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SIGNS OF DERIVATIVES AND ANALYTIC BEHAVIOR

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This is an account of some striking results, most of which are far from new but not widely known. Although many of them were quite unexpected when they were discovered, the results themselves are easily comprehended by undergraduates; also, many of the proofs are sufficiently elementary to be presented in a course in advanced calculus or elementary real analysis, or in an under-

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graduate seminar. The article is intended as a "resource paper," rather than a formal exposition, and accordingly I have omitted proofs that can be found in easily accessible sources.

1. Derivatives all positive. We are concerned with real functions that have derivatives of all orders. The field we are considering began in 1914, when S. Bernstein proved that if $f^{(k)}(x) \geqq 0$ for all x on $[a, b]$, then f is real-analytic, in fact is the restriction of a function that is analytic in a disk centered at a and of radius $b-a$. (We shall usually disregard the distinction between an analytic function and its restriction to the real axis, and simply say "f is analytic in a disk" in this case.) The rather simple proof is reproduced in a number of places, for example [24], p. 146; [6], p. 155. (It is somewhat harder to show that it is enough just to assume that $f^{(k)}(x) \geqq 0$ for $k \geqq n(x)$, where $n(x)$ may depend on x.)

A function with all derivatives nonnegative is called **absolutely monotonic**. A function whose successive derivatives alternate in sign, so that $(-1)^n f^{(n)}(x) \geqq 0$, is called **completely monotonic**; the change of variable $y=b+a-x$ converts a member of one class into a member of the other. Naturally a completely monotonic function is analytic in a disk centered at b.

Many of the familiar functions that occur in calculus are either absolutely or completely monotonic, and Bernstein's theorem then provides an immediate proof that they are represented by their power series. Obvious examples are e^x, e^{-x}, and $(1-x)^{-1}$. Although $(1-x)^r$ is not necessarily absolutely monotonic, one of its derivatives of sufficiently high order is so, and we obtain an easy proof of the binomial theorem for general real exponents. On the other hand, although $\tan x$ is absolutely monotonic on $[0, \pi/2)$, it would not be easy to establish this by direct inspection of the derivatives of $\tan x$.

A function that is absolutely monotonic on $[0, \infty)$ is the restriction of an **entire function** (one that is analytic in the whole finite complex plane). On the other hand, when a function is completely monotonic on $[0, \infty)$, as are $1/(x+1)$ and e^{-cx} $(c>0)$, the statement about where the function is analytic has to be modified; what is in fact true is that the function is analytic in a right-hand half-plane. A little thought shows that a function defined by a convergent Laplace integral of the form

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt, \qquad \phi(t) \geqq 0,$$

or more generally by a convergent Laplace-Stieltjes integral

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t), \qquad \alpha \text{ increasing,} \tag{1}$$

is completely monotonic where it converges. The converse is also true: Bernstein and Widder discovered independently about 1929 that *every* function completely monotonic on a half-line (a, ∞) is a Laplace transform of the form (1). There is a full account of the subject (which is neither elementary nor wholly germane to this article) in [**24**].

It is interesting, and sometimes useful, to know that a function, initially known only to be continuous, is absolutely monotonic if all its differences are nonnegative, that is,

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x+jh) \geqq 0 \tag{2}$$

for all nonnegative integers k, for all x and all positive h such that the points $x+jh$ that occur in (2) are in the domain of f ([**24**], Chap. 4). Actually it can be shown that (2) for $k \leqq n$ makes f have continuous derivatives of orders up to and including $n-2$; this is quite elementary, but not entirely trivial [**9**].

Bernstein's theorem on absolutely monotonic functions has been extended to functions with domain of dimension greater than 1 [**19**], and even to functions with infinite-dimensional domain [**22**].

2. Each derivative has a fixed sign. Perhaps the most natural next step is to consider functions for which each derivative is of fixed sign on $[a, b]$, without regard to how the signs are distributed. Bernstein did this; he called such functions **regularly monotonic,** and showed that a regularly monotonic function is always analytic ([**1**], pp. 196–197). However, the function does not have to be analytic in as large a region as in the absolutely monotonic case. Bernstein's proof is a remarkable application of the elements of the theory of the approximation of continuous functions by polynomials; since there does not appear to be any readily accessible account of it, I reproduce it here, giving the necessary background in an appendix.

