as well developed as in the real case, the present author [9] has presented an alternative algorithm for obtaining all proper subsystems of a given (real or complex) proper root system without any reference to extended diagrams. This algorithm has the further advantage that it simultaneously obtains a simple system which is a subset of the primary roots. Moreover, our method is more useful from a computational point of view (see [10]). In [9], we studied how to obtain the parabolic and non-parabolic proper subsystems of a given complex proper root system. There is no difficulty in determining all the parabolic proper subsystems. In [9], we also claimed that "*as we run through all the parabolic proper subsystems, we generate all the non-parabolic proper subsystems*". In Section 2 of this paper, we prove this claim by using the ideas of [9] and give an algorithm to classify the proper subsystems of complex proper root systems. In Section 3, we determine the proper subsystems of imprimitive complex proper root systems.

We first establish the basic notation and state some results which are required later. We refer the reader to [9,13] and [17] for much of the undefined terminology.

As a convention, throughout this paper, we assume that ξ_m is a fixed primitive mth root of unity.

1.1. Let V be a complex vector space of dimension n furnished with a unitary inner product $(\cdot,\cdot)$. A *reflection* in V is a linear transformation of V of finite order with exactly $n-1$ eigenvalues equal to 1. A *reflection group* G in V is a finite group generated by reflections in V. A (*base*) *root of a reflection* in V is an eigenvector (of length 1) corresponding to the unique non-trivial eigenvalue of the reflection. A (*base*) *root* of G is a (base) root of a reflection in G. Let s be a reflection in V of order $m > 1$. Then there exists a non-zero vector $\alpha \in V$ and a fixed primitive mth root of unity ξ_m for each m such that $s_{\alpha,m}(v) = v - (1-\xi_m)\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha$ for all $v \in V$, where $s = s_{\alpha,m}$. Define $o_G : V \to \mathbf{N}$ by $o_G(v) = |G_{\langle v\rangle^\perp}|$ $(v \in V)$. Then $o_G(v) > 1$ if and only if v is a root of G. In this case, $o_G(v)$ is the order of the cyclic group generated by the reflections in G with root v. If α is a root of G then the number $o_G(\alpha)$ is called the *order* of α (with respect to G).

1.2. (i) A *vector graph* is a pair (B,θ), where B is a non-empty finite subset of $\mathbf{C}^\infty$ such that for all $a, b \in B$, $|(a,b)| = 1$ if and only if $a = b$, and θ is a map from B to $\mathbf{N}\backslash\{1\}$. The set B is called the set of *points* or *nodes* or *vectors* of the vector graph, and $\theta(a)$ for $a \in B$ is the *order* of a (with respect to (B,θ)). A vector graph (B,θ) is represented by a directed valued graph by assigning to each element $a \in B$ a node a with weight $\theta(a)$ and if $(a,b) \neq 0, 1$ a directed edge from a to b with weight (a,b).

(ii) Let $\pi = (B,\theta)$ be a vector graph. Denote by $\dim(\pi)$ the *dimension* of the vector space spanned by B and by $W(\pi)$ the group generated by the *simple reflections* $s_{a,\theta(a)}$ with $a \in B$. The vector graph π is called a *root graph* if $\dim(\pi) = |B|$ (i.e., B is linearly independent over $\mathbf{C}$) and $W(\pi)$ is a finite reflection group. Let $\pi' = (B',\theta')$ be another root graph. If $B \subset B'$ and $\theta'|_B = \theta$, we say that π' is an *extension* of π, or that π is a *sub-root graph* of π'. Root graphs $\pi = (B,\theta)$ and $\pi' = (B',\theta')$ are *equivalent* if the groups $W(\pi)$ and $W(\pi')$ are conjugate. For any root graph $\pi = (B,\theta)$ and for any $w \in U(V)$—the group of all unitary transformations with respect to a unitary inner product, let $w\pi = (B_w,\theta_w)$, where $B_w = wB$ and $\theta_w(w(a)) = \theta(a)$ with $a \in B$, then $w\pi$ is also a root graph which is equivalent to π since $s_{w(a),\theta_w(w(a))} w s_{a,\theta(a)} w^{-1}$ for all $a \in B$ it follows that $W(w\pi) = wW(\pi)w^{-1}$.

(iii) A pair (R,f) is called a *pre-root system* if R is a finite subset of non-zero elements of $\mathbf{C}^\infty$ and $f : R \to \mathbf{N}\backslash\{1\}$ is a map such that for all $a, b \in R$, $s_{a,f(a)}R = R$ and $f(s_{a,f(a)}b) = f(b)$. To $\Phi = (R,f)$ is associated the reflection group $W(\Phi)$ defined by $W(\Phi) = \langle s_{a,f(a)} \mid a \in R\rangle$. A pre-root system $\Phi = (R,f)$ is called a *root system* if in addition $za \in R$ if and only if $za \in W(\Phi)a$ for all $a \in R$, $z \in \mathbf{C}$.

(iv) Every root graph defines a pre-root system, for if $\pi = (B,\theta)$ is a root graph, then the pair $\Phi = (R,f)$ where $R = W(\pi)B$ and the map $f : R \to \mathbf{N}\backslash\{1\}$ is induced by the order function $o_{W(\pi)}$ defines a pre-root system with $W(\Phi) = W(\pi)$.

(v) Every finite irreducible n-dimensional reflection group G in V that is generated by n reflections yields a root graph: Fix a base root for each of the n generating reflections in G. Let B be the set of these base roots and let $\theta : B \to$

$\mathbf{N}\backslash\{1\}$ be given by $\theta(a) = o_G(a)$, $a \in B$. Then $\pi = (B, \theta)$ is a root graph with $W(\pi) = G$.

(vi) If $\Phi = (R, f)$ is a pre-root system, then there is a root system $\Sigma = (S, g)$ with $W(\Sigma) = W(\Phi)$, $S \subset R$ and $g = f|_S$.

(vii) If a root system Φ is the pre-root system obtained from a root graph π as described in 1.2 (iv), then π is called a *simple system* in Φ. If Φ is a root system with simple system π, then the graph associated to π is called a *Cohen* (*Dynkin*) *diagram* of Φ.

