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THE SUBGROUP OF THE ELEMENTS OF FINITE ORDER OF AN ABELIAN GROUP †

BY REINHOLD BAER

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Every (additively written) Abelian group A determines two characteristic "unmixed" groups: the subgroup $F(A)$ of all the elements of finite order in A and the classgroup $I(A) = A/F(A)$. All the elements of $F(A)$ have finite order and 0 is the only element of finite order in $I(A)$. If conversely all the elements of the Abelian group F have finite order and 0 is the only element of finite order of the Abelian group J , then the direct sum $A = F + J$ satisfies $F(A) = F$ and $I(A) = J$.¹ The structure of A is therefore completely determined by the structure of $F(A)$ and of $I(A)$ if, and only if, $F(A)$ is a direct summand of every group A satisfying $F(A) = F$ and $I(A) = J$.

In the following conditions will be given for the group $F(A)$ to be a direct summand of the group A . In accordance with the above considerations we are not interested in conditions which are dependent on the structure of the whole group A , but only in such conditions which depend on the structure of $F(A)$ and (or) $I(A)$ alone.

The principal results can be found in section 8. and the concepts used for enunciating them in the sections 2 and 3.

1. Subgroups satisfying $nS = S$ for every integer n .

(1;1) *If the subgroup S of the Abelian group A satisfies $nS = S$ for every integer $n \neq 0$, then S is a direct summand of A .*

PROOF:² Let T be a greatest subgroup of A such that the intersection of S and T contains only the element 0. Then the subgroup of A which is generated by the elements of S and T is the direct sum $S + T$ of S and T . If x is any element of A , then the subgroup $T(x)$ of A , generated by x and the elements in T , contains only elements of the form $nx + t$ where n is an integer and t an element of T . By the choice of T either x is contained in T or there exists an integer m and an element t' in T such that $mx - t'$ is an element $\neq 0$ in S .

In particular to every element x in A there exists an integer $n \neq 0$ such that $nx = s + t$ is an element in $S + T$. Since $S = nS$, there exists an element s'

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¹ This is not quite correct, since the groups $I(A)$ and J are only isomorphic, but not identical. But here and in the following we identify isomorphic groups wherever that is possible without confusion, in particular always when there exists a "natural" isomorphism between a given group and a class group.

² The statement (1;1) is well known. But we prove it here, since, to the authors knowledge, it has not been published before.

in S such that $ns' = s$ and therefore the element $x' = x - s'$ will satisfy: $x' \equiv x \pmod S$ and $nx' = t$. If X is the set of all the elements y such that $y \equiv x \pmod S$ and $ny \equiv 0 \pmod T$ for a certain positive integer n , then X is not vacuous and contains elements w satisfying $vw \equiv 0 \pmod T$ with a smallest positive v . But then there exists, as noted before, an integer u and elements $a \neq 0$ in S , b in T such that $uw = a + b$ provided w is not contained in T . If (u, v) is the greatest common divisor of u and v , then there exist integers h and k such that $(u, v) = hu + kv$ and therefore $(u, v)w = ha + (kvw + hb) \equiv ha \pmod T$. It follows as before that X contains an element w' satisfying $(u, v)w' \equiv 0 \pmod T$ and therefore that $v = (u, v)$ i.e. u is a multiple of v . But that leads to a contradiction with $a \neq 0$ and therefore w is contained in T , i.e. x is congruent mod S to an element of T , i.e. $A = S + T$.

2. Groups without elements of infinite order.³ If F is an Abelian group without elements of infinite order, then to every prime number p there belongs the primary-component $C(p, F)$ of F which consists of all those elements of F whose order is a power of p . F is the direct sum of its primary-components $C(p, F)$ and therefore every element x of F is the sum of its uniquely determined primary-components $c(p, x)$ contained in $C(p, F)$ respectively.

Let now B be a primary Abelian group, belonging to the prime number p . Then $p^i B$ is the intersection of all the groups $p^i B$ for positive integers i .

(2;1) *If the orders of the elements of $B/p^i B$ are bounded, then $p^i B = p(p^i B)$. $p^i B$ is therefore (by (1;1)) a direct summand of B and B a direct sum of $p^i B$ and of cyclic groups (of bounded orders).*

(2;2) *If the orders of the elements of $B/p^i B$ are not bounded, then there exist elements $a(i)$ in B such that the orders of the elements $b(i) = \sum_{j=0}^{i-1} p^j a(j)$ tend to infinity with i and such that $b(i)$ is an element of lowest order in its class mod $p^i B$.*

Since the primary group B belonging to the prime number p satisfies $qB = B$ for every prime number $q \neq p$, the statement (1;1) implies:

(2;3) *Let $Q(A)$ be the direct sum of the groups $C(p, F(A))$ which satisfy $pC(p, F(A)) = C(p, F(A))$. Then $Q(A)$ is a direct summand of A .*

It is now easy to give an example of a group A such that $F(A)$ is not a direct summand of A .⁴ For let B be a primary group such that the orders of its elements are not bounded and such that $p^i B = 0$. The sets $K(i)$ for $i = 1, 2, \dots$ form an L -series of B , if $K(i)$ is a class of $B \pmod{p^i B}$ and $K(i) < K(i - 1)$ for every i . The intersection of all the classes of an L -series is either vacuous or contains exactly one element, since $p^i B = 0$. Therefore it is possible to identify the L -series $p^i B + x$ with the element x of B . If $K(i)$ and $H(i)$ are two L -series, then $K(i) - H(i)$ is also an L -series. The set \bar{B} of all the L -series in B is therefore an Abelian group, containing B , the p -adic closure of B .

