ABSOLUTE VALUES II: TOPOLOGIES, COMPLETIONS AND THE EXTENSION PROBLEM

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Contents

2.1. **Introduction and Reorientation.**

In this chapter we will study more explicitly the topology on a field induced by a norm. Especially interesting from this perspective are the (nontrivially) normed fields which are **locally compact** with respect to the norm topology.

But we have been studying normed fields for a little while now. Where are we going? What problems are we trying to solve?

Problem 1: Local/Global Compatibility. Arguably the most interesting results in Chapter 1 were the complete classification of all norms on a **global field** K, i.e., a finite extension of either $\mathbb{Q}$ (a number field) or $\mathbb{F}_q(t)$ for some prime power q (a function field).

We interrupt for two remarks:

Remark 1: Often when dealing with function fields, we will say "Let $K/\mathbb{F}_q(t)$ be

Thanks to John Doyle and David Krumm for pointing out typos. Section 2.9 on Big Ostrowski was written following lecture notes of David Krumm.

a finite *separable* field extension". It is not true that every finite field extension of $\mathbb{F}_q(t)$ is separable: e.g. $\mathbb{F}_q(t^{\frac{1}{q}})/\mathbb{F}_q(t)$ is an inseparable field extension. However, the following is true: if $\iota : \mathbb{F}_q(t) \hookrightarrow K$ is a finite degree field homomorphism – don't forget that this wordier description is the true state of affairs which is being elided when we speak of "a field extension K/F" – then there is always another finite degree field homomorphism $\iota' : \mathbb{F}_q(t) \hookrightarrow K$ which makes $K/\iota'(\mathbb{F}_q(t))$ into a **separable** field extension: e.g. [Eis, Cor. 16.18].

Remark 2: In the above passage we could of course have replaced $\mathbb{F}_q(t)$ by $\mathbb{F}_p(t)$. But the idea here is that for an arbitrary prime power q, the rational function field $\mathbb{F}_q(t)$ is still highly analogous to $\mathbb{Q}$ rather than to a more general number field. For instance, if K is any number field, then at least one prime ramifies in the extension of Dedekind domains $\mathbb{Z}_K/\mathbb{Z}$. However, the extension $\mathbb{F}_q[t]/\mathbb{F}_p[t]$ is everywhere unramified. Moreover, $\mathbb{F}_q[t]$ is always a PID.[1]

For a global field K, we saw that there is always a Dedekind ring R with K as its fraction field with "sufficiently large spectrum" in the sense that all but finitely many valuations on K are just the $\mathfrak{p}$-adic valuations associated to the nonzero prime ideals of R. This suggests – correctly!– that much of the arithmetic of K and R can be expressed in terms of the valuations on K.

A **homomorphism of normed fields** $\iota : (K, | \ |) \rightarrow (L, | \ |)$ is a field homomorphism ι such that for all $x \in K$, $|x| = |\iota(x)|$. We say that the norm on L **extends** the norm on K. When the normed is non-Archimedean, this has an entirely equivalent expression in the language of valuations: a **homomorphism of valued fields** $\iota : (K, v) \rightarrow (L, w)$ is a field homomorphism $\iota : K \hookrightarrow L$ such that for all $x \in K$, $v(x) = w(\iota(x))$. We say that w **extends** v or that $w|_K = v$. (Later we will abbreviate this further to $w \mid v$.)

Problem 2: The Extension Problem. Let $(K, | \ |)$ be a normed field, and let L/K be a field extension. In how many ways does v extend to a norm on L?

Theorem 1. *Let $(K, | \ |)$ be a normed field and L/K an extension field. If* either *of the following holds, then there is a norm on L extending the given norm on K:*
(i) L/K is algebraic.
(ii) $(K, | \ |)$ is non-Archimedean.

Example: Let $K = \mathbb{Q}$, $| \ | = | \ |_2$ and $L = \mathbb{R}$. Then there exists a norm on $\mathbb{R}$ which extends the 2-adic norm on $\mathbb{Q}$. This may seem like a bizarre and artifical example, but it isn't: this is the technical heart of the proof of a beautiful theorem of Paul Monsky [Mon]: it is not possible to dissect a square into an odd number of triangles such that all triangles have the same area. In fact, after 40 years of further work on this and similar problems, to the best of my knowledge no proof of Monsky's theorem is known which *does not* use this valuation-theoretic fact.

Exercise 2.1: Let $(K, | \ |)$ be an Archimedean norm.
a) Suppose that L/K is algebraic. Show that $| \ |$ extends to a norm on L.

[1] Somewhat embarrassingly, the question of whether there exist infinitely many number fields of class number one remains open!

b) Give an example where L/K is transcendental and the norm on K does extend to a norm on L.
c) Give an example where L/K is transcendental and the norm on K does not extend to a norm on L.
Hint for all three parts: use the Big Ostrowski Theorem.

In view of Exercise 2.1, we could restrict our attention to non-Archimedean norms and thus to valuations. Nevertheless it is interesting and useful to see that the coming results hold equally well in the Archimedean and non-Archimedean cases.

Theorem 1 addresses the existence of an extended norm but not the number of extensions. We have already seen examples to show that if L/K is transcendental, the number of extensions of a norm on K to L may well be infinite. The same can happen for algebraic extensions of infinite degree: e.g., as we will see later, for any prime p, there are uncountably many extensions of the p-adic norm to $\overline{\mathbb{Q}}$.

Exercise 2.2T[2]: Let K be a field and $\{K_i\}_{i \in I}$ be a family of subfields of K such that: (i) for all $i, j \in I$ there exists $k \in I$ such that $K_i \cup K_j \subset K_k$ and (ii) $\bigcup_i K_i = K$. (Thus the family of subfields is a directed set under set inclusion, whose direct limit is simply K.) Suppose that for each i we have a norm $|\ |_i$ on K_i, compatibly in the following sense: whenever $K_i \subset K_j$, $|\ |_j$ extends $|\ |_i$. Show that there is a unique norm $|\ |$ on K extending each norm $|\ |$ on K_i.