LEMMA 1 (see p. 1090). *If* $f^{(n)}(x) \geqq N > 0$ *on an interval* I *of length* $2h$, *and* M *is the maximum of* $|f(x)|$ *on* I, *then* $M \geqq 2N(h/2)^n/n!$.

Suppose now that $f^{(n+1)}(x)$ has constant sign on I; then $f^{(n)}(x)$ is monotonic (either increasing or decreasing). Let t be any point of I; we may suppose $f^{(n)}(t) > 0$ (otherwise consider $-f(x)$). Since $f^{(n)}$ is monotonic there is either an interval $(t, t+\epsilon)$ or an interval $(t-\epsilon, t)$ on which $f^{(n)}(x) > f^{(n)}(t) > 0$ (where ϵ can be taken as the distance from t to the nearer endpoint of I). Hence by Lemma 1

$$M > 2|f^{(n)}(t)|(\epsilon/4)^n/n!$$

for each t in I. That is,

$$|f^{(n)}(t)| \leqq \tfrac{1}{2} n! M (4/\epsilon)^n. \tag{3}$$

If we expand $f(x)$ in a Taylor series about a point s of I, the usual estimate for the remainder after n terms yields

$$|R_n| \leqq |x-s|^n \max|f^{(n)}(t)|/n!,$$

where the maximum is taken for t between s and x. If then $|x-s|$ is less than one-fourth the smaller of the distances from s and from x to the endpoints of I, we obtain $R_n \to 0$.

Bernstein next asked what happens if not all $f^{(n)}(x)$ have constant sign on I, but infinitely many of them do. He showed that f then always has a "quasi-analytic" property, namely that f is determined throughout I by its values on an arbitrarily short subinterval. If enough of the $f^{(n)}$ have constant sign, f is still analytic; more precisely, this happens when $f^{(n_k)}(x) \geqq 0$ with n_{k+1}/n_k bounded; for example, if $f^{(2k)}(x) \geqq 0$ or even if $f^{(2^k)}(x) \geqq 0$, but not if $f^{(k!)}(x) \geqq 0$. (See [**4**].)

3. Sequence of derivatives of fixed sign. These results of Bernstein's are far from simple to establish. In the summer of 1940, Widder asked me if I knew a simple proof that f is analytic when $f^{(2n)}(x) \geqq 0$. I was not immediately able to provide one, but I suggested that he try to use Lidstone series, which are series of polynomials, two of each odd degree, with coefficients $f^{(2n)}(1)$ and $f^{(2n)}(0)$. This seems, in retrospect, not a very sensible suggestion because what is "really" involved in Lidstone series is $(-1)^n f^{(2n)}(1)$ and $(-1)^n f^{(2n)}(0)$. It did not, in fact, lead to a proof of the theorem in question; but it led to the quite unexpected result that if the even derivatives of f alternate in sign on (0, 1), i.e., if $(-1)^n f^{(2n)}(x) \geqq 0$, then f is represented by a Lidstone series and consequently is not only the restriction of an analytic function, but of one that is entire and of slow growth. (See [**24**], pp. 177–179.) More precisely, it satisfies $|f^{(n)}(x)| \leqq A\pi^n$, where A does not depend on n, and hence is what is known as an entire function of **exponential type,** satisfying $|f(z)| \leqq Ae^{\pi|z|}$. A function f with $(-1)^n f^{(2n)}(x) \geqq 0$ in an interval is now called **completely convex.** The discovery of completely convex functions led immediately to a few years of intense development of related results, after which the field became rather inactive, although a number of open problems remain.