³ This section is a compilation of mostly well-known facts.

⁴ A first example of such a group has been given by F. Levi: *Abel-sche Gruppen mit abzählbaren Elementen*, Hab.-Schrift, Leipzig, 1919.

Since the orders of the elements of B are not bounded, it follows from (2;2) that there exist elements of infinite order in B . Furthermore the only direct summand of B which contains B is B . Since $F(B)$ contains B , it follows therefore that $F(B)$ is not a direct summand of B .

3. Groups without elements of finite order. Suppose that 0 is the only element of finite order in the (Abelian) group⁵ J . If p is a prime number, then J is said to be p -complete, if $J = pJ \neq 0$. If J is p -complete for every prime number p , then J is complete.

By (1;1) every complete subgroup of J is a direct summand of J . Since the joingroup of two complete subgroups is also a complete subgroup it follows that there exists a uniquely determined greatest complete subgroup, provided there exist complete subgroups at all.

(3;1) *Every group J without elements of finite order is contained in one and essentially only one smallest complete group.*

PROOF: Let P be the set of all the pairs (x, n) for x in J and n a positive integer. Define equality in P by $x = (x, 1)$, $(x, n) = (y, m)$ if, and only if, $mx = ny$, and addition by $(x, n) + (y, m) = (mx + ny, mn)$.

This group P is a smallest complete group, containing J , and if P' is another smallest complete group, containing J , then there exists an isomorphism between P and P' which leaves all the elements in J invariant.

If G is a greatest linearly independent subset of the complete group J , (x) the subgroup of all the elements of J which are dependent on x , then J is the direct sum of all the groups (g) for g in G and every group (g) is isomorphic with the additive group of all the rational numbers. As a consequence of these facts and of (3;1) it follows:

(3;2) *If J is a group without elements of finite order, then any two greatest linearly independent subsets of J contain the same number of elements. This number is the rank of J .*

The subgroup S of the group J is closed if J/S does not contain elements of finite order, i.e. if all the elements of J which are dependent on S are contained in S .

DEFINITION 3;2: *Let J be an Abelian group without elements of finite order. If J is countable, then $D(J) = 1$. If v is any (finite or infinite) positive ordinal, then $D(J) = v$, if $D(J) \prec v$ and if there exists a closed subgroup S of finite rank such that J/S is a direct sum of groups J' with $D(J') < v$.*

Note that there exist groups J such that $D(J)$ is not defined e.g. the group of all the sequences of integers.

(3;3) *Suppose that the group J without elements of finite order contains a closed subgroup S of finite rank such that $D(J/S)$ exists. Then $D(J)$ exists and satisfies $D(J) \leq D(J/S)$.*

PROOF: If $J^* = J/S$, $D(J^*) = v$, then there exists a closed subgroup T^* of J^*

⁵ Then we say that J is a group without elements of finite order.

of finite rank such that $J^{**} = J^*/T^*$ is the direct sum of groups $J^{**}(u)$ satisfying $D(J^{**}(u)) < v$. Let $J^*(u)$ be the subgroup of J^* which corresponds to $J^{**}(u)$, T and $J(u)$ the subgroups of J corresponding to $J^*(u)$ and T^* respectively. Then T is a closed subgroup of finite rank of J and, since $J(u)/T$ and $J^{**}(u)$ are isomorphic, it follows that $D(J(u)/T) < v$. Hence J/T is the direct sum of the groups $J(u)/T$ of smaller D and therefore $D(J) \leq v$.

The following concepts will be needed later on:

Let P be an (infinite) set of prime numbers. Then the element x is *P-infinite* (in J), if $x \neq 0$ and if for an infinity of prime numbers p in P there exist solutions of the equation $py = x$ in J . If the group J contains a *P-infinite* element, then J itself is called *P-infinite* and J is called *almost P-infinite* if there exists a closed subgroup S of finite rank such that J/S is *P-infinite*.

Similarly: if p is a prime number, then J is called *almost p-infinite* if there exists a closed subgroup S of finite rank such that J/S contains a *p-complete* subgroup.

If e.g. J is the additive group of all the integer p -adic numbers, then $D(J) = 2$ and J is almost p -infinite but does not contain any p -complete subgroup.

4. Extension of groups.

(4;1) *If A is any Abelian group, $x^* \neq 0$ an element of $A/F(A)$ and p a prime number, then there exists a group A' such that*

- (a) $A \leq A'$,
- (b) $F(A) = F(A')$,
- (c) *the equation $py^* = x^*$ has a solution y^* in $A'/F(A)$,*
- (d) *either $A = A'$ or A'/A is a group of order p .*

PROOF: If the equation $py^* = x^*$ has a solution y^* in $A/F(A)$, then $A' = A$ satisfies the conditions (a) – (d). If on the other hand the equation $py^* = x^*$ has no solution y^* in $A/F(A)$, let x be any element in A such that $x^* = F(A) + x$ and let A' denote the group which contains A and an element y such that $py = x$. Then every element of A' has the form $a + cy$ for a in A and $0 \leq c < p$ where a, c are uniquely determined by the element $a + cy$. Suppose now that n is a positive integer such that $n(a + cy) = 0$. If $c = 0$, then a is an element of $F(A)$. If $c \neq 0$, then c and p are relatively prime and, since $-na = ncy$, it follows that nc and therefore n is a multiple of p , i.e. $n = pm$ and hence $0 = m(pa + cx)$. Therefore $pa + cx$ is (as an element of A) contained in $F(A)$ or $p(-a) \equiv cx \pmod{F(A)}$. But since p is prime to c , this implies that $py^* = x^*$ has a solution y^* in $A/F(A)$ and this is impossible. Therefore $F(A) = F(A')$, i.e. A' satisfies the conditions (a) – (d).