Exercise 2.3: Let $(k, |\ |)$ be a non-Archimedean normed field. Let $R = k[t]$ and $K = k(t)$. For $P(t) = a_n t^n + \ldots + a_1 t + a_0 \in R$, define the **Gauss norm** $|P| = \max_i |a_i|$. Show that this is indeed a norm on $k[t]$ and thus induces a norm on the fraction field $K = k(t)$ extending the given norm on k. Otherwise put, this shows that every valuation on a field k extends to a valuation on $k(t)$.

Exercise 2.4: Let (K, v) be a valued field, and let L/K be a purely transcendental extension, i.e., the fraction field of a polynomial ring over K (in any number of indeterminates, possibly infinite or uncountable). Use the previous Exercise to show that v extends to a valuation on L. (Suggestion: this is a case where a transfinite induction argument is very clean.)

Exercise 2.4 and basic field theory reduces Theorem 1 to the case of an algebraic extension L/K. As we will see, this can be further reduced to the case of finite extensions. Moreover, when (K, v) is a valued field and L/K is a finite extension, we wish not only to show that an extension w of v to L exists but to classify (in particular, to count!) all such extensions. We saw in Chapter 1 that this recovers one of the core problems of algebraic number theory. Somewhat more generally, if v is discrete, then the valuation ring R is a DVR – in particular a Dedekind domain – and then its integral closure S in L is again a Dedekind domain, and we are asking how the unique nonzero prime ideal $\mathfrak{p}$ of R splits in S: i.e., $\mathfrak{p}S = \mathcal{P}_1^{e_1} \cdots \mathcal{P}_r^{e_r}$. With suitable separability hypotheses, we get the fundamental relation $\sum_{i=1}^r e_i f_i = [L : K]$.

[2]The letter T will be used to denote an exercise that is – despite appearances, perhaps – trivial to prove, but useful to apply later. I do not guarantee that an exericse not so marked will be nontrivial.

The key idea that makes this bookkeeping automatic – and has many other virtues besides – is that of the **completion** $\hat{K}$ of a normed field $(K, | \, |)$. This is indeed a special case of the completion of a metric space – a concept which we will review – but bears further scrutiny in this case because we wish $\hat{K}$ to itself have the structure of a normed field. Here are some fundamental results:

Theorem 2. *Let $(K, | \, |)$ be a normed field.*
a) There is a complete normed field $(\hat{K}, | \, |)$ and a homomorphism of normed fields $\iota : (K, | \, |) \to (\hat{K}, | \, |)$ such that $\iota(K)$ is dense in $\hat{K}$.
b) The homomorphism ι is universal for norm-preserving homomorphisms of K into complete normed fields.
c) In particular, $\hat{K}$ is unique up to canonical isomorphism.
d) It follows that any homomorphism of normed fields extends uniquely to a homomorphism on the completions.

Remark: In categorical language, these results amount to the following: completion is a functor from the category of normed fields to the category of complete normed fields which is left adjoint to the forgetful functor from the category of complete normed fields to the category of normed fields. We stress that, for our purposes here, it is absolutely not necessary to understand what the previous sentence means.

Theorem 3. *Let $(K, | \, |)$ be a complete normed field and let L/K be algebraic.*
a) There exists a ***unique*** *norm $| \, |_L$ on L such that $(K, | \, |) \to (L, | \, |_L)$ is a homomorphism of normed fields.*
b) If L/K is finite, then $(L, | \, |_L)$ is again complete.

Corollary 4. *If $(K, | \, |)$ is a normed field and L/K is an algebraic extension, then there is at least one norm on L extending the given norm on K.*

Proof. We may as well assume that $L = \overline{K}$. The key step is to **choose** a field embedding $\Phi : \overline{K} \hookrightarrow \overline{\hat{K}}$. This is always possible by basic field theory: any homomorphism from a field K into an algebraically closed field F can be extended to any algebraic extension L/K. Since this really is the point, we recall the proof. Consider the set of all embeddings $\iota_i : L_i \hookrightarrow F$, where L_i is a subextension of L/K. This set is partially ordered by inclusion. Moreover the union of any chain of elements in this poset is another element in the poset, so by Zorn's Lemma we are entitled to a maximal embedding $\iota_i : L_i \hookrightarrow F$. If $L_i = L$, we're done. If not, there exists an element $\alpha \in L \setminus L_i$, but then we could extend ι_i to $L_i[\alpha]$ by sending α to any root of its $\iota_i(L_i)$-minimal polynomial in F. By Theorem 3, there is a unique norm on $\overline{\hat{K}}$ extending the given norm on K. Therefore we may define a norm on L by $x \mapsto |\Phi(x)|$. □

Exercise 2.5: Use Corollary 4 and some previous exercises to prove Theorem 1.

Theorem 5. *Let $(K, | \, |)$ be a normed field and L/K a finite extension. Then there is a bijective correspondence between norms on L extending the given norm on K and prime ideals in the $\hat{K}$-algebra $L \otimes_K \hat{K}$.*

There is a beautiful succinctness to the expression of the answer in terms of tensor products, but let us be sure that we understand what it means in more down-to-earth terms. Suppose that there exists a primitive element $\alpha \in L$ i.e., such that

$L = K(\alpha)$. Recall that this is always the case when L/K is separable or $[L : K]$ is prime. In fact, the existence of primitive elements is often of mostly psychological usefulness: in the general case we can of course write $L = K(\alpha_1, \ldots, \alpha_n)$ and decompose L/K into a finite tower of extensions, each of which has a primitive element.

