TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 216, 1976

FINITE GROUPS AS ISOMETRY GROUPS

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ABSTRACT. We show that given any finite group G of cardinality $k + 1$, there is a Riemannian sphere S^{k-1} (imbeddable isometrically as a hypersurface in $\mathbf{R}^k$) such that its full isometry group is isomorphic to G. We also show the existence of a finite metric space of cardinality $k(k + 1)$ whose full isometry group is isomorphic to G.

Let G be a finite group of $k + 1$ elements $\{1, g_1, \ldots, g_k\}$.

THEOREM. *There exists a Riemannian metric on the sphere S^{k-1} such that the isometry group is isomorphic to G.*

PROOF. Label the $k + 1$ vertices of a regular k-simplex Δ_k by the names $1, g_1, \ldots, g_k$ of the elements of G. Assume Δ_k to be inscribed in a standard S^{k-1} sitting in $\mathbf{R}^k$ as usual. $T_y(S^{k-1})$ denotes the tangent space at y.

Now in $T_1(S^{k-1})$ pick an orthonormal frame $(v_1, \ldots, v_{k-1})$. Pick $\epsilon > 0$ small and let

$$w_i = \epsilon(1 + (i - 1)/4k^2)v_i, \qquad 1 \leqslant i \leqslant k - 1.$$

Let

$$Q = \{\exp_1(w_i) \mid 1 \leqslant i \leqslant k - 1\} \cup \{\exp_1(0)\} \cup \{w_1/10\}.$$

$\exp_1$ is the exponential map $\exp_1\colon T_1(S^{k-1}) \longrightarrow S^{k-1}$.

Think of G as acting on S^{k-1} by the isometries induced from the permutation representation on the vertices of Δ_k. Let $X = \{gQ \mid g \in G\}$.

PROPOSITION. *With the induced metric from $\mathbf{R}^k$, the metric space X has its group of isometries isomorphic to G.*

PROOF. Clearly G acts as a group of isometries of X, since $X = h\{gQ \mid g \in G\} = \{hgQ \mid g \in G\} = \{gQ \mid g \in G\} = X$.

Conversely, any isometry of X must take the point 1 to some point g, since the points g are characterized by being the only points in X having their

Received by the editors May 8, 1975.

AMS (MOS) subject classifications (1970). Primary 53C20, 54E40, 20C99; Secondary 20D99, 54E35, 53C40.

Key words and phrases. Group, isometry, Riemannian, manifold, the metric space.

two nearest neighbors at distance of $\epsilon/10$ and ϵ respectively. Once we know that $1 \mapsto g$, the configuration gQ determines the image of the frame $(w_1, \ldots, w_{k-1})$ at 1, and hence determines the unique isometry of X defined by the element $g \in G$. Of course ϵ must be chosen small enough so that the configurations gQ, $g \in G$ do not "interfere" with one another.

Now we add bumps to S^{k-1} at the points of X using scalar multiplication in $\mathbf{R}^k$. Let

$$\delta = (1/3)\min\{\mathrm{dist}_{S^{k-1}}(x, y) \mid x, y \in X\}.$$

Let $f\colon [0, \delta] \longrightarrow \mathbf{R}$ be a smooth function satisfying

(a) $f(s) = 100,\ 0 \leqslant s \leqslant \delta/2$,

(b) $f(\delta) = 1; f^{(k)}(\delta) = 0,\ k = 1, 2, \ldots,$

(c) $f^{(k)}(\delta/2) = 0,\ k = 1, 2, \ldots,$ and

(d) $f'(s) < 0$ if $\delta/2 < s < \delta$.

Now for each point $x \in X$ we remove the disk $\exp_x(D_\delta)$ from S^{k-1} and replace it by the point set $B_x = \{(f(|v|))\exp_x(v) \mid v \in D_\delta\}$, where D_δ is the (δ)-disk about the origin of $T_x(S^{k-1})$. Clearly the set $S^{k-1} - \bigcup_{x\in X}\exp_x(D_\delta) \cup \bigcup_{x\in X} B_x$ is a smooth S^{k-1} imbedded in $\mathbf{R}^k$. We give it the induced Riemannian metric from $\mathbf{R}^k$ and denote it by M.

CLAIM: $\mathrm{Isom}(M) \approx G$.

PROOF. First we notice that the points of $100 \cdot X \subset M$ must be taken to themselves by any isometry I of M, by the choice of the function f. Clearly the same arguments above for X hold for $100 \cdot X$, hence the isometry $I\colon M \longrightarrow M$ restricted to $100 \cdot X$ comes from the action of G.

Let us now consider the "bump" B_1 above the point 1. Let us define for $r \geqslant 0$, $S_r = \{f(r) \cdot \exp_1(v) \mid |v| = r,\ v \in T_1(S^{k-1})\}$. In other words, S_r is the $(k-2)$-sphere of B_1 lying above the $(k-2)$-sphere about 1 of radius r, for $0 < r \leqslant \delta$, and for $r = 0$ we set $S_0 = p$, the peak point of B_1.

Now it is easy to show that the orthogonal trajectories of the S_r's are geodesics of M and as such must be preserved under any isometry taking p to p.

Thus any isometry I of M which takes p to p (and which must thus leave all points of $100 \cdot X$ fixed) must be a "rotation" on all of B_1, determined by $I \mid \partial B_1$, carrying each S_r into itself by the "same" element of $O(k-2)$. Similarly, this I must rotate each bump B_x, $x \in X$.

Also this rotation must extend past the boundary of the bumps for some ways, so we can easily extend $I \mid (M - \bigcup_x B_x)$ to an isometry $\widetilde{I}$ of S^{k-1} to itself, by simply "coning" I over $\exp_x(D_\delta)$, $x \in X$. Clearly we will have $\widetilde{I}(x) = x$ for $x \in X$, and it follows easily that $\widetilde{I}\colon S^{k-1} \longrightarrow S^{k-1}$ is the identity. Hence $I\colon M \longrightarrow M$ must have been the identity.

Now it is clear that for each $g \in G$ there is one isometry of M determined by the action of g on S^{k-1}, extended to R^k, restricted to M. Now if there is another isometry $I: M \longrightarrow M$ such that $I \mid X = g \mid X$, then $I \circ g^{-1}: M \longrightarrow M$ must leave points of X fixed, so by the above discussion must be the identity. This establishes $\mathrm{Isom}(M) \approx G$.

COROLLARY. *Any finite group G is isomorphic to the (full) isometry group of a finite subset X_G of euclidean space. If* $\mathrm{card}(G) = k$ *then the X_G can be found with* $\mathrm{card}(X_G) = k^2 - k$ *in euclidean space of dimension $k - 1$.*

PROOF. Simply take $X_G = X$ in the proof of the Theorem, and count (noting that we initially took $\mathrm{card}(G) = k + 1$).

REMARK. Further considerations can very likely reduce the necessary cardinality for X_G to $k(k - 3)$. The various numbers

$$d = \min\{\mathrm{card}(X) \mid G \approx \mathrm{Isom}(X)\} \quad \text{and}$$

$$e = \min\{N \mid G \text{ has a faithful representation into } O(N)\}$$

seem to be interesting invariants of a finite group G.

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