(4;2) *If A is any Abelian group, then there exists an Abelian group A' such that*

- (a) $A \leq A'$,
- (b) $F(A) = F(A')$
- (c) $A'/F(A')$ *is complete, [or 0],*
- (d) *all the elements of A'/A are of finite order.*

PROOF: By well-ordering the elements of $A/F(A)$ and by applying (4:1) successively on all the elements of $A/F(A)$ it follows that there exists a group A^* such that

- (a) $A \leq A^*$,
- (b) $F(A) = F(A^*)$,
- (c) the equation $py^* = x^*$ has a solution y^* in $A^*/F(A)$ for every element $x^* \neq 0$ in $A/F(A)$ and every prime number p ,
- (d) A^*/A contains only elements of finite order.

To A^* there exists an analogous group $(A^*)^* = A^{**}$ and so on and the join A' of all these groups satisfies the conditions of (4:2).

(4:3) *Suppose that A is an Abelian group, J a group without elements of finite order and $A/F(A)$ a subgroup of J . Then there exists a group A' such that*

- (a) $A \leq A'$,
- (b) $F(A) = F(A')$,
- (c) $A'/F(A) = J$ (i.e. there exists an isomorphism between $A'/F(A)$ and J which leaves invariant the elements of $A/F(A)$).

PROOF: By (4:2) there exists a group A'' such that $A \leq A''$, $F(A) = F(A'')$, $A''/F(A)$ is complete and is the (by (3:1) essentially uniquely determined) smallest complete group which contains $A/F(A)$. Furthermore there exists by (3:1) an essentially uniquely determined smallest complete group C which contains J . Since $A/F(A)$ is a subgroup of J , there exists exactly one smallest complete subgroup S of C which contains $A/F(A)$ which is therefore essentially identical with $A''/F(A)$. By (1:1) S is a direct summand of C , i.e. $C = S + T$. Now it is easy to see that there exists a group A' between A'' and $A'' + T$ which satisfies the conditions (a) – (c).

(4:4) *If F is a group without elements of infinite order, J a group without elements of finite order and $F(A)$ is a direct summand of every group A such that $F(A) = F$ and $A/F(A) = J$, then $F(B)$ is a direct summand of every group B such that $F(B) = F$ and $B/F(B) \leq J$.*

PROOF: By (4:3) B is contained in a group A such that $F(A) = F(B) = F$ and $A/F(A) = J$. $F(A)$ is therefore from the assumption a direct summand of A , i.e. $A = H + F(A)$. If K denotes the intersection of H and B , then $B = K + F(B)$, since $F(A) = F(B)$.

(4:5) *If F is a group without elements of infinite order, J a group without elements of finite order, and if $F(A)$ is a direct summand of every group A such that $F(A) = C(p, F)$ and $A/F(A) = J$, then $C(p, F(B))$ is a direct summand of every group B such that $F(B) = F$ and $B/F(B) = J$.*

PROOF: If Q is the direct sum of all the $C(q, F)$ with $q \neq p$, $A = B/Q$, then $F(A)$ is essentially $= C(p, F)$, since every class of $F(B)/Q$ contains exactly one element of $C(p, F)$. Furthermore $A/F(A) = J$. $F(A)$ is therefore by the assumption a direct summand of A , i.e. $A = H + F(A)$. Let K be the subgroup of B corresponding to H (such that $Q \leq K$ and $K/Q = H$). The intersection of K and $C(p, F)$ equals the intersection of Q and $C(p, F)$ and consists

therefore of the 0-element alone. B is generated by the elements in $C(p, F)$ and in K , since they represent all the classes of B/Q . Hence $B = C(p, F) + K = C(p, F(B)) + K$.

5. The first necessary condition.

(5;1) *Suppose that the group J without elements of finite order contains an almost p -complete subgroup. Then J contains two subgroups S, T and elements $e(i)$ such that*

- (a) S is a closed subgroup of finite rank of T ,
- (b) S does not contain any almost p -infinite subgroup,
- (c) T is generated by the elements $e(i)$ and the elements in S ,
- (d) $e(0) \not\equiv 0 \pmod S, pe(i) \equiv e(i - 1) \pmod S$.

PROOF: If V is an almost p -complete subgroup of J , then there exists a closed subgroup W of V which is of finite rank such that V/W is p -complete. Therefore V contains an element x which is not contained in W and the smallest closed subgroup of V which contains x and W is of finite rank and almost p -infinite. The existence of an almost p -infinite subgroup of J implies therefore the existence of an almost p -infinite subgroup of finite rank. Therefore there exists an almost p -infinite subgroup U of smallest (finite) rank. Since U is almost p -infinite, it contains a subgroup S (closed in U) such that U/S is p -complete. The rank of S is smaller (and therefore finite) than the rank of U , since $U/S \not\equiv 0$, and therefore S cannot contain any almost p -infinite subgroup. If $e(0)$ is an element of U which is not contained in S (and such an element exists) then there exist elements $e(i)$ such that $pe(i) \equiv e(i - 1) \pmod S$ for $i = 1, 2, \dots$, since U/S is p -complete. The subgroup T of J which is generated by S and the elements $e(i)$ clearly satisfies (together with S and the elements $e(i)$) the conditions (a) - (d).