Now let $P(t) \in K[t]$ be the minimal polynomial of α over K, so $P(t)$ is irreducible and $L \cong K[t]/(P(t))$. In this case, for any field extension F/K, we have isomorphisms

$$L \otimes_K F \cong K[t]/(P(t)) \otimes_K F \cong F[t]/(P(t)).$$

Thus, $L \otimes_K F$ is an F-algebra of dimension $d = \deg P = [L : K]$. It need not be a field, but its structure is easy to analyze using the Chinese Remainder Theorem in the Dedekind ring $F[t]$. Namely, we factor $P(t)$ into irreducibles: say $P(t) = P_1^{e_1} \cdots P_r^{e_r}$. Then CRT gives an isomorphism

$$L \otimes_K F \cong F[t]/(P(t)) \cong \bigoplus_{i=1}^{r} F[t]/(P_i^{e_i}).$$

Let us put $A_i = F[t]/(P_i^{e_i})$. This is a local Artinian F-algebra with unique prime ideal $P_i/P_i^{e_i}$. Thus the number of prime ideals in $L \otimes_K F$ is r, the number of distinct irreducible factors of F. Moreover, suppose that L/K is separable. Then $P(t)$ splits into distinct linear factors in the algebraic closure of K, which implies that when factored over the extension field F (algebraic or otherwise), it will have no multiple factors. In particular, if L/K is separable (which it most often will be for us, in fact, but there seems to be no harm in briefly entertaining the general case), then all the e_i's are equal to 1 and $A_i = F[t]/(P_i)$ is a finite, separable field extension of F.

Example: We apply this in the case $(K, | \ |)$ is the rational numbers equipped with the standard Archimedean norm. Then the number of extensions of $| \ |$ to $L \cong K[t]/(P(t))$ is equal to the number of (necessarily distinct) irreducible factors of $P(t)$ in $\mathbb{R} = \hat{\mathbb{Q}}$. How does a polynomial factor over the real numbers? Every irreducible factor has degree either 1 – corresponding to a real root – or 2 – corresponding to a conjugate pair of complex roots. Thus $L \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ and the number of extensions is $r_1 + r_2$, as advertised – but not proved! – in the Remark following Theorem 1.16.

It remains to prove Theorems 2, 3 and 5. Before proving Theorem 1, we give several short sections of "review" on topics which are probably somewhat familiar from previous courses but are important enough to revisit from a slightly more sophisticated perspective. In §2.7 we give the proof of Theorem 2.

2.2. **Reminders on metric spaces.**

Let X be a set. A function $\rho : X \times X \to \mathbb{R}^{\geq 0}$ is a **metric** on X if it satisfies all of the following:

(M1) (positive definiteness) $\forall x, y \in X,\ \rho(x, y) = 0 \iff x = y$.
(M2) (symmetry) $\forall x, y \in X,\ \rho(x, y) = \rho(y, x)$.
(M3) (triangle inequality) $\forall x, y, z \in X,\ \rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

A **metric space** is a pair (X, d) where d is a metric on X.

For x an element of a metric space X and $r \in \mathbb{R}^{>0}$, we define the **open ball**

$$B(x, r) = \{y \in X \mid \rho(y, x) < r\}.$$

The open balls form the base for a topology on X, the **metric topology**. With your indulgence, let's check this. What we must show is that if $z \in B(x, r_1) \cap B(y, r_2)$, then there exists $r_3 > 0$ such that $B(z, r_2) \subset B(x, r_1) \cap B(y, r_2)$. Let $r_3 = \min(r_1 - \rho(x, z), r_2 - \rho(y, z))$, and let $w \in B(z, r_3)$. Then by the triangle inequality $\rho(x, w) \leq \rho(x, z) + \rho(z, w) < \rho(x, z) + (r_1 - \rho(x, z)) = r_1$, and similarly $\rho(y, w) < r_2$.

Given a finite collection of metric spaces $\{(X_i, \rho_i)\}_{1 \leq i \leq n}$, we define the **product metric** on $X = \prod_{i=1}^n X_i$ to be $\rho(x, y) = \max_i \rho_i(x_i, y_i)$.[3]

Remark: As is typical, instead of referring to "the metric space (X, ρ)", we will often say instead "the metric space X", i.e., we allow X to stand both for the set and for the pair (X, ρ).

Exercise 2.6 (pseudometric spaces): Let X be a set. A function $\rho : X \times X \to \mathbb{R}^{\geq 0}$ satisfying (M2) and (M3) is called a **pseudometric**, and a set X endowed with a pseudometric is called a **pseudometric space**.
a) Show that all of the above holds for pseudometric spaces – in particular, the open balls form the base for a topology on X, the **pseudometric topology**.
b) Show that for a pseudometric space (X, ρ), the following are equivalent:
(i) ρ is a metric.
(ii) The topological space X is Hausdorff.
(iii) The topological space X is separated (i.e., T_1: points are closed).
(iv) The topological space X is Kolmogorov (i.e., T_0: no two distinct points have exactly the same open neighborhoods).
c) Define an equivalence relation $\sim$ on X by $x \sim y \iff \rho(x, y) = 0$. Let $\overline{X} = X/\sim$ be the set of equivalence classes. Show that ρ factors through a function $\overline{\rho} : \overline{X} \times \overline{X} \to \mathbb{R}^{\geq 0}$ and that $\overline{\rho}$ is a metric on $\overline{X}$. Show that the map $q : X \to \overline{X}$ is the **Kolmogorov completion** of the topological space X, i.e., it is the universal continuous map from X into a T_0-space.

A **Cauchy sequence** in a metric space (X, ρ) is a sequence $\{x_n\}$ in X such that for all $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $m, n \geq N \implies \rho(x_m, x_n) < \epsilon$. Every convergent sequence is convergent. Conversely, we say that a metric space X is **complete** if every Cauchy sequence converges.

Let X and Y be metric spaces. A function $f : X \to Y$ is **uniformly continuous** if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall x, y \in X, \rho_X(x, y) < \delta \implies \rho_Y(f(x), f(y)) < \epsilon$.

[3]This is just one of many possible choices of a product metric. The non-canonicity in the choice of the product is a clue that our setup is not optimal. But the remedy for this, namely **uniform spaces**, is not worth our time to develop.

Exercise 2.7: Let (X, ρ) be a metric space. Show that $\rho : X \times X \to \mathbb{R}$ is a uniformly continuous function: here $\mathbb{R}$ is endowed with the standard Euclidean metric $\rho(x, y) = |x - y|$.