(viii) Cohen [13] proves that all finite irreducible imprimitive reflection groups in V are of the form $G(m, p, n)$ for some $m, p \in \mathbf{N}$ with $p \mid m$ and $n \geqslant 2$. The reflection group $G(m, 1, n)$ has the following presentation (see [14]):

$$\begin{aligned} G(m, 1, n) = \langle r_1, \ldots, r_{n-1}, w_1, \ldots, w_n \mid r_i^2 = (r_i r_{i+1})^3 = (r_i r_j)^2 = e,\ |i - j| \geqslant 2, \\ w_i^m = e,\ w_i w_j = w_j w_i,\ r_i w_i = w_{i+1} r_i,\ r_i w_j = w_j r_i,\ j \neq i, i+1 \rangle. \end{aligned}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of V. We may identify r_i $(i = 1, \ldots, n-1)$ with the reflection $s_i = s_{\alpha_i, \theta(\alpha_i)}$ of order 2 with root $\alpha_i = e_i - e_{i+1}$ $(i = 1, \ldots, n-1)$, and therefore the group generated by $\{r_1, \ldots, r_{n-1}\}$ is the Weyl group $W(A_{n-1})$. The w_i $(i = 1, \ldots, n)$ may be identified with the reflection $s_{e_i, \theta(e_i)}$ of order m. Now, if $s_n = s_{\alpha_n, \theta(\alpha_n)}$ is the reflection of order m with root $\alpha_n = e_n$, then w_n is the reflection s_n and consequently w_i $(i = 1, \ldots, n-1)$ is the reflection $s_i s_{i+1} \ldots s_{n-1} s_n s_{n-1} \ldots s_i$ $(i = 1, \ldots, n-1)$ since $e_i = s_i s_{i+1} \ldots s_{n-1}(\alpha_n)$ and $\theta(e_i) = \theta(e_n)$ for $i = 1, \ldots, n-1$ it follows that

$$\begin{aligned} s_{e_i, \theta(e_i)} &= s_{s_i s_{i+1} \ldots s_{n-1}(\alpha_n), \theta(\alpha_n)} \\ &= s_i s_{i+1} \ldots s_{n-1} s_n s_{n-1} \ldots s_i \quad (i = 1, \ldots, n-1). \end{aligned}$$

Note that not all finite irreducible complex n-dimensional reflection groups are generated by n reflections. On the other hand, we do not have root graphs for n-dimensional reflection groups generated by $n+1$ reflections, i.e., the groups $G(m, p, n)$ for $p \neq 1, m$ (see [13]). So we do not have a simple system (in the manner of 1.2 (vii)) for the root system associated with $G(m, p, n)$, $p \neq 1, m$.

Now we make the following definitions. If G is a finite irreducible complex reflection group of dimension n generated by n reflections, then we say that G is a *properly generated finite complex reflection group*. Furthermore, if Φ is a root system associated with a properly generated finite complex reflection group, then we say that Φ is a *proper root system*. Clearly, every proper root system has a simple system by 1.2 (v) and (vii). In our discussion of root systems, we consider the proper root systems (and their proper subsystems) only. If Φ is a proper root system with simple system π, then we say that Φ is *irreducible* if $W(\Phi)$ is irreducible on V, and we also call π *irreducible* if $W(\Phi) = W(\pi)$ is irreducible, or equivalently, if π is connected (see [13, 4.2]).

In this paper, we only study the properly generated reflection subgroups of a properly generated finite complex reflection group. We shall do this by means of proper root systems and their simple systems.

1.3. Now, as in Can [9] and Hughes [17], primary systems for proper root system Φ with simple system $\pi = (B, \theta)$ are defined. These play the role of positive systems for real reflection groups. Let $B = \{a_1, \ldots, a_n\}$, and put $r_i = s_{a_i, \theta(a_i)}$ with $a_i \in B$, $i = 1, \ldots, n$, then the corresponding primary system is defined inductively as follows:

(i) Let $\Phi_1^+ = B$.
(ii) Let $\Phi_2^+ = \{r_i(a_j) \mid i \neq j,\ i, j = 1, \ldots, n,\ a_j \in \Phi_1^+,\ r_i(a_j) \notin \Phi_1^+\}$. (Here, we require that $r_i(a_j) \neq \mu b$ for all $b \in \Phi_1^+$, where μ is a root of unity.)
(iii) For $k \geqslant 3$, let $\Phi_k^+ = \{r_i(a) \mid i = 1, \ldots, n,\ a \in \Phi_{k-1}^+,\ r_i a \neq \mu b$ for all $b \in \Phi_l^+$, $l < k\}$, where μ is a root of unity.

A *primary system* in Φ is defined to be the union of all Φ_k^+ $(k \geqslant 1)$ and will be denoted by Φ^+. By the construction of each Φ_k^+ $(k \geqslant 1)$, it is clear that $\Phi^+ = \biguplus_{k \geqslant 1} \Phi_k^+$ with $\Phi_i^+ \cap \Phi_j^+ = \emptyset$ whenever $i \neq j$. The elements of Φ^+ are called *primary roots*. This algorithm says that a primary root is a single root in each 1-dimensional space spanned by a root.

The primary system is not unique in that there is an element of choice at each step. However, having fixed a primary system Φ^+ for the simple system π of the root system Φ, if the simple system π is replaced by another simple system $w\pi$, $w \in W(\pi)$, then the corresponding primary system obtained by making the same choices in the above algorithm is the conjugate $w\Phi^+$ of Φ^+, namely, any two primary systems in Φ are conjugate under $W(\Phi)$ (see [9, Lemma 2.1]). Thus, this fact shows that it makes no great difference which Φ^+ we choose. In fact, different choices in the above algorithm would result in conjugate primary systems. In the case of real reflection groups, the primary systems are positive systems.

2. CLASSIFICATION OF PROPER SUBSYSTEMS

Let $\Phi = (R, f)$ be a proper root system with a fixed simple system $\pi = (B, \theta)$. Denote by $W(\Phi) = W(\pi)$ the properly generated finite reflection group generated by the simple reflections $s_{a,\theta(a)}$ with $a \in B$. If S is a subset of R and $g = f|_S$, then the pair $\Psi = (S, g)$ is called a *proper subsystem* of Φ if Ψ is itself a proper root system. The corresponding *properly generated reflection subgroup* $W(\Psi)$ is the subgroup of $W(\Phi)$ generated by the $s_{a,g(a)}$ with $a \in S$.

The proper subsystems $\Psi_1 = (S_1, g_1)$ and $\Psi_2 = (S_2, g_2)$ of Φ are *conjugate under* $W(\Phi)$ if $S_2 = wS_1$ and $g_2(w(a)) = g_1(a)$ for some $w \in W(\Phi)$ and for all $a \in S_1$; in which case $W(w\Psi_1) = wW(\Psi_1)w^{-1}$, that is, $W(\Psi_1)$ and $W(\Psi_2)$ are conjugate subgroups in $W(\Phi)$. If $\pi = (B, \theta)$ (resp., $\Phi = (R, f)$) is a simple system (resp., proper root system), by abuse of notation we sometimes say $\pi = B$ (resp., $\Phi = R$).

The proper subsystems of Φ fall into two categories: Proper subsystems whose simple systems $J = (B_\pi, \theta_\pi)$ are such that $B_\pi \subset B$ and $\theta_\pi = \theta|_{B_\pi}$ are called *parabolic* proper subsystems (for a similar argument in the real case, see, for example [20, pp. 18–19]). A proper subsystem of Φ which is not parabolic is called a *non-parabolic* proper subsystem.