There is an elementary proof of Widder's theorem ([**24**], p. 177). It involves repeated integration by parts in

$$\int_0^1 f(x) \sin \pi x dx;$$

this leads to the necessary estimates, but only for even n. For the transition to odd n, one needs a lemma of Hadamard's:

If on $[-h, h]$ *we have* $|f(x)| \leqq A$ *and* $|f''(x)| \leqq B$, *and* $B/A > 4/h^2$, *then* $|f'(x)| \leqq 2(AB)^{1/2}$.

Proof: By Taylor's theorem with remainder of order 2,

$$f'(x) = \frac{f(x+\delta) - f(x)}{\delta} - \frac{1}{2}\delta f''(x+\theta\delta), |\theta| < 1; |f'(x)| \leqq 2A/|\delta| + |\delta|B/2.$$

If $B/A > 4/h^2$ we can take $\delta = 2(A/B)^{1/2}$.

Incidentally this lemma is one of a family of results in which one infers something about f' from information about f and f''; some references are [**14**], p. 36; [**3**]. Many generalizations of completely convex functions rely on more general (and much deeper) inequalities of the following form: Let $M_k = \max |f^{(k)}(x)|$; then if $M_n \neq 0$, there are numbers $C_{n,k}$ such that

$$|f^{(k)}(x)| \leqq C_{n,k} M_0^{1-1/n} M_n^{k/n} \qquad (0 < k < n);$$

fairly precise estimates for $C_{n,k}$ are required. At present, the $C_{n,k}$ are known explicitly for functions whose domain is $(-\infty, \infty)$ [**12**], and for those whose domain is $[0, \infty)$ [**20**]; various estimates are known when the domain is a finite interval ([**8**], and references given there).

The idea of the elementary proof of Widder's theorem was used by Pólya [**15**] to show that f is an entire function of exponential type if $f^{(k)}(x) \sin (k+1)\alpha \geqq 0$ for some α $(0<\alpha<\pi)$ and all k; it seems that more general results of this kind have not been studied. The elementary proof, however, does not work even for such a regular case as $(-1)^k f^{(4k)}(x) \geqq 0$, and it does not give any insight into why completely monotonic and completely convex functions behave so differently.

At this point we should observe something that had been forgotten in 1940, namely that about 1928 Bernstein had already found that the distribution of the signs of successive derivatives has a decisive influence on the behavior of a regularly monotonic function. More precisely, the significant property in his work is the distribution of successive blocks of either constant or alternating sign. If each derivative has a fixed sign, each derivative is monotonic; the property is easier to state in terms of whether $|f^{(n)}(x)|$ increases or decreases (as was suggested by Pólya). Since the derivative of $[f^{(n)}(x)]^2$ is $2f^{(n)}(x)f^{(n+1)}(x)$, we have $|f^{(n)}(x)|$ increasing if $f^{(n)}(x)$ and $f^{(n+1)}(x)$ have the same sign, $|f^{(n)}(x)|$ decreasing if $f^{(n)}(x)$ and $f^{(n+1)}(x)$ have opposite signs. We consider successive blocks where $|f^{(n)}(x)|$ increases or decreases; for example, when the signs of successive derivatives are $+++-+-+----+-+-+++$, etc., the lengths of the successive blocks are 2, 5, 2, 5, etc. For $\sin x$ on $(0, \pi/2)$, the signs are $++--++--$, etc., and all blocks are of length 1. For an absolutely or completely monotonic function, there is just one block, of infinite length. A convenient way to see where one block ends and the next begins is to notice that $f^{(n)}$ and $f^{(n+1)}$ belong to different blocks if and only if $f^{(n)}(x)f^{(n+2)}(x)<0$. (See [**16**], p. 185.) (Functions with the signs of successive derivatives distributed periodically, for example like those of $\sin x$ on $(0, \pi/2)$, are called **cyclically monotonic**; they have a substantial literature of their own.) The general lesson of Bernstein's results (which will be stated in greater detail below) is that the presence of many blocks makes the function behave regularly, and more regularly when the blocks are short.

However, Bernstein's results do not establish Widder's theorem since that theorem has no hypothesis at all about the derivatives of odd order. The following general theorem, which includes both Bernstein's and Widder's results, appeared two years after Widder's theorem [**8**].