Exercise 2.8: Let X and Y be metric spaces, and let $f : X \to Y$ be a continuous function.
a) If f is uniformly continuous and $\{x_n\}$ is a Cauchy sequence in X, show that $\{f(x_n)\}$ is a Cauchy sequence in Y.
b) Give an example to show that a merely continuous function need not map Cauchy sequences to Cauchy sequences.

A Hausdorff topological space is **compact** if every open covering has a finite subcovering.[4] A Hausdorff topological space is **locally compact** if every point admits a compact neighborhood. This is equivalent (thanks to the Hausdorff condition!) to the apparently stronger condition that every point has a local base of compact neighborhoods.

A metric space (X, ρ) is **ball compact**[5] if every closed bounded ball is compact.

Exercise 2.9: Consider the following properties of a metric space (X, ρ):
(i) X is compact.
(ii) X is ball compact.
(iii) X is locally compact.
(iv) X is complete.
Show that (i) $\implies$ (ii) $\implies$ (iii) and (ii) $\implies$ (iv), but none of the other implications hold.

2.3. **Ultrametric spaces.**

An **ultrametric space** is a metric space (X, ρ) in which the following stronger version of the triangle inequality holds:

$$\forall x, y, z \in X, \ \ \rho(x, z) \leq \max(\rho(x, y), \rho(y, z)).$$

Exercise 2.10: a) Suppose that x, y, z are points in an ultrametric space such that $\rho(x, y) \neq \rho(y, z)$. Show that $\rho(x, z) = \max(\rho(x, y), \rho(y, z))$.
b) In particular, every triangle in an ultrametric space is isosceles.

c) Let $B = B(x, r)$ be an open ball in an ultrametric space (X, ρ) and let $y \in B(x, r)$. Show that y is also a center for B: $B = B(y, r)$. Does the same hold for closed balls?

Exercise 2.11: Let B_1, B_2 be two balls (each may be either open or closed) in an ultrametric space (X, ρ). Show that B_1 and B_2 are either disjoint or concentric: i.e., there exists $x \in X$ and $r_1, r_2 \in (0, \infty)$ such that $B_i = B(x, r_i)$ or $B_c(x, r_i)$.

Exercise 2.12: Let (X, ρ) be an ultrametric space.

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[4]Note that I am sidestepping the issue of whether a non-Hausdorff space should be called "compact" or just "quasi-compact" as is standard e.g. in algebraic geometry. The point is that all our spaces will be metrizable, hence Hausdorff, so no worries.

[5]I made up the term.

a) Let $r \in (0, \infty)$. Show that the set of open (resp. closed) balls with radius r forms a partition of X.
b) Deduce from part a) that every open ball is also a closed subset of X and that every closed ball of positive radius is also an open subset of X.
c) A topological space is **zero-dimensional** if there exists a base for the topology consisting of clopen (= closed and open) sets. Thus part b) shows that an ultrametric space is zero-dimensional. Show that a zero-dimensional Hausdorff space is totally disconnected. In particular, an ultrametric space is totally disconnected.

Exercise 2.13: Prove or disprove: it is possible for the same topological space (X, τ) to have two compatible metrics ρ_1 and ρ_2 (i.e., each inducing the given topology τ on X) such that ρ_1 is an ultrametric and ρ_2 is not.

Exercise 2.14: Let Ω be a nonempty set, and let $\mathcal{S} = \prod_{i=1}^{\infty} \Omega$, i.e., the space of infinite sequences of elements in Ω, endowed with the metric $\rho(x, y) = 2^{-N}$ if $x_n = y_n$ for all $n < N$ and $x_N \neq y_N$. (If $x_n = y_n$ for all n, then we take $N = \infty$.)
a) Show that $(\mathcal{S}, \rho)$ is an ultrametric space, and that the induced topology coincides with the product topology on $\mathcal{S}$, each copy of Ω being given the discrete topology.
b) Show that $\mathcal{S}$ is a complete[6] metric space without isolated points.
c) Without using Tychonoff's theorem, show that $\mathcal{S}$ is compact iff Ω is finite. (Hint: since $\mathcal{S}$ is metrizable, compact is equivalent to sequentially compact. Show this via a diagonalization argument.)
d) Suppose Ω_1 and Ω_2 are two finite sets, each containing more than one element. Show that the spaces $\mathcal{S}(\Omega_1)$ and $\mathcal{S}(\Omega_2)$ are homeomorphic.

2.4. Normed abelian groups.

Let G be an abelian group, written additively. By a **norm** on G we mean a map $| \ | : G \to \mathbb{R}^{\geq 0}$ such that:

(NAG1) $|g| = 0 \iff g = 0$.
(NAG2) $\forall g \in G, \ |-g| = |g|$.
(NAG3) $\forall g, h \in G, \ |g + h| \leq |g| + |h|$.

For example, an absolute value on a field k is (in particular) a norm on $(k, +)$. By analogy to the case of fields, we will say that a norm is **non-Archimedean** if $\forall g, h \in G, \ |g + h| \leq |g| + |h|$.

For a normed abelian group $(G, | \ |)$, define $\rho : G^2 \to \mathbb{R}^{\geq 0}$ by $\rho(x, y) = |x - y|$.

Exercise 2.15: Show that ρ defines a **metric** on G. Show that the norm is non-Archimedean iff ρ is an ultrametric.

Exercise 2.16: Show that the norm $| \ | : X \to \mathbb{R}$ is uniformly continuous.

The metric topology on X is Hausdorff and first countable, so convergence can be described in terms of sequences: a sequence $\{x_n\}$ in X **converges** to $x \in X$

[6] Completeness is formally defined in the next section.

if for all $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for all $n \geq N$, $\rho(x_n, x) < \epsilon$. A sequence is said to be **convergent** if it converges to some x. Since X is Hausdorff, a sequence converges to at most one point.

Exercise (semi-normed group): A **semi-norm** on an abelian group is a map $| \; | : G \to \mathbb{R}^{\geq 0}$ which satisfies (NAG2) and (NAG3). Show that a semi-norm induces a pseudometric on G.