(5;2) *Suppose that the group J is of finite rank and not almost p -infinite. Then to every greatest linearly independent subset G of J there exists an integer w such that an equation $p^{w+k}x = \sum_{g \text{ in } G} c(g)g$ with integer coefficients $c(g), 0 < k, (x \text{ in } J)$ implies that $c(g) \equiv 0 \pmod{p^k}$ for every g in G .*

PROOF: G is finite, since J is of finite rank. Furthermore it is sufficient to prove the statement for $k = 1$. If this special case of (5;2) would not be true, then there would exist to every positive integer i an element $x(i)$ in J and integers $w(g, i)$ not all of them $\equiv 0 \pmod p$ such that

$$p^i x(i) = \sum_{g \text{ in } G} w(g, i)g.$$

Since G is finite, there exists an element h in G such that $w(h, i) \not\equiv 0 \pmod p$ for an infinity of numbers i . Since for these values of i the prime number p and $w(h, i)$ are relatively prime, it follows easily that there exist elements $y(i)$ in J and integers $v(g, i)$ such that

$$p^i y(i) = h + \sum_{\substack{g \text{ in } G \\ g \neq h}} v(g, i)g.$$

If H is the smallest closed subgroup of J which contains the elements $g \neq h$ in G , then $H < J$ and $p^i y(i) \equiv h \pmod H$, i.e. J is almost p -infinite and this is impossible.

THEOREM 5;3: *Suppose that g is a prime number, that F is a group without elements of infinite order, that J^* is a group without elements of finite order and that*

- (1) *the orders of the elements of $C(g, F)/g^*C(g, F)$ are not bounded,*
- (2) *J^* is almost g -infinite.*

Then there exists a group A such that $F(A) = F$, $A/F(A) = J^$ and such that $C(g, F(A))$ is not a direct summand of A .*

Since $C(g, F(A))$ is a direct summand of $F(A)$, this implies that $F(A)$ is not a direct summand of A .

PROOF: Because of (2) and (5;1) there exists a subgroup T^* of J^* , a subgroup S^* of T^* and elements $e^*(i)$ in T^* such that

- S^* is closed in T^* and of finite rank,
- S^* is not almost g -infinite,
- $e^*(0)$ is not contained in S^* ,
- $s^*(i) = ge^*(i) - e^*(i - 1)$ is contained in S^* (for $0 < i$),
- S^* and the elements $e^*(i)$ generate T^* .

Let S be a group isomorphic with S^* and let always s and s^* correspond under this isomorphism.

By (1) and (2;1) there exist in $C(g, F)$ elements $a(i)$ such that the elements $b(i) = \sum_{j=0}^{i-1} g^j a(j + 1)$ are elements of lowest order in their class mod $g^i C(g, F)$ but such that their orders tend to infinity with i .

Now let T be a group containing the direct sum $F + S$ and elements $e(i)$ such that

$$ge(i) - e(i - 1) = s(i) + a(i)$$

for every positive i .

This (Abelian) group T satisfies: $F(T) = F$ and $T/F(T) = T^*$.

Suppose now that $C(g, F(T)) = C(g, F)$ is a direct summand of T . Then there exists a subgroup H of T such that $T = H + C(g, F)$. Let Q be the direct sum of all the primary-components $C(p, F)$ with $p \neq g$. Then there exists to every element x in $Q + S$ an element $f(x)$ in $C(g, F)$ and to every element $e(i)$ an element $f(i)$ in $C(g, F)$ such that $x + f(x)$ and $e(i) + f(i)$ are contained in H and these elements f in $C(g, F)$ are uniquely determined by this condition. Since H and $Q + S$ are subgroups of T , it follows therefore that

$$f(x + y) = f(x) + f(y) \text{ for } x \text{ and } y \text{ in } Q + S.$$

If x is an element in Q , the orders of x and of $f(x)$ are relatively prime and the equality for the function $f(z)$ implies therefore that $f(x) = 0$, if x is contained in Q .

(*) *The orders of the elements $f(x)$ for x in $Q + S$ are bounded.*

In order to prove this statement consider a greatest linearly independent sub-

set G of S . There exists an integer w by (5;2) such that $g^{w+k}x = \sum_{h \text{ in } \sigma} c(h)h$ for x in S and integers $c(h)$, $0 < k$, implies that all the $c(h)$ are divisible by g^k .

If now x is any element in $Q + S$, then $x = y + s$ for y in Q and s in S and therefore as noted before: $f(x) = f(s)$. If $s \neq 0$, then there exists an integer v and integers $v(h)$ such that $vs = \sum_{h \text{ in } \sigma} v(h)h$ and v and the numbers $v(h)$ have no common divisor. But this implies that v is not divisible by g^{w+1} and since G is finite and $vf(s) = \sum_{h \text{ in } \sigma} v(h)f(h)$, we have proved (*).

Since furthermore

$$\begin{aligned} g(e(i) + f(i)) &= e(i - 1) + s(i) + a(i) + gf(i) \\ &= e(i - 1) + f(i - 1) + s(i) + f(s(i)), \end{aligned}$$

we have:

$$(**) \quad gf(i) - f(i - 1) = f(s(i)) - a(i).$$

By (*) there exists a number v such that $g^v f(x) = 0$ for x in $Q + S$. Then it follows from (**) that

$$\begin{aligned} g^{v+i}f(v + i) - f(0) &= \sum_{j=0}^{v+i-1} g^j(f(s(j + 1)) - a(j + 1)) \\ &= \sum_{j=0}^{v-1} g^j f(s(j + 1)) - \sum_{j=0}^{v+i-1} g^j a(j + 1) \end{aligned}$$

since the elements $s(j)$ are contained in $Q + S$. But this equation implies

$$f(0) + \sum_{j=0}^{v-1} g^j f(s(j + 1)) \equiv b(v + i - 1) \pmod{g^{v+i}C(g, F)}$$

and this congruence contradicts the choice of the elements $a(i)$, since its left side does not depend on i . Hence $C(g, F(T))$ is not a direct summand of T .