Let $\{n_k\}$ *and* $\{q_k\}$ *be increasing sequences of positive integers such that* $q_1+q_2+\cdots+q_k=O(n_k)$, *and suppose that* $f^{(n_k)}(x)$ *and* $f^{(n_k+2q_k)}(x)$ *have opposite signs (so that it is derivatives with orders differing by an even integer that have opposite signs). Then if*:

$n_k-n_{k-1}=o(n_k)$ *and* $q_k=o(n_k)$, f *is entire*;
$n_k-n_{k-1}=O(n_k^{(\rho-1)/\rho})$ *and* $q_k=o(n_k)$, f *is entire and of order at most* ρ;
$n_k-n_{k-1}=O(1)$ *and* $q_k=O(1)$, f *is entire and of exponential type.*

For example, when $n_k=2k$ and $q_k=1$ we have completely convex functions; when $n_k=4k$ and $q_k=2$, we have functions such that $(-1)^kf^{(4k)}(x)\geqq 0$; when $l_1+l_2+\cdots+l_k=n_k$ (l_k is the length of the kth block for a regularly monotonic function), and $q_k=1$, we get Bernstein's results on blocks of signs. When $n_k=k^2$ and $q_k=k$ the theorem says that if $f^{(k^2)}(x)$ and $f^{((k+1)^2-1)}(x)$ always have opposite signs, for example if $f'(x), f^{(4)}(x), f^{(9)}(x), \cdots \geqq 0$ and $f(x), f'''(x), f^{(8)}(x), \cdots \leqq 0$, then f is an entire function of order at most 2.

It is interesting that in spite of the apparent generality of this theorem, there still are theorems with simple statements that it does not cover. Indeed, Leeming and Sharma [**13**] have shown that f is entire and of exponential type if $(-1)^kf^{(pk)}(x)\geqq 0$ and $(-1)^kf^{(pk+l)}(a)\geqq 0$ for $l=1, 2, \cdots, p-2$. Note that nothing is said about the sign of $f^{(pk-1)}(x)$, and nothing about the intermediate derivatives except at one point; we are again outside the domain of Bernstein's results. Leeming and Sharma base their proof on generalized Lidstone series; it would be interesting to have a direct elementary proof.

So far we have dealt with functions such that each derivative, or each of a sequence of derivatives, has no zeros on an interval. Suppose instead that no derivative changes sign more than a prescribed number of times, say that $f^{(n)}(x)$ has at most N_n zeros. It is reasonable to suppose that f will be more well-behaved when N_n is small. In fact, in 1943 Schaeffer [**18**] showed that if N_n is bounded for $a\leqq x\leqq b$, then f is analytic there. A considerable number of further results were obtained between 1940 and 1943; see [**16**]. Since 1943 the field has been rather inactive. See, however, [**17**] for some generalizations of completely convex functions, and [**5**] for a representation of completely convex functions. There are still a number of open questions. For example, suppose that $f^{(n_k)}(x)$ has at most N zeros for a sequence $\{n_k\}\neq\{k\}$? Suppose that $f^{(k)}(x)$ has at most $N(k)$ zeros? Suppose that $f^{(n)}(x)\geqq 0$ only in an interval I_n, where the length of I_n does not decrease too fast? Completely monotonic sequences $\{\mu_n\}$ are defined by having their differences of alternating sign; they have a complete theory (see [**24**], Chap. 3). Nothing seems to be known about completely convex sequences.

Appendix. *Proof of Lemma 1.* Except for Chebyshev's theorem on best approximation, the proof is entirely elementary, although rather exacting. Our version is expanded from the outline given in [**1**], pp. 8–10.