Exercise 2.17: Suppose G is an arbitrary (i.e., not necessarily abelian) group – with identity element e and group law written multiplicatively – endowed with a function $| \; | : G \to \mathbb{R}^{\geq 0}$ satisfying:
(NG1) $|g| = 0 \iff g = e$.
(NAG2) $\forall g \in G, \; |g^{-1}| = |g|$.
(NAG3) $\forall g, h \in G, |gh| \leq |g| + |h|$.
a) Show that $d : G \times G \to \mathbb{R}, \; (g, h) \mapsto |gh^{-1}|$ defines a metric on G.
b) If $| \; |$ is a norm on G and $C \in R^{>0}$, show that $C| \; |$ is again a norm on G. Let us write $| \; |_1 \approx | \; |_2$ for two norms which differ by a constant in this way.
c) Define on any group G a trivial norm; show that it induces the discrete metric.

In any topological abelian group, it makes sense to discuss the convergence of infinite series $\sum_{n=1}^{\infty} a_n$ in G: as usual, we say $\sum_{n=1}^{\infty} a_n = S$ if the sequence $\{\sum_{k=1}^{n} a_k\}$ of partial sums converges to S.

A series $\sum_{n=1}^{\infty} a_n$ is **unconditionally convergent** if there exists $S \in G$ such that for every permutation σ of the positive integers, the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges to S.

In a normed abelian group G we may speak of **absolute convergence**: we say that $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if the real series $\sum_{n=1}^{\infty} |a_n|$ converges.

Exercise 2.18: For a normed group G, show that TFAE:
(i) Every absolutely convergent series is unconditionally convergent.
(ii) G is complete.

Whether unconditional convergence implies absolute convergence is more delicate. If $G = \mathbb{R}^n$ with the standard Euclidean norm, then it follows from the **Riemann Rearrangement Theorem** that unconditional convergence implies absolute convergence. On the other hand, it is a famous theorem of Dvoretsky-Rogers that in any infinite dimensional real Banach space (i.e., a complete, normed $\mathbb{R}$-vector space) there exists a series which is unconditionally convergent but not absolutely convergent.

The theory of convergence in complete non-Archimedean normed groups is in fact much simpler:

Proposition 6. *Let G be a complete, non-Archimedean normed group, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence in G. TFAE:*
(i) The series $\sum_{n=1}^{\infty} a_n$ is unconditionally convergent.

(ii) The series $\sum_{n=1}^{\infty} a_n$ *is convergent.*
(iii) $\lim_{n\to\infty} a_n = 0$.

Exercise 2.19: Prove Proposition 6.

Exercise 2.20: Use Proposition 6 to give an explicit example of a series in $\mathbb{Q}_p$ which is unconditionally convergent but not absolutely convergent.

Exercise 2.21: Let $(G, |\ |)$ be a normed abelian group. Suppose that G is locally compact in the norm topology.
a) Show that G is complete.
b) Must G be ball compact?

2.5. **The topology on a normed field.**

Let k be a field and $|\ |$ an Artin absolute value on k. We claim that there is a unique metrizable topology on k such that a sequence $\{x_n\}$ in k converges to $x \in k$ iff $|x_n - x| \to 0$. To see this, first note that the condition $|x_n - x| \to 0$ depends only on the equivalence class of the Artin absolute value, since certainly $|x_n - x| \to 0 \iff |x_n - x|^\alpha \to 0$ for any positive real number α. So without changing the convergence of any sequence, we may adjust $|\ |$ in its equivalence class to get an absolute value (i.e., with Artin constant $C \leq 2$) and then we define the topology to be the metric topology with respect to $\rho(x, y) = |x - y|$ as above. Of course this recovers the given notion of convergence of sequences. Finally, we recall that a metrizable topological space is first countable and that there exists at most one first countable topology on a set with a given set of convergent sequences. We call this topology the **valuation topology**.

Exercise 2.22: Show that the trivial valuation induces the discrete topology.

Exercise 2.23: Let $(k, |\ |)$ be a valued field, and let $\{x_n\}$ be a sequence in k. Show that $x_n \to 0$ iff $|x_n| \to 0$.

Proposition 7. *Let* $|\ |_1$ *and* $|\ |_2$ *be norms on a field* k*. TFAE:*
(i) $|\ |_1 \sim |\ |_2$ *in the sense of Theorem 1.4.*
(ii) The topologies induced by $|\ |_1$ *and* $|\ |_2$ *coincide.*

Proof. The direction (i) $\implies$ (ii) follows from the discussion given above. Assume (ii). Let $x \in k$. Then $|x|_1 < 1 \iff x^n \to 0$ in the $|\ |_1$-topology iff $x^n \to 0$ in the $|\ |_2$-metric topology $\iff |x|_2 < 1 \iff |\ |_1 \sim |\ |_2$. □

An equivalent topological statement of Artin-Whaples approximation is:

Theorem 8. *(Artin-Whaples Restated) Let* k *be a field and, for* $1 \leq i \leq n$*, let* $|\ |_i$ *be inequivalent nontrivial norms on* k*. Let* (k, τ_i) *denote* k *endowed with the* $|\ |_i$*-norm topology, and let* $k^n = \prod_{i=1}^n (k, \tau_i)$*. Then the diagonal map* $\Delta : k \hookrightarrow k^n$, $x \mapsto (x, \ldots, x)$ *has dense image.*

Exercise 2.24: Convince yourself that this is equivalent to Theorem 1.5.

Exercise 2.25: Show that any two closed balls of finite radius in a normed field are homeomorphic. Deduce that a locally compact normed field is ball compact. (In particular, it is complete, although we knew that already by Exercise X.X.)

2.6. Completion of a metric space.

Lemma 9. *Let (X,ρ_X) be a metric space, (Y,ρ_Y) be a complete metric space, $Z \subset X$ a dense subset and $f : Z \to Y$ a continuous function.*
a) There exists at most one extension of f to a continuous function $F : X \to Y$. (N.B.: This holds for for any topological space X and any Hausdorff space Y.)
b) f is uniformly continuous $\implies$ f extends to a uniformly continuous $F : X \to Y$.
c) If f is an isometric embedding, then its extension F is an isometric embedding.