By (4;3) T is contained in a group A such that $F(A) = F(T) = F$ and $A/F(A) = J$. Then $C(g, F(A))$ is not a direct summand of A , since otherwise $C(g, F(A)) = C(g, F(T))$ would be a direct summand of T and this is impossible as proved before. This completes the proof of our Theorem.

6. The second necessary condition.

(6;1) *Suppose that P is an infinite set of prime numbers and that the group J without elements of finite order is almost P -infinite. Then there exists an infinite subset P^* of P , a pair of subgroups S and T of J and to every prime number p in P^* an element $e(p)$ such that*

- (a) S is a closed subgroup of finite rank of T ,
- (b) S is not almost P -infinite,
- (c) T is generated by the elements $e(p)$ and by the elements in S ,
- (d) $pe(p) \equiv qe(q) \not\equiv 0 \pmod{S}$ for every pair p, q of prime numbers in P^* .

PROOF: Since the group J is almost P -infinite, there exists a closed subgroup U of finite rank of J , an infinite subset P' of P and elements $v(p)$ for every p in P'

such that $pv(p) \equiv qw(q) \not\equiv 0 \pmod V$ for every pair p, q of prime numbers in P' . The group V , generated by U and the elements $v(p)$, is an almost P -infinite subgroup of finite rank of J and therefore J contains an almost P -infinite subgroup W of lowest rank. W contains a closed subgroup S such that W/S is P -infinite (and of rank 1). Since the rank of S is lower than the rank of W , S is not almost P -infinite. Since W is almost P -infinite, there exists an infinite subset P^* of P and elements $e(p)$ for p in P^* such that $e(p)$ is contained in W and satisfies: $pe(p) \equiv qe(q) \not\equiv 0 \pmod S$ for every pair p, q of prime numbers in P^* . S, P^* , the elements $e(p)$ and the subgroup T , generated by S and the elements $e(p)$, satisfy (a) – (d).

(6;2) *Suppose that the group J without elements of finite order is not almost P -infinite, that J is of finite rank and that G is a greatest linearly independent subset of J . Then G is independent mod pJ for almost every prime number p in P .*

PROOF: Let W be the set of all the prime numbers p in P such that G is not independent mod pJ . Then to every prime number p in W there exists an element $w(p)$ in J and integers $w(p, g)$ such that

$$pw(p) = \sum_{g \text{ in } G} w(p, g)g, 0 \leq w(p, g) < p, \text{ not every } w(p, g) = 0.$$

If W is infinite, then there exists, since G is finite, an element h in G and an infinite subset H of W such that $w(p, h) \not\equiv 0$ for every p in H . Since $w(p, h)$ and p are relatively prime, we can assume without loss of generality that $w(p, h) = 1$ for every p in H . If V is the smallest closed subgroup of J generated by the elements $\neq h$ in G , then

$$pw(p) \equiv h \not\equiv 0 \pmod V,$$

i.e. J would be almost H -infinite and therefore almost P -infinite and this is impossible. Hence W is finite and this proves (6;2).

THEOREM 6;3: *Suppose that F is a group without elements of finite order that the set P of all the prime numbers p with $pC(p, F) \neq C(p, F)$ is infinite and that the group J^* without elements of finite order is almost P -infinite. Then there exists a group A such that $F(A) = F, A/F(A) = J^*$ and such that $F(A)$ is not a direct summand of A .*

PROOF: By (6;1) there exists an infinite subset W of P and a pair S^*, T^* of subgroups of J^* and elements $e^*(p)$ for p in W such that

- S^* is a closed subgroup of finite rank of T^* ,
- S^* is not almost P -infinite,
- T^* is generated by the elements $e^*(p)$ and by the elements of S^* ,
- $pe^*(p) \equiv qe^*(q) \not\equiv 0 \pmod{S^*}$ for every pair p, q of prime numbers in W .

Then there exists an element e^* in T^* such that

$$pe^*(p) - e^* = s^*(p) \text{ is an element of } S^*.$$

Furthermore there exists to every prime number p in W an element $f(p)$ in $C(p, F)$ which is not contained in $pC(p, F)$.

Let S be a group isomorphic with S^* and let always s and s^* correspond under this isomorphism.

T is the group containing the direct sum $F + S$ and elements $e(p)$, e for p in W such that

$$pe(p) - e = s(p) + f(p).$$

This (Abelian) group T satisfies: $F(T) = F$ and $T/F(T) = T^*$.

Suppose now that $F(T)$ is a direct summand of T . Then $T = F(T) + H$ for a suitable subgroup H of T and therefore there exists to every element x in S an element $g(x)$ in F and to every p in W an element $g(p)$ in F and an element g in F such that $x + g(x)$ and $e(p) + g(p)$, $e + g$ are contained in H . Since S and H are subgroups of T it follows that

$$g(x + y) = g(x) + g(y) \text{ for } x \text{ and } y \text{ in } S.$$

(*) For almost every prime number p in W the relation $c(p, g(x)) = 0$ holds for every element x in S .