Lemma 2.1 [9, 2.2]. *If $\Phi = (R, f)$ is a proper root system with a fixed simple system $\pi = (B, \theta)$ then the pair $J = (B_\pi, \theta_\pi)$, where $B_\pi \subset B$ and $\theta_\pi = \theta|_{B_\pi}$, is a sub-root graph of π. Furthermore, J yields a parabolic proper subsystem of Φ.*

Lemma 2.2 [9, 2.3]. *Let $\Phi = (R, f)$ be a proper root system with a fixed simple system $\pi = (B, \theta)$ and Φ^+ be a primary system determined by π. Let $\Psi = (S, g)$ be a parabolic proper subsystem of Φ with simple system $J = (B_\pi, \theta_\pi)$, where $B_\pi \subset B$ and $\theta_\pi = \theta|_{B_\pi}$, and let Ψ^+ be a primary system determined by J. Define $\Phi^+_\Psi = \Phi^+ \backslash \Psi^+$, and let B_Ψ be a subset of Φ^+_Ψ such that $B_\pi \cup B_\Psi$ is linearly independent over $\mathbf{C}$. Then the pair $J_0 = (B_0, \theta_0)$, where $B_0 = B_\pi \cup B_\Psi$ and $\theta_0 = f|_{B_0}$, is a root graph which is an extension of J. If $B_0 \not\subset wB$ for all $w \in W(\pi)$, then J_0 yields a non-parabolic proper subsystem of Φ. Furthermore, if $B_0 \subset wB$ for some $w \in W(\pi)$, then J_0 yields a parabolic proper subsystem of Φ.*

If Ψ is a parabolic (or non-parabolic) proper subsystem of Φ, recall that its conjugates $w\Psi$, $w \in W(\pi)$, are also parabolic (or non-parabolic) proper subsystems of Φ.

Now, there is a question to be considered: If we start with a non-parabolic proper subsystem, can we form it from a parabolic proper subsystem? In that case we say that a non-parabolic proper subsystem Ψ_0 of Φ can be formed from a parabolic proper subsystem Ψ_1 of Φ if a simple system J_0 of Ψ_0 can be chosen as an extension of a simple system J_1 of Ψ_1.

In the following lemma we prove that the non-parabolic proper subsystems of Φ are obtained from the nonempty parabolic proper subsystems.

Lemma 2.3. *Let $\Phi = (R, f)$ be a proper root system with a fixed simple system $\pi = (B, \theta)$ and Φ^+ be a primary system determined by π. If Ψ_0 is a non-parabolic proper subsystem of Φ, then Ψ_0 can be formed from a nonempty parabolic proper subsystem Ψ_1 of Φ, where $\Psi_1 \neq \Phi$.*

Proof. Let $W(\Psi_0)$ be the properly generated reflection subgroup of $W(\Phi)$ corresponding to the non-parabolic proper subsystem Ψ_0. Referring to 1.2 (v) and 1.3, choose and fix a primary base root $a_s \in \Psi_0 \cap \Phi^+$ for each of the generating reflections s in $W(\Psi_0)$. Let B_0 be the set of elements $a_s \in \Psi_0 \cap \Phi^+$ obtained in this way, and define a map $\theta_0 : B_0 \to \mathbf{N}\backslash\{1\}$ by $\theta_0(a) = o_{W(\Psi_0)}(a)$ for all $a \in B_0$. Then by 1.2 (v) the pair $J_0 = (B_0, \theta_0)$ is a root graph with $W(J_0) = W(\Psi_0)$. Now, put $S_0 = W(J_0)B_0$, and define the extension $g_0 : S_0 \to \mathbf{N}\backslash\{1\}$ of θ_0 by $g_0(w(a)) = \theta_0(a)$ for $a \in B_0$ and $w \in W(J_0)$. Then we have $\Psi_0 = (S_0, g_0)$ by 1.2 (iv), and J_0 is a simple system for Ψ_0 by 1.2 (vii). By definition of the non-parabolic proper subsystem, we have also $B_0 \not\subset wB$ for all $w \in W(\Phi)$.

We decompose the set B_0 into two nonempty parts as follows. $B_0 = B_1 \uplus B_2$ such that $B_1 = wB_\pi$ for some $B_\pi \subset B$ but $B_\pi \neq \emptyset$ and $w \in W(\Phi)$, and $B_2 = B_0 \backslash B_1$. (Here, the symbol $\uplus$ denotes the disjoint union. This is possible, for if $a \in B_0$ then $a = w(\alpha)$ for some $\alpha \in B$ and $w \in W(\Phi)$. This will be illustrated in Example 2.5.) If we put $\theta_\pi = \theta|_{B_\pi}$ then the pair $J = (B_\pi, \theta_\pi)$ is a nonempty sub-root graph of π and

yields a nonempty parabolic proper subsystem Ψ of Φ by Lemma 2.1. Since $B_\pi \neq \emptyset$ and $B_1 = wB_\pi \subset wB$, the pair $J_1 = (B_1, \theta_1)$, where $\theta_1(w(\alpha)) = \theta_\pi(\alpha)$ with $\alpha \in B_\pi$, is a nonempty sub-root graph of $w\pi$ and yields a nonempty parabolic proper subsystem Ψ_1 of Φ which is conjugate to Ψ by Lemma 2.1.

Now, let Ψ_1^+ be a primary system determined by J_1. Then, it is clear that $B_2 \not\subset \Psi_1^+$, for if so, then J_0 is a simple system for Ψ_1. But since J_0 corresponds to the non-parabolic proper subsystem Ψ_0, we have $\Psi_0 = \Psi_1$, contradicting the choice of Ψ_1. To obey the notation in Lemma 2.2 we write just B_{Ψ_1} instead of B_2. Since $B_{\Psi_1} \subset \Phi^+$, we have $B_{\Psi_1} \subset \Phi^+_{\Psi_1} = \Phi^+ \backslash \Psi_1^+$ such that $B_0 = wB_\pi \uplus B_{\Psi_1}$ is linearly independent over $\mathbf{C}$. Thus, we have verified the hypotheses of Lemma 2.2, so the root graph J_0 is an extension of J_1 and so the non-parabolic proper subsystem Ψ_0 is formed from the nonempty parabolic proper subsystem Ψ_1, and the proof is complete. □

We recall that the previous lemma is trivially true with Ψ_1 the empty system; we reach this case by taking B_π (and so B_1) to be the empty set in the proof of Lemma 2.3. On the other hand, we must take that $\Psi_1 \neq \Phi$, for if we take $\Psi_1 = \Phi$ then a simple system J_0 of Ψ_0 cannot be chosen as an extension of the simple system $J_1 = \pi$ of $\Psi_1 = \Phi$ (for if J_0 is chosen as an extension of $J_1 = \pi$ then J_0 is linearly dependent over $\mathbf{C}$, contradicting the definition of simple system).