LEMMA 2 (Chebyshev's theorem). *If f is a real continuous function on a finite interval $[a, b]$, there is a unique polynomial P_n of degree at most n that approximates f most closely on $[a, b]$, in the sense that $\max_x|f(x)-P_n(x)|$ is as small as possible. The minimum is denoted by $E_n[f]$. The polynomial P_n is characterized by the property that $f(x)-P_n(x)$ has at least $n+2$ extrema on $[a, b]$, where $|f(x)-P_n(x)| = E_n[f]$ and the signs of $f(x)-P_n(x)$ alternate at successive extrema.*

A proof of Lemma 2 can be found in almost any book on the theory of approximation, for example [**10**] or [**21**]. The proof of Lemma 1 exploits all the information furnished by Lemma 2.

LEMMA 3. *If $g^{(n+1)}(x)$ is continuous and strictly positive in $[a, b]$, and $g(x)$ has exactly $n+1$ changes of sign in (a, b), then $g(b)>0$.*

Since $g(x)$ has $n+1$ changes of sign, $g'(x)$ has at least n, $g''(x)$ has at least $n-1$, and so on. Finally, $g^{(n)}(x)$ has at least one, and it cannot have more, since $g^{(n+1)}(x)$ has none. Let $g^{(n)}(x)$ change sign at y_1, where of course $g^{(n)}(y_1)=0$. If $x>y_1$,

$$g^{(n)}(x) = \int_{y_1}^{x} g^{(n+1)}(t)dt > 0.$$

Similarly, $g^{(n-1)}(x)$ has two changes of sign, say at z_1, z_2 with $z_1<y_1<z_2$; and so, for $x>z_2$, $g^{(n-1)}(x)>0$. This clearly starts an induction that winds up with $g(b)>0$.

LEMMA 4. *If ϕ and f have continuous $(n+1)$-th derivatives on $[a, b]$ and $0<\phi^{(n+1)}(x)<f^{(n+1)}(x)$ on $[a, b]$, then $E_n[\phi]<E_n[f]$.*

Let P_n, Q_n be the polynomials of degree n of best approximation to ϕ and f, respectively; write $D_n(x)=\phi(x)-P_n(x)$. Then D_n has at least $n+2$ extrema with alternating signs and so changes sign at $n+1$ points at least. If D_n had more than $n+1$ changes of sign, $D_n^{(n+1)}$ would have at least one, but $D_n^{(n+1)}(x)=\phi^{(n+1)}(x)>0$. Hence D_n has exactly $n+1$ changes of sign and D_n' has exactly n; and D_n has exactly $n+2$ extrema on the closed interval. Next, two of the extrema of D_n are at a and b, since otherwise D_n would have at least $n+1$ interior extrema, each of which would be a zero of D_n'. (None of these points could be anything but a simple zero of D_n', since otherwise D_n' would have at least $n+2$ zeros (counting multiplicity), and $(D_n')^{(n)}=D_n^{(n+1)}=\phi^{(n+1)}$ would have a zero, contrary to hypothesis.) Then D_n' would have at least $n+1$ changes of sign, whereas we know that it has exactly n.

Now suppose, contrary to what we want to prove in Lemma 4, that $0<\phi^{(n+1)}(x)<f^{(n+1)}(x)$ on $[a, b]$ and $|f(x)-Q_n(x)|<E_n[\phi]$ on $[a, b]$, where Q_n is the polynomial (of degree at most n) of best approximation to f. Then at the points where $|\phi(x)-P_n(x)|=E_n[\phi]$, the function

$$F(x) = \phi(x) - P_n(x) - f(x) + Q_n(x) = D_n(x) - [f(x) - Q_n(x)]$$

has the sign of $\phi(x)-P_n(x)$. We already know that $D_n(x)$ has exactly $n+1$ changes of sign on (a, b); hence so does $F(x)$. Now

$$F^{(n+1)}(x) = \phi^{(n+1)}(x) - f^{(n+1)}(x) < 0$$

on $[a, b]$ by hypothesis, and so by Lemma 3 (applied to $g=-F$), $F(b)<0$. But $F(b)$ has the sign of $\phi(b)-P_n(b)=D_n(b)$, since b is a point where $|D_n(x)| = E_n[\phi]$. By Lemma 3 applied to $D_n(x)$, which has $D^{(n+1)}(x)=\phi^{(n+1)}(x)>0$, we must have $D_n(b)>0$, contradicting the facts that $F(b)<0$ and that $F(b)$ and $D_n(b)$ have the same sign.