In order to prove this statement consider a greatest linearly independent subset G of S . Since G is finite, there exists only a finite number of prime numbers p such that

$$c(p, g(h)) \neq 0 \text{ for at least one element } h \text{ in } G.$$

Therefore it follows from (6;2) that the set V of all the prime numbers p in W satisfying

$$c(p, g(h)) = 0 \text{ for every element } h \text{ of } G$$

and

$$G \text{ is linearly independent mod } pS$$

contains almost every prime number of W .

Let now s be any element in S and p a prime number contained in V . Then there exist integers n , $d(h)$ for h in G such that

$$ns = \sum_{h \text{ in } G} d(h)h$$

and since G is independent mod pS , it can be assumed without loss of generality that n and p are relatively prime. But this relation implies

$$nc(p, g(s)) = c(p, ng(s)) = c(p, g(ns)) = \sum_{h \text{ in } G} d(h)c(p, g(h)) = 0$$

and therefore

$c(p, g(s)) = 0$ for every p in V . This proves (*), since V contains almost every element of W .

An obvious consequence of (*) is

(**) For almost every prime number p in W the relations

$$c(p, g(x)) = 0 \text{ for every } x \text{ in } S \text{ and } c(p, g) = 0 \text{ hold.}$$

Since $T = F(T) + H = F + H$, it follows that

$$\begin{aligned} p(e(p) + g(p)) - (e + g) &= s(p) + g(s(p)) \\ &= s(p) + f(p) + pg(p) - g \end{aligned}$$

or

$$f(p) + pg(p) = g(s(p)) + g$$

or

$$c(p, f(p)) + pc(p, g(p)) = f(p) + pc(p, g(p)) = c(p, g(s(p))) + c(p, g).$$

By (**) this implies for almost every p in W :

$$f(p) + pc(p, g(p)) = 0$$

and this is impossible, since $f(p)$ is contained in $C(p, F)$ but not in $pC(p, F)$. Hence $F = F(T)$ is not a direct summand of T .

The Theorem 6;3 is now an obvious consequence from the statement (4;4).

7. Sufficient conditions.

(7;1) Suppose that the subgroup B of the group A satisfies the following conditions:

- (a) A/B is a group of rank 1 without elements of finite order;⁶
- (b) If the orders of the elements of $C(p, F(B))/p^m C(p, F(B))$ are not bounded, then A/B is not p -complete;
- (c) If the set P of all the prime numbers p such that $C(p, F(B)) \neq pC(p, F(B))$ is infinite, then A/B is not P -infinite.

Then to every subgroup B' of B such that $B = F(B) + B'$ there exists⁷ a subgroup A' of A such that

$$B' \leq A' \text{ and } A = F(A) + A'.$$

PROOF: There exists a greatest subgroup C of $F(B)$ such that $pC = C$ for every prime number p . Furthermore there exists a greatest subgroup H of A such that 0 is the only element contained in both the subgroups C and H , and it is possible to choose H in such a way that H contains B' . Then as proved in (1;1) it follows that $A = C + H$. If K is the intersection of B and H , then $B = C + K$, and the pair H, K satisfies the conditions (a) – (c) exactly as the pair A, B and therefore we can assume without loss of generality:

⁶ This condition implies that $F(A) = F(B)$ and that $B/F(B)$ is a closed subgroup of $A/F(A)$.

⁷ This statement is of course vacuous if $F(B)$ is not a direct summand of the group B .

(d) If C is any subgroup of $F(B)$ such that $pC = C$ for every prime number p , then $C = 0$.

Let now $x^* \neq 0$ be any element of the group $R^* = A/B$. Condition (c) implies that there exists only a finite number of prime numbers p in P such that x^* is the p -fold of an element in R^* and therefore there exists an element $e^* \neq 0$ in R^* such that:

for every prime number p in P either R^* is p -complete or e^* is not a p -fold.

There are three classes of prime numbers to distinguish:

The finite set U of the prime numbers u in P such that R^* is u -complete.

The set V of prime numbers v which are not contained in P such that R^* is v -complete.

The set W of prime numbers w which are not contained in P such that R^* is not w -complete.

Note that some of these classes may be vacuous and that the prime numbers of P which are not contained in U are not attributed to any one of these classes.

If r is a prime number either in U or in V , then there exist (uniquely determined) elements $e^*(r, i)$ in R^* such that

$$e^* = e^*(r, 0), re^*(r, i + 1) = e^*(r, i).$$

If w is a prime number in W , then there exists a (uniquely determined) element $e^*(w)$ and a number $h(w)$ such that

$$e^*(w) \text{ is not a } w\text{-fold of an element in } R^* \text{ and satisfies } w^{h(w)} e^*(w) = e^*.$$

The elements e^* , $e^*(w)$ and $e^*(r, i)$ generate the group R^* .

Now let e , $e(w)$ and $e(r, i)$ be elements contained in the classes e^* , $e^*(w)$ and $e^*(r, i)$ of A/B respectively, and since $e^* = e^*(r, 0)$ we choose $e = e(r, 0)$. Then

$$w^{h(w)} e(w) - e = f(w) + b(w),$$

$$re(r, i + i) - e(r, i) = f(r, i) + b(r, i),$$

where $f(w), f(r, i)$ are uniquely determined elements in $F(B)$, $b(w), b(r, i)$ uniquely determined elements in B' , (since $B = F(B) + B'$).