Corollary 2.4. *Let $\Phi = (R, f)$ be a proper root system with a fixed simple system $\pi = (B, \theta)$ and Φ^+ be a primary system determined by π. If Ψ is a non-empty proper subsystem of Φ, then we have a decomposition $\Psi = \biguplus_{i=1}^k \Psi_i$ $(k \geqslant 1)$ of Ψ, where each Ψ_i is an irreducible proper subsystem of Φ.*

Proof. Construct a simple system $J = (B', \theta')$ for Ψ as in the proof of Lemma 2.3. If J is connected, then Ψ itself is irreducible and we are done ($k = 1$ and $\Psi = \Psi_1$). If not, then suppose that J splits into k connected components $J_1 = (B_1, \theta_1)$, $\ldots$, $J_k = (B_k, \theta_k)$, where $B' = \biguplus_{i=1}^k B_i$ with $B_i \neq \emptyset$ and $B_i \cap B_j = \emptyset$ whenever $i \neq j$, and $\theta_i = \theta'|_{B_i}$ for all $i = 1, \ldots, k$. Thus each $J_i = (B_i, \theta_i)$ is a sub-root graph of J and yields a subsystem Ψ_i of Φ by 1.2 (iv) and (vi). Furthermore, since each $J_i = (B_i, \theta_i)$ is connected then each proper subsystem Ψ_i of Φ is irreducible.

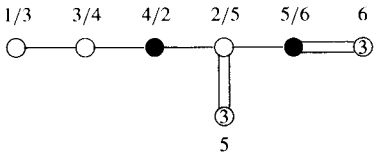
Now, if $\alpha \in B_i$ and $\beta \in B_j$ $(i \neq j)$, then we have $(\alpha, \beta) = 0$ and therefore $s_{\alpha,\theta_i(\alpha)} s_{\beta,\theta_j(\beta)} = s_{\beta,\theta_j(\beta)} s_{\alpha,\theta_i(\alpha)}$, and so the corresponding properly generated reflection subgroup $W(J)$ is the direct product $W(J) = W(J_1) \times \cdots \times W(J_k)$ of properly generated reflection subgroups $W(J_i)$, where $W(J_i) \cap W(J_j) = \{e\}$ $(i \neq j)$. Then, $W(J_i)B_i \cap W(J_j)B_j = \emptyset$ $(i \neq j)$, that is, $\Psi_i \cap \Psi_j = \emptyset$ $(i \neq j)$. Thus,

$$\big(W(J_1) \times \cdots \times W(J_k)\big) \biguplus_{i=1}^k B_i = \biguplus_{i=1}^k \Psi_i = \Psi,$$

as desired. □

Example 2.5. Let Φ be the proper root system of type B_6^3 with simple system $\pi = (B, \theta)$, where $B = \{\alpha_i = e_i - e_{i+1}\ (i = 1, \ldots, 5), \alpha_6 = e_6\}$, and corresponding primary system Φ^+. Referring to 1.2 (viii), denote by r_i $(i = 1, \ldots, 5)$ and w_i $(i = 1, \ldots, 6)$ the reflection $s_{\alpha_i, \theta(\alpha_i)}$ of order 2 $(i = 1, \ldots, 5)$ and the reflection $s_{e_i, \theta(e_i)}$ of order 3 $(i = 1, \ldots, 6)$ respectively. Now, put $s_i = s_{\alpha_i, \theta(\alpha_i)}$ for $i = 1, \ldots, 6$, then w_6 is the reflection s_6 and consequently w_i $(i = 1, \ldots, 5)$ is the reflection $s_i s_{i+1} \ldots s_5 s_6 s_5 \ldots s_i$ $(i = 1, \ldots, 5)$.

Let Ψ_0 be a proper subsystem of Φ of type $A_2 + B_2^3 + B_1^3$ with simple system $J_0 = \{e_1 - e_3, e_3 - \xi_3^2 e_4, e_2 - e_5, e_5, e_6\}$. (Here, ξ_3 is a fixed primitive cube root of unity.) Since $J_0 \not\subset w\pi$ for all $w \in W(\pi)$, then Ψ_0 is non-parabolic. The Cohen diagram Δ for Ψ_0 is



where the nodes corresponding to $e_1 - e_3$, $e_3 - \xi_3^2 e_4$, $e_2 - e_5$, e_5, e_6 are denoted by 1/3, 3/4, 2/5, 5, 6 respectively, the nodes 4/2 and 5/6 have been deleted.

Now, write $J_0 = J_1 \uplus B_{\Psi_1}$, where $B_{\Psi_1} = \{e_5\}$, $J_1 = \{e_1 - e_3, e_3 - \xi_3^2 e_4, e_2 - e_5, e_6\} = w_4^2 r_2 r_3 J$ for $w_4^2 r_2 r_3 \in W(\pi)$ where w_4^2 is the reflection $s_4 s_5 s_6^2 s_5 s_4$, and where $J = \{e_1 - e_2, e_2 - e_3, e_4 - e_5, e_6\} \subset \pi$. Then Ψ is a parabolic proper subsystem of Φ of type $A_2 + A_1 + B_1^3$ with simple system J, by Lemma 2.1. Since $J_1 = w_4^2 r_2 r_3 J \subset w_4^2 r_2 r_3 \pi$, J_1 is a sub-root graph of $w_4^2 r_2 r_3 \pi$ and yields the parabolic proper subsystem $\Psi_1 = w_4^2 r_2 r_3 \Psi$ of Φ of type $A_2 + A_1 + B_1^3$. The primary system of Ψ_1 determined by J_1 is $\Psi_1^+ = \{e_1 - e_3, e_1 - \xi_3^2 e_4, e_3 - \xi_3^2 e_4, e_2 - e_5, e_6\}$, and so $B_{\Psi_1} \not\subset \Psi_1^+$. Thus, we have $B_{\Psi_1} \subset \Phi_{\Psi_1}^+ = \Phi^+ \setminus \Psi_1^+$ such that $J_0 = J_1 \uplus B_{\Psi_1}$ is linearly independent over $\mathbf{C}$. Therefore, by Lemma 2.2 the root graph J_0 is an extension of J_1 and so the Ψ_0 is formed from the parabolic proper subsystem Ψ_1.

We now have all the ingredients to give an algorithm to classify all the proper subsystems of a given (real or complex) proper root system.

Algorithm 2.6. Let Φ be a proper root system. Choose a simple system π in Φ and corresponding Cohen (Dynkin) diagram Δ, and keep them fixed. Let Φ^+ be a primary system determined by π.

(1) Obtain the parabolic proper subsystems of Φ by removing one or more nodes in all possible ways from the Cohen (Dynkin) diagram Δ of Φ.
(2) To generate a non-parabolic proper subsystem of Φ, take a parabolic proper subsystem Ψ of Φ obtained in (1) with simple system $J_\pi \subset \pi$ and corresponding primary system Ψ^+.

Define $\Phi_\Psi^+ = \Phi^+ \setminus \Psi^+$, and choose a non-empty subset B_Ψ of Φ_Ψ^+ such that

(i) $J_\pi \cup B_\Psi \not\subset w\pi$ for all $w \in W(\pi)$,
(ii) $J_\pi \cup B_\Psi$ is linearly independent over $\mathbf{C}$.