Exercise 2.26: Prove Lemma 9.

Let us say that a map $f : X \to Y$ of topological spaces is **dense** if $f(X)$ is dense in Y. An **isometric embedding** is a map $f : (X,\rho_X) \to (Y,\rho_Y)$ such that for all $x_1,\ x_2 \in X$, $\rho_Y(f(x_1), f(x_2)) = \rho_X(x_1,x_2)$. An **isometry** is a surjective isometric embedding.

Exercise 2.27: Let f be an isometric embedding of metric spaces.
a) Show that f is uniformly continuous with $\delta = \epsilon$.
b) Show that f is injective. Therefore an isometry is bijective. Show that if f is an isometry, then f^{-1} is also an isometry.

Theorem 10. *let (X,ρ) be a metric space.*
a) There is a complete metric space $\hat{X}$ and a dense isometric embedding $\iota : X \to \hat{X}$.
b) The completion ι satisfies the following universal mapping property: if (Y,ρ) is a complete metric space and $f : X \to Y$ is a uniformly continuous map, then there exists a unique uniformly continuous map $F : \hat{X} \to Y$ such that $f = F \circ \iota$.
c) If $\iota' : X \hookrightarrow \hat{X}'$ is another isometric embedding into a complete metric space with dense image, then there exists a unique isometry $\Phi : \hat{X} \to \hat{X}'$ such that $\iota' = \Phi \circ \iota$.

Proof. a) Let $X^\infty = \prod_{i=1}^\infty X$ be the set of all sequences in X. Inside X, we define $\mathcal{X}$ to be the set of all Cauchy sequences. We introduce an equivalence relation on $\mathcal{X}$ by $x_\bullet \sim y_\bullet$ if $\rho(x_n,y_n) \to 0$. Put $\hat{X} = \mathcal{X}/\sim$. For any $x \in X$, define $\iota(x) = (x,x,\ldots)$, the constant sequence based on x. This of course converges to x, so is Cauchy and hence lies in $\mathcal{X}$. The composite map $X \xrightarrow{\iota} \mathcal{X} \xrightarrow{\sim} \hat{X}$ (which we continue to denote by ι) is injective, since $\rho(x_n,y_n) = \rho(x,y)$ does not approach zero. We define a map $\hat{\rho} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ by
$$\hat{\rho}(x_\bullet, y_\bullet) = \lim_{n\to\infty} \rho(x_n,y_n).$$
To see that this limit exists, we may reason (for instance) as follows: the sequence $x_\bullet \times y_\bullet$ is Cauchy in $X \times X$, hence its image under the uniformly continuous function ρ is Cauchy in the complete metric space $\mathbb{R}$, so it is convergent. It is easy to see that $\hat{\rho}$ factors through to a map $\hat{\rho} : \hat{X} \to \hat{X} \to \mathbb{R}$. The verification that $\hat{\rho}$ is a metric on $\hat{X}$ and that $\iota : X \to \hat{X}$ is an isometric embedding is straightforward and left to the reader. Moreover, if $x_\bullet = \{x_n\}$ is a Cauchy sequence in X, then the sequence of constant sequences $\{\iota(x_n)\}$ is easily seen to converge to $x_\bullet$ in $\hat{X}$.
b) Let $x_\bullet \in \mathcal{X}$ be a Cauchy sequence in X. As above, since f is uniformly continuous and Y is complete, $f(x_\bullet)$ is convergent in Y to a unique point, say y, and we put $y = F(x_\bullet)$. Since X is dense in $\hat{X}$ this is the only possible choice, and by Lemma 9 it does indeed give a well-defined uniformly continuous function $F : X \to Y$.
c) Isometric embeddings are uniformly continuous, so we may apply the universal mapping property of part b) to the map $\iota' : X \hookrightarrow \hat{X}'$ to get a map $\Phi : \hat{X} \to \hat{X}'$.

Similarly, we get a map $\Phi' : \hat{X}' \to \hat{X}$. The compositions $\Phi' \circ \Phi$ and $\Phi' \circ \Phi$ are uniformly continuous maps which restrict to the identity on the dense subspace X, so they must each by the identity map, i.e., Φ and Φ' are mutually inverse bijections. By Lemma 9c), Φ is an isometric embedding, and therefore it is an isometry. □

We refer to $\hat{X}$ as the **completion** of X.[7]

Corollary 11. *(Functoriality of completion) a) Let $f : X \to Y$ be a uniformly continuous map between metric spaces. Then there exists a unique map $F : \hat{X} \to \hat{Y}$ making the following diagram commute:*

$$X \xrightarrow{f} Y$$
$$\hat{X} \xrightarrow{F} \hat{Y}.$$

b) If f is an isometric embedding, so is F.
c) If f is an isometry, so is F.

Proof. a) The map $f' : X \to Y \hookrightarrow \hat{Y}$, being a composition of uniformly continuous maps, is uniformly continuous (check this if you haven't seen it before!). Applying the universal property of completion to f' gives a unique extension $\hat{X} \to \hat{Y}$.
Part b) follows immediate from Lemma 9b). As for part c), if f is an isometry, so is its inverse f^{-1}. The extension of f^{-1} to a mapping from $\hat{Y}$ to $\hat{X}$ is easily seen to be the inverse function of F. □

Exercise 2.28: For a metric space (X, ρ), define the **distance set** $\mathcal{D}(X) = \rho(X \times X)$, i.e., the set real numbers which arise distances between points in X.
a) Prove or disprove: if $\mathcal{D}$ is a discrete subset of $\mathbb{R}$, then ρ is ultrametric.
b) Prove or disprove: if ρ is an ultrametric, then $\mathcal{D}$ is discrete.
c) Let $\tilde{X}$ be the completion of X. Show that $\mathcal{D}(\tilde{X}) = \overline{\mathcal{D}(X)}$ (closure in $\mathbb{R}$).
d)(U) Determine which subsets of $\mathbb{R}^{\geq 0}$ arise as distance sets of some metric space.