There exists a subgroup A' of A which contains B' and satisfies $A = F(A) + A'$ if, and only if, there exist solutions $g, g(w), g(r, i)$ of the equations

$$w^{h(w)} (e(w) + g(w)) - (e + g) = b(w),$$

$$r(e(r, i + 1) + g(r, i + 1)) - (e(r, i) + g(r, i)) = b(r, i), g(r, 0) = g,$$

in $F(B)$. Equivalent with these equations are the equations:

$$w^{h(w)} g(w) - g = -f(w),$$

$$rg(r, i + 1) - g(r, i) = -f(r, i), g(r, 0) = g.$$

If the prime number q is not contained in P , i.e. if q is contained in V or W , then $C(q, F(B)) = qC(q, F(B))$ and therefore by condition (d): $C(q, F(B)) = 0$,

i.e. the orders of all the elements in $F(B)$ are relatively prime to q . Therefore there exist to every given element g uniquely determined solutions $g(u)$ and $g(v, i)$ in $F(B)$.

Similarly it follows from the finiteness of U that it is sufficient to solve the equations $uh(u, i + 1) - h(u, i) = -c(u, f(u, i))$, $h(u, 0) = h$ in $C(u, F(B))$.

From (b), (d) and (2;1) it follows that the orders of the elements in $C(u, F(B))$ are bounded, i.e. that there exists a number $m = m(u)$ such that $u^m x = 0$ for every element x in $C(u, F(B))$. But then the elements

$$h(u, i) = \sum_{j=0}^{m-1} u^j c(u, f(u, i + j))$$

are contained in $C(u, F(B))$ and satisfy:

$$uh(u, i + 1) - h(u, i) = -c(u, f(u, i))$$

and this completes the proof of (7;1).

REMARK: By an argument similar to that used in proving the Theorems 5;3 and 6;3 it can be shown that the conditions (b) and (c) are also necessary for the existence of the direct summands A' containing B' for every possible B' .

(7;2) Suppose that the group A satisfies the following conditions:

- (a) $A/F(A)$ is countable;
- (b) If the orders of the elements of $C(p, F(A))/p^n C(p, F(A))$ are not bounded, then $A/F(A)$ is not almost p -infinite;
- (c) If the set P of all the prime numbers p such that $C(p, F(A)) \neq pC(p, F(A))$ is infinite, then $A/F(A)$ is not almost P -infinite.

Then there exists to every subgroup B of A which exactly represents^s a closed subgroup of finite rank of $A/F(A)$ a subgroup A' of A such that

$$B \cong A' \text{ and } A = F(A) + A'.$$

PROOF: Let B^* be the subgroup of $A^* = A/F(A)$ which is represented by the elements in B . Then there exists because of condition (a) a sequence $B^*(i)$ of subgroups of A^* such that

$$B^* = B^*(0),$$

$B^*(i)$ is a closed subgroup of finite rank of A^* ,

$$B^*(i) < B^*(i + 1),$$

$$B^*(i + 1)/B^*(i) \text{ is a group of rank 1,}$$

either $A^* = B^*(m)$ for a certain integer m or A^* is the join of all the $B^*(i)$.

Assume that there exist groups $A(i)$ for $0 \leq i \leq k$ such that $B = A(0)$, $A(i) < A(i + 1)$, $A(i)$ represents exactly $B^*(i)$. Let $B(i)$ be the subgroup of A which contains $F(A)$ and satisfies $B(i)/F(A) = B^*(i)$. Then $B(k) = F(A) + A(k)$ and it follows from (b), (c) and (7;1) applied on $B(k)$, $B(k + 1)$ and $A(k)$ that there

^s I.e. B contains at most one element of a class of $A/F(A)$.

exists a subgroup $A(k + 1)$ of $B(k + 1)$ such that $A(k) < A(k + 1)$ and $B(k + 1) = F(A) + A(k + 1)$. Hence these subgroups $A(i)$ exist for every i . If A' is the join of all the $A(i)$, then

$$B \leq A' \text{ and } A = F(A) + A'.$$

(7;3) Suppose that the group A satisfies the following conditions:

- (a) If the orders of the elements of $C(p, F(A))/p^*C(p, F(A))$ are not bounded, then $A/F(A)$ is not almost p -infinite.
- (b) If the set P of all the prime numbers p such that $pC(p, F(A)) \neq C(p, F(A))$ is infinite, then $A/F(A)$ is not almost P -infinite.

Suppose that the subgroup B of A satisfies the following conditions:

- (c) The intersection of B and $F(A)$ is 0.
- (d) The subgroup B^* of $A^* = A/F(A)$ which is represented by elements of B is closed in A^* and of finite rank.
- (e) $D(A^*/B^*)$ exists.⁹

Then there exists a subgroup A' of A such that $B \leq A'$ and $A = F(A) + A'$.

PROOF: If $D(A^*/B^*) = 1$, then A^* is countable, since B^* is a subgroup of finite rank. This case of (7;3) is therefore a consequence of (7;2). Hence we can prove (7;3) by complete (transfinite) induction with regard to $v = D(A^*/B^*)$.

Since $D(A^*/B^*)$ exists, there exists a closed subgroup S^{**} of $A^{**} = A^*/B^*$ of finite rank such that $A^{***} = A^{**}/S^{**}$ is the direct sum of groups $A^{***}(w)$ satisfying $D(A^{***}(w)) < D(A^{**}) = v$. Then let $A^{**}(w)$ be the subgroup of A^{**} which contains S^{**} and satisfies $A^{**}(w)/S^{**} = A^{***}(w)$, S^* the subgroup of A^* corresponding to S^{**} , $A^*(w)$ the subgroup of A^* corresponding to $A^{**}(w)$, finally $A(w)$ and S the subgroups of A which contain $F(A)$ and correspond to $A^*(w)$ and S^* respectively.