Then $J_0 = J_\pi \cup B_\Psi$ is a root graph which is an extension of J_π which yields a non-parabolic proper subsystem Ψ_0 of Φ.

(3) Repeat the process (2) with the parabolic proper subsystems obtained in (1), and continue any number of times.
(4) To obtain all the proper subsystems (both parabolic and non-parabolic) of Φ up to conjugacy, replace π by another simple system $\sigma\pi$, $\sigma \in W(\pi)$, and repeat the processes (1), (2) and (3) step by step.

Remark 2.7. In the previous algorithm, if hypothesis (2) (i) is dropped then, at any stage of the above construction, we may have $J_0 \subset w\pi$ for some $w \in W(\pi)$ which implies that Ψ_0 is parabolic. Therefore, the hypothesis (2) (i) merely enables us to construct the non-parabolic proper subsystems of Φ. For a precise information, see Lemma 2.2. Furthermore, If Φ is a real root system, then we may replace the hypothesis (2) (ii) of Algorithm 2.6 by $(a, b) \leqslant 0$ for all pairs $a \neq b$ in $J_0 = J_\pi \cup B_\Psi$ (see [9, Corollary 2.4]).

Indeed, the following theorem states that Algorithm 2.6 does everything we want.

Theorem 2.8. *Let Φ be a proper root system. Choose a simple system π in Φ and corresponding Cohen (Dynkin) diagram Δ, and keep them fixed. Then:*

(i) *The parabolic proper subsystems of Φ are obtained by removing one or more nodes in all possible ways from the Cohen (Dynkin) diagram Δ of Φ.*
(ii) *As we run through all the parabolic proper subsystems obtained in* (i), *we generate the non-parabolic proper subsystems of Φ.*
(iii) *A simple system for a proper subsystem of Φ can be chosen as a subset of a primary system of Φ.*
(iv) *All the proper subsystems (both parabolic and non-parabolic) of Φ are obtained up to conjugacy.*
(v) *Let G be a finite complex reflection group of dimension n generated by n reflections, with proper root system Φ. Then every proper subsystem Ψ of Φ, with simple system J is contained in a proper root system which has simple system consisting of n nodes.*

Proof. The statement (i) follows immediately from Lemma 2.1. The assertion (ii) follows from Lemma 2.2 and Lemma 2.3. (iii) follows from Theorem 1 of [9]. To prove (iv), replace π by another simple system $\sigma\pi$, $\sigma \in W(\pi)$, and apply (i) and (ii).

Now we consider the last assertion. If J consists of fewer than n nodes, then there are roots in Φ not expressible linearly in terms of the roots of J. We adjoin one of these roots to J to obtain a new proper subsystem with new graph which is equivalent to a root graph. We continue this process until we obtain a proper root system with simple system consisting of n nodes. This proves part (v) of the theorem. □

We now give the following example to illustrate the fact stated in the part (v) of Theorem 2.8.

Example 2.9. Let us consider the Weyl group $G = W(\Phi)$ with proper root system Φ of type B_4. Let $\pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_4\}$ be a fixed simple system in Φ. Then G is a finite reflection group of dimension 4 generated by 4 simple reflections of order 2 with roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

Now, consider the proper subsystem Ψ of Φ of type A_3 with simple system $J = \{\alpha_1, \alpha_2, \alpha_3\}$. Let $-\tilde{\alpha} \in \Phi$, where $\tilde{\alpha}$ is the highest root of Φ. Since $\tilde{\alpha} = e_1 + e_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$ then $-\tilde{\alpha}$ cannot be written as a linear combination of roots in J. If we adjoin $-\tilde{\alpha}$ to J by attaching $-\tilde{\alpha}$ to α_2, then $J \cup \{-\tilde{\alpha}\}$ is a root graph of type D_4 and yields a proper root system Σ of type D_4 by 1.2 (iv). Then by 1.2 (vii) $J \cup \{-\tilde{\alpha}\}$ is a simple system in Σ consisting of 4 nodes. Furthermore, Ψ is contained in Σ.

Now, consider $\alpha_4 \in \Phi$. Then α_4 cannot be written as a linear combination of the roots of J. If α_4 is adjoined to J, then $J \cup \{\alpha_4\} = \pi$. Thus, for Ψ the proper root system Φ itself satisfies part (v) of Theorem 2.8. In fact, we have a chain $\Psi \subset \Sigma \subset \Phi$ for the proper subsystem Ψ of Φ.

Theorem 2.8 gives us a direct way for finding a certain proper subsystem. Furthermore, the part (v) of Theorem 2.8 says that a given system is a proper subsystem of a proper root system. The corresponding results for real root systems are well known (see [1,15]). Hence, in order to classify all the proper subsystems of a given (real or complex) proper root system it is sufficient to apply the above algorithm. Now, as an application of this section, we shall construct the proper subsystems in the imprimitive case in the following section. (For the primitive reflection groups, the proper subsystems of each individual proper root system need to be listed one by one.)

3. PROPER SUBSYSTEMS IN THE IMPRIMITIVE CASE

Let $V = \mathbf{C}^n$, the complex vector space of dimension n with standard unitary inner product $(\cdot, \cdot)$ and the standard basis $\{e_1, e_2, \ldots, e_n\}$. A group G of unitary automorphisms of V is called *imprimitive* if V is a direct sum $V = V_1 \oplus V_2 \oplus \cdots \oplus V_t$ of non-trivial proper linear subspaces V_i $(1 \leqslant i \leqslant t)$ of V such that $\{V_i \mid i = 1, \ldots, t\}$ is invariant under G. If such a direct splitting of V does not exist, then G is called *primitive*.

Let $\mathcal{S}_n$ be the group of all $n \times n$ permutation matrices, and let $A(m, p, n)$, where $p \mid m$ $(m, p \in \mathbf{N})$, be the group of all diagonal $n \times n$ matrices with $\xi_m^{s_i}, s_i \in \mathbf{Z}$ in the (i, i) position and $\sum_{i=1}^n s_i \equiv 0 \pmod{p}$. Define $G(m, p, n) = A(m, p, n) \times \mathcal{S}_n$ (semi-direct product), then the imprimitive reflection groups in V are of the form $G(m, p, n)$, where $p \mid m$ (see [13]). Furthermore, $G(m, m, 2)$ is conjugate to $W(I_2(m))$, $G(1, 1, n) = W(A_{n-1})$, $G(2, 1, n) = W(B_n) = W(C_n)$ and $G(2, 2, n) = W(D_n)$.

If $p = 1$ or m, it is possible to choose n generating reflections for $G(m, p, n)$. Take the reflections of order 2 with roots $e_1 - e_2$, $e_2 - e_3$, ..., $e_{n-1} - e_n$ and if

$p = m$ also the reflection of order 2 with root $e_{n-1} - \xi_m e_n$; and if $p = 1$ also the reflection of order m with root e_n. If $p \neq 1, m$, take the n generating reflections for $G(m, m, n)$ together with the reflection of order m/p with root e_n, to obtain $n + 1$ generating reflections for $G(m, p, n)$. Clearly, the groups $G(m, m, n)$ and $G(m, 1, n)$ are properly generated groups.