Exercise 2.29: The notion of a metric space and a completion seems to presuppose knowledge of $\mathbb{R}$, the set of real numbers. In particular, it is *a priori* logically unacceptable to *define* $\mathbb{R}$ to be the completion of $\mathbb{Q}$ with respect to the Archimedean norm $| \, |_\infty$. (Apparently for such reasons, Bourbaki's influential text *General Topology* avoids mention of the real numbers until page 329, long after a general discussion of uniform spaces and topological groups.) Show that this is in fact not necessary and that the completion of a metric space *can* be used to construct the real numbers. (Hint: first define a $\mathbb{Q}$-valued metric and its completion.)

2.7. Completions of normed abelian groups and normed fields.

When G is a normed abelian group (or a field with an absolute value) we wish to show that the completion $\tilde{G}$ is, in a natural way, again a normed abelian group (or a field with an absolute value). This follows readily from the results in the previous section, but we take the opportunity to point out a simplification in the construction of $\hat{G}$ in this case.

As above, we put $G^\infty = \prod_{i=1}^\infty$ and $\mathcal{G}$ the subset of Cauchy sequences. But this time

[7] This is a standard abuse of terminology: really we should refer to the *map* $\iota : X \hookrightarrow \hat{X}$ as the completion, but one rarely does so.

G^∞ is an abelian group and $\mathcal{G}$ is a subgroup of G^∞ (easy exercise). Furthermore, we may define $\mathfrak{g}$ to be the set of sequences converging to 0, and then $\mathfrak{g}$ is a subgroup of $\mathcal{G}$. Thus in this case we may define $\hat{G}$ simply to be the quotient group $\mathcal{G}/\mathfrak{g}$, so by its provenance it has the structure of an abelian group. Moreover, if $x_\bullet$ is a Cauchy sequence in G, then by Exercise X.X $|x_\bullet|$ is a Cauchy sequence in $\mathbb{R}$, hence convergent, and we may define

$$|x_\bullet| = \lim_{n\to\infty} |x_n|.$$

We leave it to the reader to carry through the verifications that this factors to give a norm on $\hat{G}$ whose associated metric is the same one that we constructed in the proof of Theorem XX.

Now suppose that $(k, |\ |)$ is a normed field. Then the additive group $(k, +)$ is a normed abelian group, so the completion $\hat{k}$ exists at least as a normed abelian group. Again though we want more, namely we want to define a multiplication on $\hat{k}$ in such a way that it becomes a field and that the norm satisfies $|xy| = |x||y|$. Again the porduct map on k is uniformly continuous, so that it extends to $\hat{k}$, but to see that $\hat{k}$ is a field the algebraic construction is more useful. Indeed, it is not hard to show that k^∞ is a ring, the Cauchy sequences $\mathcal{K}$ form a subring. But more is true:

Lemma 12. *The set $\mathfrak{k}$ of sequences converging to 0 is a maximal ideal of the ring $\mathcal{K}$ of Cauchy sequences. Therefore the quotient $\mathcal{K}/\mathfrak{k} = \hat{k}$ is a field.*

Proof. Since a Cauchy sequence is bounded, and a sequence which converges to 0 multiplied by a bounded sequence again converges to 0, it follows that $\mathfrak{k}$ is an ideal of $\mathcal{K}$. To show that the quotient is a field, let $x_\bullet$ be a Cauchy sequence which does not converge to 0. Then we need to show that $x_\bullet$ differs by a sequence converging to 0 from a unit in $\mathcal{K}$. But since $x_\bullet$ is Cauchy and not convergent to 0, then (e.g. since it converges to a nonzero element in the abelian group $\hat{k}$) we have $x_n \neq 0$ for all sufficiently large n. Since changing any finite number of coordinates of $x_\bullet$ amounts to adding a sequence which is ultimately zero hence convergent to 0, this is permissible as above, so after adding an element of $\mathfrak{k}$ we may assume that for all $n \in \mathbb{Z}^+$, $x_n \neq 0$, and then the inverse of $x_\bullet$ in $\mathcal{K}$ is simply $\{\frac{1}{x_n}\}$. $\square$

Exercise 2.30*: Find all maximal ideals in the ring $\mathcal{K}$.

Exercise 2.30.5: Let $(k, |\ |)$ be a nontrivially normed field.
a) Show that $\#\{x \in k \mid 0 < |x| < 1\} = \#k$.
b) Show that the cardinality of the set of all convergent sequences in k is $(\#k)^{\aleph_0}$. Deduce that the same holds for the set of all Cauchy seqeunces of k.
c) Show that the cardinality of the completion of k is $(\#k)^{\aleph_0}$. (Hint: consider separately the cases in which $\#k = (\#k)^{\aleph_0}$ and $\#k < (\#k)^{\aleph_0}$.

Thus for a field k to be complete with respect to a nontrivial norm, it must satisfy a rather delicate cardinality requirement: $(\#k)^{\aleph_0} = \#k$. This certainly implies $\#k \geq 2^{\aleph_0} = \#\mathbb{R}$, i.e., k has at least continuum cardinality. Conversely, there are certainly complete fields of continuum cardinality, and indeed have $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}$. However, there are sets S with $2^{\aleph_0} < \#S < (\#S)^{\aleph_0}$.

Let $(k, | \ |)$ be a normed field, and let $\sigma : k \to k$ be a field automorphism. We say that σ is an automorphism of the normed field $(k, | \ |)$ if $\sigma^* | \ | = | \ |$.

Exercise 2.31: Let $(k, | \ |)$ be a normed field and σ an automorphism of k. Show that σ is an automorphism of $(k, | \ |)$ iff it is continuous in the norm topology on k.

Exercise 2.32: Let $(k, | \ |)$ be a complete normed field, and let σ be an automorphism of k. Put $| \ |' = \sigma^* | \ |$. Show that k is also complete with respect to $| \ |'$.