Since S^* is of finite rank (and therefore countable), there exists from (a), (b) and (7;2) a subgroup S' of S such that $B \leq S'$ (for B is a subgroup of S) and $S = F(A) + S'$.

Since $A^*(w)/S^*$ and $A^{***}(w)$ are isomorphic, $D(A^*(w)/S^* = D(A^{***}(w)) < v$ and there exists therefore by the hypothesis of the induction a subgroup $A'(w)$ of $A(w)$ such that $S' \leq A'(w)$ and $A(w) = F(A) + A'(w)$.

Let A' be the subgroup of A , generated by the subgroups $A'(w)$. Then every class of $A/F(A)$ is represented by elements in A' , since A/S is the direct sum of the groups $A(w)/S$. Furthermore every element of A' has the form $\sum_{i=1}^k x(i)$ where the elements $x(i)$ belong to different groups $A'(w)$. If this element is contained in $F(A)$, then $\sum_{i=1}^k x(i) \equiv 0 \pmod{S}$, i.e. $x(i) \equiv 0 \pmod{S}$, since A/S is the direct sum of the groups $A(w)/S$. Since $x(i)$ is an element of S and of an $A'(w)$, $x(i)$ is an element of S' and therefore $\sum_{i=1}^k x(i)$ is an element of S' . But this implies that $\sum_{i=1}^k x(i) = 0$, since the intersection of $F(A)$ and S' is 0, i.e. the intersection of A' and $F(A)$ is 0 and therefore $A = A' + F(A)$. Furthermore $B \leq S' \leq A'$ and this completes the proof.

⁹ As a consequence of (3;3) it follows from this condition (e) that $D(A^*)$ exists and satisfies $D(A^*) \leq D(A^*/B^*)$.

8. Final conclusions.

THEOREM 8;1: *Suppose that F is a group without elements of infinite order, that J is a group without elements of finite order and that $D(J)$ exists. Then $F(A)$ is a direct summand of every group A such that $F(A) = F$ and $A/F(A) = J$ if, and only if, the following two conditions are satisfied:*

- (a) *If the orders of the elements of $C(p, F)/p^a C(p, F)$ are not bounded, then J is not almost p -infinite.*
- (b) *If the set P of all the prime numbers p such that $C(p, F) \neq pC(p, F)$ is infinite, then J is not almost P -infinite.*

PROOF: The necessity of condition (a) is a consequence of Theorem 5;3 and the necessity of condition (b) is a consequence of Theorem 6;3.

That the conditions are sufficient is a consequence of (7;3), if we choose as the subgroup B of A the 0-group which satisfies all the conditions.

Since the existence of $D(J)$ has not been used in the proof of the necessity of the conditions, we have even proved the

COROLLARY 8;2: *If $F(A)$ is a direct summand of every group A such that $F(A) = F$ and $A/F(A) = J$, then the groups F and J satisfy the conditions (a) and (b).*

COROLLARY 8;3: *Suppose that F is a group without elements of infinite order, that the group J is contained in a group J' without elements of finite order such that $D(J')$ exists and the pair F, J' satisfies the conditions (a) and (b).¹⁰ Then $F(A)$ is a direct summand of every group A such that $F(A) = F$ and $A/F(A) = J$.*

This Corollary is a consequence of the Theorem 8;1 and of (4;4).

COROLLARY 8;4: *Suppose that F is a group without elements of infinite order, that J is a group without elements of finite order and that $D(J)$ exists. Then $C(p, F(A))$ is a direct summand of every group A such that $F(A) = F$ and $A/F(A) = J$ if, and only if, the condition (a) is satisfied for this particular prime number p .*

This is a consequence of the Theorem 8;1 and of (4;5), since for primary groups F the condition (b) becomes void.

THEOREM 8;5: *Let F be a group without elements of infinite order. Then $F(A)$ is a direct summand of every group A such that $F(A) = F$ if, and only if,*

- (1) *the orders of the elements of $C(p, F)/p^a C(p, F)$ are bounded for every prime number p ;*
- (2) *$C(p, F) = pC(p, F)$ for almost every prime number p .*

PROOF: The necessity of the conditions is a consequence of the Corollary 8;2, since we can choose as a group J the additive group of all the rational numbers. That the conditions are sufficient, is a consequence of the Corollary 8;3, since by (3;1) every group J is contained in a complete group J' , and since complete groups are direct sums of countable groups, i.e. $D(J') \leq 2$.

THEOREM 8;6: *Suppose that J is a group without elements of finite order and that $D(J)$ exists. Then $F(A)$ is a direct summand of every group A such that $A/F(A) = J$ if, and only if,*

¹⁰ This implies that the pair of groups F, J satisfies also the conditions (a) and (b).

for every closed subgroup S of J of finite rank all the (closed) subgroups of rank 14 of J/S are cyclic groups.¹¹

PROOF: It is easily inferred from Theorem 8;1 that $F(A)$ is a direct summand of every group A such that $A/F(A) = J$ if, and only if, J is neither almost p -infinite for any prime number p nor almost P -infinite for any infinite set P of prime numbers. But this condition is clearly equivalent with the condition of the Theorem.

Note that by Corollary 8;2 the condition of the Theorem is also necessary if $D(J)$ does not exist.

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¹¹ It can be proved that this condition together with the existence of $D(J)$ imply that J is a direct sum of cyclic groups. But there exist groups which satisfy this condition and which are not direct sums of cyclic groups (e.g. the additive group of all the sequences of integers) and for these groups the problem whether the condition of the Theorem 8;6 is a sufficient one is still unsettled.