A root system for $G(m, p, n)$ may be defined as follows (see [13]). Let $\mu_m = \{\xi_m^l \mid l \in \mathbf{N},\ \xi_m$ is a fixed primitive mth root of unity$\}$. For $W(D_n^m) = G(m, m, n)$, take

$$R(m, m, n) = \mu_m\{\pm(e_i - \xi_m^l e_j) \mid i, j, l \in \mathbf{N},\ i \neq j,\ 1 \leqslant i,\ j \leqslant n\}$$

with $f_{m,m,n} : R(m, m, n) \to \mathbf{N}\backslash\{1\}$ being the constant map 2; then we have that $\Phi(m, m, n) = (R(m, m, n), f_{m,m,n})$ is a root system with $W(\Phi(m, m, n)) = G(m, m, n)$.

Now let $q = m/p \in \mathbf{N}\backslash\{1\}$. Put $R(m, p, n) = R(m, m, n) \cup \mu_q\{e_k \mid 1 \leqslant k \leqslant n\}$, and let $f_{m,p,n} : R(m, p, n) \to \mathbf{N}\backslash\{1\}$ be defined by

$$f_{m,p,n}(a) = \begin{cases} q & \text{if } a \in \mu_q\{e_k \mid 1 \leqslant k \leqslant n\}, \\ 2 & \text{otherwise.} \end{cases}$$

Then $\Phi(m, p, n) = (R(m, p, n), f_{m,p,n})$ is a root system with $W(\Phi(m, p, n)) = G(m, p, n)$. If $p = 1$, then we write $W(\Phi(m, 1, n)) = W(B_n^m)$.

If $\Phi(m, p, n)$ is a root system associated with an imprimitive reflection group $G(m, p, n)$, then we say that $\Phi(m, p, n)$ is an *imprimitive root system.*

We now determine the proper subsystems of $\Phi(m, p, n)$ $(p = 1, m)$ by means of Algorithm 2.6. These proper subsystems are now used to construct some irreducible modules of $G(m, p, n)$ (see [22,11]).

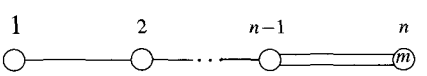
Let $\Phi(m, p, n)$ $(p = 1, m)$ be an imprimitive proper root system with a fixed simple system $\pi(m, p, n) = (B, \theta)$, where

$$B = \begin{cases} \{\alpha_i = e_i - e_{i+1}\ (i = 1, \ldots, n-1),\ \alpha_n = e_n\} & \text{if } p = 1, \\ \{\beta_i = e_i - e_{i+1}\ (i = 1, \ldots, n-1),\ \beta_n = e_{n-1} - \xi_m e_n\} & \text{if } p = m, \end{cases}$$

and corresponding primary system $\Phi^+(m, p, n)$ $(p = 1, m)$.

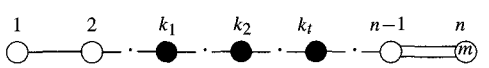
3.1. Type B_n^m

If $p = 1$, then $\Phi(m, 1, n)$ is a proper root system of type B_n^m and the corresponding Cohen diagram for $\Phi(m, 1, n)$ is



where the node corresponding to α_i $(i = 1, \ldots, n)$ is denoted by i.

By following Algorithm 2.6 (1), for $1 \leqslant k_1 < k_2 < \cdots < k_t \leqslant n$, let Ψ be a parabolic proper subsystem of $\Phi(m, 1, n)$ with Cohen diagram



where the nodes $k_1, k_2, \ldots, k_t$ have been deleted. If we put $l_j = k_j - k_{j-1} - 1$ ($j = 1, \ldots, t$) with $k_0 = 0$, then Ψ is of type $\sum_{j=1}^{t} B_{l_j}^1 + B_{n-k_t}^m$ with simple system $J = \sum_{j=1}^{t} J_{k_j} + J_{n-k_t}$, where $J_{k_j} = \{\alpha_{k_{j-1}+1}, \alpha_{k_{j-1}+2}, \ldots, \alpha_{k_j-1}\}$ and $J_{n-k_t} = \{\alpha_{k_t+1}, \alpha_{k_t+2}, \ldots, \alpha_n\}$ are simple systems for types $B_{l_j}^1$ and $B_{n-k_t}^m$ respectively.

By considering Algorithm 2.6 (2), for $1 < t_1 < t_2 < \cdots < t_m < t_{m+1} = t$, let $J_\Psi = \sum_{j=t_1+1}^{t_{m+1}} J_{\Psi,k_j}$ be a subset of $\Phi_\Psi^+(m, 1, n)$, where

$$J_{\Psi,k_j} = \begin{cases} \{e_{k_j-1} - \xi_m e_{k_j}\} & \text{if } t_1 + 1 \leqslant j \leqslant t_2, \\ \{e_{k_j}\} & \text{if } t_2 + 1 \leqslant j \leqslant t_{m+1}, \end{cases}$$

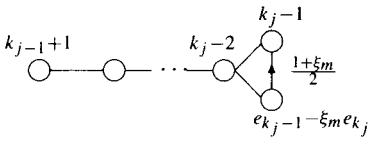
and let

$$J_0 = J \cup J_\Psi = \sum_{j=1}^{t_1} J_{k_j} + \sum_{j=t_1+1}^{t_2} \{J_{k_j} + J_{\Psi,k_j}\}$$
$$+ \sum_{i=2}^{m} \sum_{j=t_i+1}^{t_{i+1}} \{J_{k_j} + J_{\Psi,k_j}\} + J_{n-k_t}.$$

Then $J_0 \not\subset w\pi(m, 1, n)$ for all $w \in W(\pi(m, 1, n))$.