This shows that we cannot have $E_n[f]<E_n[\phi]$; to complete the proof of Lemma 4, we need to know that $E_n[f]=E_n[\phi]$ is also impossible. Suppose the contrary, and let $0<\phi^{(n+1)}(x)<f^{(n+1)}(x)$ and $E_n[f]=E_n[\phi]$. Let λ be a positive number and consider $\phi(x)+\lambda f(x)=g(x)$. Since $g^{(n+1)}(x)=\phi^{(n+1)}(x)+\lambda f^{(n+1)}(x)$, we have $(1+\lambda)f^{(n+1)}(x)>g^{(n+1)}(x)>(1+\lambda)\phi^{(n+1)}(x)$, and hence by what has already been proved, $(1+\lambda)E_n[\phi]\leqq E_n[g]\leqq(1+\lambda)E_n[f]$. By assumption, $E_n[f]=E_n[\phi]$, so

$$E_n[g] = (1+\lambda)E_n[\phi] = (1+\lambda)E_n[f] = E_n[\phi] + E_n[f].$$

Let P_n and Q_n be the polynomials realizing $E_n[\phi]$ and $E_n[f]$; then

$$\begin{aligned} |P_n(x) + \lambda Q_n(x) - g(x)| &= |P_n(x) - \phi(x) + \lambda[Q_n(x) - f_n(x)]| \\ &\leqq E_n[\phi] + \lambda E_n[f]. \end{aligned}$$

But $E_n[g]=E_n[\phi]+E_n[f]$, so that no polynomial S_n of degree n (or less) can make $\max_x|S_n(x)-g(x)|$ less than this value. Since the best approximating polynomial for g is unique, it must therefore be $P_n(x)+\lambda Q_n(x)$, and consequently $|\phi(x)+\lambda f(x)-[P_n(x)+\lambda Q_n(x)]|$ attains its maximum value $E_n[\phi]+\lambda E_n[f]$ at $n+2$ points. This is possible only if $|\phi(x)-P_n(x)|$ and $|f(x)-Q_n(x)|$ attain their maximum values, namely $E_n[\phi]$ and $E_n[f]$, at the same $n+2$ points. This means that $F(x)=\phi(x)-P_n(x)-[f(x)-Q_n(x)]$ has $n+2$ double zeros, and so $F^{(n+1)}(x)$ has at least one zero; but we had $F^{(n+1)}(x)<0$ by hypothesis. This completes the proof of Lemma 4.

LEMMA 5 (Another theorem of Chebyshev). *On any interval of length* $2h$' $E_n[x^{n+1}]=2^nh^{n+1}$.

Another way of stating this is to say that if a polynomial of degree $n+1$ has its absolute value bounded by 1 on an interval of length $2h$, the absolute value of its leading coefficient is at most $2^{-n}h^{-n-1}$ (and can attain this value). This can be proved in a quite elementary way; see, for example, [21], p. 24, or [7].

We can now prove Lemma 1. Suppose that $f^{(n+1)}(x)>N>0$. Take $\phi(x) = Nx^{n+1}/(n+1)!$, so that $\phi^{(n)}(x)=N$; we then have $f^{(n+1)}(x)>\phi^{(n+1)}(x)>0$. By Lemma 4, $E_n[f]>E_n[\phi]$. By Lemma 5, $E_n[\phi]=2N(h/2)^{n+1}/(n+1)!$, and therefore $E_n[f]>2N(h/2)^{n+1}/(n+1)!$. If P_n is the polynomial of degree at most n that realizes $E_n[f]$, and R_n is any other polynomial of degree at most n, we have

$$E_n[f] = \max_x|f(x) - P_n(x)| \leqq \max_x|f(x) - R_n(x)|.$$

Take $R_n(x) \equiv 0$; then

$$2N(h/2)^{n+1}/(n+1)! < E_n[f] \leqq \max_x |f(x)|.$$

This establishes Lemma 1.

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