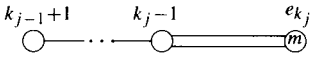
Since the corresponding Cohen diagrams for $J_{k_j} + J_{\Psi,k_j}$ ($t_1 + 1 \leqslant j \leqslant t_2$) and $J_{k_j} + J_{\Psi,k_j}$ ($t_2 + 1 \leqslant j \leqslant t_{m+1}$) are respectively

$D_{l_j+1}^m$



and

$B_{l_j+1}^m$



then J_0 is linearly independent over $\mathbf{C}$. Hence, by Algorithm 2.6 (2), J_0 is a root graph which is an extension of J and so Ψ_0 is a non-parabolic proper subsystem of $\Phi(m, 1, n)$ of type

$$\sum_{j=1}^{t_1} B_{l_j}^1 + \sum_{j=t_1+1}^{t_2} D_{l_j+1}^m + \sum_{i=2}^{m} \sum_{j=t_i+1}^{t_{i+1}} B_{l_j+1}^m + B_{n-k_t}^m,$$

with simple system J_0 and

$$\sum_{j=1}^{t_1} (l_j + 1) + \sum_{j=t_1+1}^{t_2} (l_j + 1) + \sum_{i=2}^{m} \sum_{j=t_i+1}^{t_{i+1}} (l_j + 1) + n - k_t = n.$$

On the other hand, we can rewrite the type of Ψ_0 as follows:

$$\sum_{j=1}^{t_1} B^1_{l_j} + \sum_{j=1}^{t_2-t_1} D^m_{l_{t_1+j}+1} + \sum_{i=2}^{m} \sum_{j=1}^{t_{i+1}-t_i} B^m_{l_{t_i+j}+1} + B^m_{n-k_t}.$$

By setting $s = t_2 - t_1$ and $\mu_j = l_{t_1+j} + 1$ $(j = 1, \ldots, s)$, and for $i = 1, 2, \ldots, m$

$$s_i = \begin{cases} t_1 & \text{if } i = 1, \\ t_{i+1} - t_i & \text{if } i = 2, \ldots, m-1, \\ t_{m+1} - t_m + 1 & \text{if } i = m, \end{cases}$$

$$\lambda^{(i)}_j = \begin{cases} l_j \ (j = 1, 2, \ldots, s_1) & \text{if } i = 1, \\ l_{t_i+j} + 1 \ (j = 1, 2, \ldots, s_i) & \text{if } i = 2, \ldots, m-1, \\ l_{t_m+j} + 1 \ (j = 1, 2, \ldots, s_m - 1) & \text{if } i = m, \\ n - k_t \ (j = s_m) & \text{if } i = m, \end{cases} \quad \text{and}$$

$$m_i = \begin{cases} 1 & \text{if } i = 1, \\ m & \text{if } i = 2, \ldots, m, \end{cases}$$

we can refine the type of Ψ_0 as follows:

$$\text{(I)} \qquad B^m_n\colon \quad \sum_{i=1}^{m} \sum_{j=1}^{s_i} B^{m_i}_{\lambda^{(i)}_j} + \sum_{j=1}^{s} D^m_{\mu_j} \quad \text{with}$$

$$\sum_{j=1}^{s_1} (\lambda^{(1)}_j + 1) + \sum_{i=2}^{m} \sum_{j=1}^{s_i} \lambda^{(i)}_j + \sum_{j=1}^{s} \mu_j = n.$$

In the formula (I), if we take $m = 2$ then the formula for type B_n is

$$\text{(II)} \qquad B_n\colon \quad \sum_{i=1}^{2} \sum_{j=1}^{s_i} B^{m_i}_{\lambda^{(i)}_j} + \sum_{j=1}^{s} D^m_{\mu_j}.$$

But Hughes [16] has proved that there is a one-to-one correspondence between conjugacy classes of $W(B^m_n)$ and proper subsystems of the type

$$\text{(III)} \qquad \sum_{i=1}^{m} \sum_{j=1}^{s_i} B^{m_i}_{\lambda^{(i)}_j} \quad \text{with } m_i = \begin{cases} 1 & \text{if } i = 1, \\ m & \text{if } i = 2, \ldots, m, \end{cases} \quad \text{and}$$

$$\sum_{j=1}^{s_1} (\lambda^{(1)}_j + 1) + \sum_{i=2}^{m} \sum_{j=1}^{s_i} \lambda^{(i)}_j = n.$$

Of course the choice of the set J_Ψ is arbitrary. Therefore, in the construction of the formula (I), if we take the empty set instead of $\{e_{k_j-1} - \xi_m e_{k_j}\}$ for $t_1 + 1 \leqslant j \leqslant t_2$ then the last term of the formula (I) disappears completely, and so we directly obtain the formula (III).

In the formula (III), if we take $m = 1, 2$ then the formulas for types A_{n-1} and C_n are respectively

Table 1

Φ	Types of proper subsystems	
A_n	$\sum_{i=1}^{p} A_{\lambda_i}$	$\sum_{i=1}^{p}(\lambda_i+1)=n+1$
B_n	$\sum_{i=1}^{p} A_{\lambda_i}+\sum_{j=1}^{r} B_{\rho_j}+\sum_{\nu=1}^{s} D_{\mu_\nu}$	$\sum_{i=1}^{p}(\lambda_i+1)+\sum_{j=1}^{r}\rho_j+\sum_{\nu=1}^{s}\mu_\nu=n$
C_n	$\sum_{i=1}^{p} A_{\lambda_i}+\sum_{j=1}^{r} C_{\mu_j}$	$\sum_{i=1}^{p}(\lambda_i+1)+\sum_{j=1}^{r}\mu_j=n$
B_n^m	$\sum_{i=1}^{m}\sum_{j=1}^{s_i} B^{m_i}_{\lambda_j^{(i)}}+\sum_{j=1}^{s} D^m_{\mu_j}$	$\sum_{j=1}^{s_1}(\lambda_j^{(1)}+1)+\sum_{i=2}^{m}\sum_{j=1}^{s_i}\lambda_j^{(i)}$ $+\sum_{j=1}^{s}\mu_j=n$

(IV) $\quad A_n$: $\displaystyle\sum_{j=1}^{s_1} B^{m_1}_{\lambda_j^{(1)}}$ and

(V) $\quad C_n$: $\displaystyle\sum_{i=1}^{2}\sum_{j=1}^{s_i} B^{m_i}_{\lambda_j^{(i)}}$,

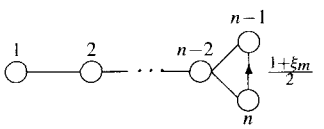
and so we recover the results of Dynkin [15] for types A_{n-1} and C_n. Thus we have the following.

Theorem 3.1. *The types of proper subsystems (up to conjugacy) of the proper root systems of types A_n, B_n, C_n and B_n^m are of the forms shown in Table 1, where*

$$m_i=\begin{cases}1 & \text{if } i=1,\\ m & \text{if } i=2,\dots,m.\end{cases}$$

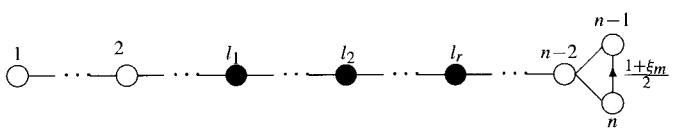
3.2. Type D_n^m

If $p=m$, then $\Phi(m,m,n)$ is a proper root system of type D_n^m and the corresponding Cohen diagram for $\Phi(m,m,n)$ is



where the node corresponding to β_i $(i=1,\dots,n)$ is denoted by i.

By following Algorithm 2.6 (1), for $1\leqslant l_1<l_2<\cdots<l_r\leqslant n$, let Ψ be a parabolic proper subsystem of $\Phi(m,m,n)$ with Cohen diagram

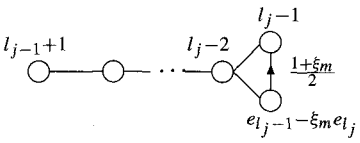


where the nodes $l_1,\dots,l_r$ have been deleted. If we put $k_j=l_j-l_{j-1}-1$ $(j=1,\dots,r)$ with $l_0=0$, then Ψ has type $\sum_{j=1}^{r} D^1_{k_j}+D^m_{n-l_r}$ with simple system $J=\sum_{j=1}^{r} J_{l_j}+J_{n-l_r}$, where $J_{l_j}=\{\beta_{l_{j-1}+1},\beta_{l_{j-1}+2},\dots,\beta_{l_j-1}\}$ and $J_{n-l_r}=\{\beta_{l_r+1},\beta_{l_r+2},\dots,\beta_n\}$ are simple systems for types $D^1_{k_j}$ and $D^m_{n-l_r}$ respectively.

By considering Algorithm 2.6 (2), for $1 < r_1 < r_2 < \cdots < r_m < r_{m+1} = r$, let $J_\Psi = \sum_{j=r_2+1}^{r_{m+1}} J_{\Psi,l_j}$ be a subset of $\Phi_\Psi^+(m,m,n)$, where $J_{\Psi,l_j}\{e_{l_j-1} - \xi_m e_{l_j}\}$ for $j = r_2+1, \ldots, r_m+1$, and let

$$J_0 = J \cup J_\Psi = \sum_{j=1}^{r_2} J_{l_j} + \sum_{i=2}^{m} \sum_{j=r_i+1}^{r_{i+1}} \{J_{l_j} + J_{\Psi,l_j}\} + J_{n-l_r}.$$

Then $J_0 \not\subset w\pi(m,m,n)$ for all $w \in W(\pi(m,m,n))$. Since the corresponding Cohen diagrams for $J_{l_j} + J_{\Psi,l_j}$ $(r_2+1 \leqslant j \leqslant r_m+1)$ are

$D^m_{k_j+1}$ 

then J_0 is linearly independent over $\mathbf{C}$. Thus, by Algorithm 2.6 (2), J_0 is a root graph which is an extension of J and so Ψ_0 is a non-parabolic proper subsystem of $\Phi(m,m,n)$ of type

$$\sum_{j=1}^{r_2} D^1_{k_j} + \sum_{i=2}^{m} \sum_{j=r_i+1}^{r_{i+1}} D^m_{k_j+1} + D^m_{n-l_r},$$

with simple system J_0 and

$$\sum_{j=1}^{r_2}(k_j+1) + \sum_{i=2}^{m} \sum_{j=r_i+1}^{r_{i+1}} (k_j+1) + n - l_r = n.$$

By using similar arguments as in the B^m_n type, we can reformulate the type of proper subsystem Ψ_0 as follows:

(VI) $\quad D^m_n$: $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{u_i} D^{m_i}_{\mu_j^{(i)}}$ with $m_i = \begin{cases} 1 & \text{if } i = 1 \\ m & \text{if } i = 2, \ldots, m \end{cases}$ and

$$\sum_{j=1}^{u_1}(\mu_j^{(1)}+1) + \sum_{i=2}^{m}\sum_{j=1}^{u_i} \mu_j^{(i)} = n.$$

If we take $m = 2$ in (VI) then the formula for type D_n is

(VII) $\quad D_n$: $\displaystyle\sum_{i=1}^{2}\sum_{j=1}^{u_i} D^{m_i}_{\mu_j^{(i)}},$

and so we recover the result of Dynkin [15] for type D_n. Thus we have the following theorem.

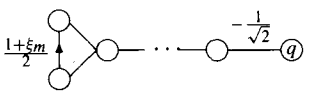
Table 2

Φ	Types of proper subsystems	
D_n	$\sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} D_{\mu_j}$	$\sum_{i=1}^{r}(\lambda_i + 1) + \sum_{j=1}^{s} \mu_j = n$
D_n^m	$\sum_{i=1}^{m} \sum_{j=1}^{u_i} D^{m_i}_{\mu_j^{(i)}}$	$\sum_{j=1}^{u_1}(\mu_j^{(1)} + 1) + \sum_{i=2}^{m} \sum_{j=1}^{u_i} \mu_j^{(i)} = n$

Theorem 3.2. *The types of proper subsystems (up to conjugacy) of the proper root systems of types D_n and D_n^m are of the forms shown in Table 2, where*

$$m_i = \begin{cases} 1 & \text{if } i = 1, \\ m & \text{if } i = 2, \ldots, m. \end{cases}$$

Remark 3.3. We shall now make a few remarks on the root system $\Phi(m, p, n)$ ($p \mid m$ and $p \neq 1, m$) associated with the groups $G(m, p, n)$. The vector graph (see [23]) for $\Phi(m, p, n)$ is



($n + 1$ points, $n \geqslant 2$),

where $q = m/p$. If we denote this vector graph by $\pi(m, p, n)$ ($p \neq 1, m$), then $W(\pi(m, p, n)) = G(m, p, n)$. But for $p \neq 1, m$, $\Phi(m, p, n)$ is not a proper root system and $\pi(m, p, n)$ is linearly dependent over $\mathbf{C}$, and so we do not have a simple system for the root system $\Phi(m, p, n)$ ($p \neq 1, m$) associated with $G(m, p, n)$. Deleting a node from $\pi(m, p, n)$ leaves us one of the vector graphs of the type $\pi(m, m, n) = D_n^m$, $\pi(q, 1, n) = B_n^q$, $\pi(m, m, r) + \pi(q, 1, n - r) = D_r^m + B_{n-r}^q$, which turn out to be root graphs. Let Σ be a proper root system such that its simple system is one of the root graphs obtained as above. Thus to obtain the proper subsystems of $\Phi(m, p, n)$, we apply Algorithm 2.6 (1) and (2) to the Σ. Since we have already dealt with these types, the types of proper subsystems of $\Phi(m, p, n)$ ($p \mid m$ and $p \neq 1, m$) are of the form

$$\text{(VIII)} \quad \sum_{i=1}^{s} \sum_{j=1}^{u_i} D^{m_i}_{\gamma_j^{(i)}} + \sum_{i=1}^{t} \sum_{j=1}^{v_i} B^{q_i}_{\rho_j^{(i)}} \quad \text{with}$$

$$m_i = \begin{cases} 1 & \text{if } i = 1, \\ m & \text{if } i = 2, \ldots, s, \end{cases} \qquad q_i = \begin{cases} 1 & \text{if } i = 1, \\ q & \text{if } i = 2, \ldots, t, \end{cases} \quad \text{and}$$

$$\sum_{j=1}^{u_1}(\gamma_j^{(1)} + 1) + \sum_{i=2}^{s} \sum_{j=1}^{u_i} \gamma_j^{(i)} + \sum_{j=1}^{v_1}(\rho_j^{(1)} + 1) + \sum_{i=2}^{t} \sum_{j=1}^{v_i} \rho_j^{(i)} = n.$$

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