Exercise 2.33: Let k be either $\mathbb{R}$ or $\mathbb{Q}_p$ for some prime p. We will show that k is **rigid**, i.e., has no automorphisms other than the identity.
Let $\sigma : k \to k$ be a field automorphism.
a) Suppose that σ is continuous. Show that $\sigma = 1_k$.
b) Show that any automorphism σ of k is continuous with respect to the norm topology. (Hint: Ostrowski's Theorem.)

Exercise 2.34: Let k be a field complete with respect to a discrete, nontrivial valuation. Let R be its valuation ring.
a) Show that k is homeomorphic to the infinite disjoint union $\coprod_{i=1}^{\infty} R$.
b) Let $k_1, \ k_2$ be two fields complete with respect to discrete, nontrivial valuations, with valuation rings $R_1, \ R_2$. Suppose that R_1 and R_2 are compact. Show that k_1 and k_2 are homeomorphic, locally compact topological spaces.

We now give an alternate, more algebraic construction of the completion in the special case of a discretely valued, non-Archimedean norm on k. Namely, the norm is equivalent to a $\mathbb{Z}$-valued valuation v, with valuation ring

$$R = \{x \in k \mid v(x) \geq 0\}$$

and maximal ideal

$$\mathfrak{m} = \{x \in k \mid v(x) > 0\} = \{x \in k \mid v(x) \geq 1\}.$$

Lemma 13. *With notation above, suppose that k is moreover complete. Then the ring R is $\mathfrak{m}$-adically complete. Explicitly, this means that the natural map*

$$R \to \lim_n R/\mathfrak{m}^n$$

is an isomorphism of rings.

Proof. This is straightforward once we unpack the definitions.
Injectivity: this amounts to the claim that $\bigcap_{n \in \mathbb{Z}^+} \mathfrak{m}^n = 0$. In fact this holds for any nontrivial ideal in a Noetherian domain (Krull Intersection Theorem), but it is obvious here, because $\mathfrak{m}^n = (\pi^n) = \{x \in R \mid v(x) \geq n$, and the only element of R which has valuation at least n for all positive integers n is 0.
Surjectivity: Take any element $\mathbf{x}$ of the inverse limit, and lift each coordinate arbitarily to an element $x_n \in R$. It is easy to see that $\{x_n\}$ is a Cauchy sequence, hence convergent in R – since k is assumed to be complete and R is closed in k, R is complete). Let x be the limit of the sequence x_n. Then $x \mapsto \mathbf{x}$. $\square$

Exercise 2.35: Suppose now that v is a discrete valuation on a field k. Let $\hat{R} = \lim_n R/\mathfrak{m}^n$.
a) Show that $\hat{R}$ is again a discrete valuation ring – say with valuation $\hat{v}$ – whose

maximal ideal $\hat{m}$ is generated by any uniformizer π of R.
b) Let $\mathbb{K}$ be the fraction field of $\hat{R}$. Show that $\mathbb{K}$ is canonically isomorphic to $\hat{k}$, the completion of k in the above topological sense.
c) Let $n \in \mathbb{Z}^+$. Explain why the natural topology on the quotient $R/\mathfrak{m}^n$ is the discrete topology.
d) Show that the following topologies on $\hat{R}$ all coincide: (i) the topology induced from the valuation $\hat{v}$; (ii) the topology $\hat{R}$ gets as a subset of $\prod_n R/\mathfrak{m}^n$ (the product of discrete topological spaces); (iii) the topology it inherits as a subset of $\hat{k}$ under the isomorphism of part b).

2.8. **Non-Archimedean Functional Analysis: page 1.**

K-Banach spaces: Let $(K, | \ |)$ be a complete normed field. In this context we can define the notion of a normed linear space in a way which directly generalizes the more familiar cases $K = \mathbb{R}$, $K = \mathbb{C}$. Namely:

A normed K-linear space is is a K-vector space V and a map $| \ | : V \rightarrow \mathbb{R}^{\geq 0}$ such that:

(NLS1) $\forall x \in V$, $x = 0 \iff |x| = 0$.
(NLS2) $\forall \alpha \in K, \ x \in V$, $|\alpha x| = |\alpha||x|$.
(NLS3) $\forall x, y \in V$, $|x + y| \leq |x| + |y|$.

If K is non-Archimedean, we require the stronger inequality $|x + y| \leq \max(|x|, |y|)$ in (NLS3).

Remark: Weakening (NLS1) to $\implies$, we get the notion of a **seminormed space**.

Note that a normed linear space is a normed abelian group under addition. In particular it has a metric. A **K-Banach space** is a complete normed linear space over K.

The study of K-Banach spaces (and more general topological vector spaces) for a non-Archimedean field K is called **non-Archimedean functional analysis**. This exists as a mathematical field which has real applications, e.g., to modern number theory (via spaces of p-adic modular forms). The theory is similar but not identical to that of functional analysis over $\mathbb{R}$ or $\mathbb{C}$. (Explain that the weak Hahn-Banach theorem only holds for spherically complete fields...)

Recall that two norms $| \ |_1$, $| \ |_2$ on a K-vector space V are **equivalent** if there exists $\alpha \in \mathbb{R}^{>0}$ such that for all $v \in V$,

$$\frac{1}{\alpha}|v|_1 \leq |v|_2 \leq \alpha |v|_1.$$

Equivalent norms induce the same topology.

Theorem 14. *Let $(K, | \ |)$ be a complete normed field, and let V be a finite dimensional K-vector space.*
a) Choose a basis $v_1, \ldots, v_n$ of V, and define a map $| \ |_\infty : V \rightarrow \mathbb{R}^{\geq 0}$ by $|\alpha_1 v_1 + \ldots + \alpha_n v_n|_\infty = \max_i |\alpha_i|$. Then $| \ |_\infty$ is a norm on V. The metric topology on V

is the one induced by pulling back the product topology on K^n via the isomorphism $V \cong K^n$.
b) Any two norms on V are equivalent.
c) It follows that for any norm $|\ |$ on V, $|\ |$ is complete and the induced topology coincides with the topology obtained by pulling back the product topology on K^n via any isomorphism $V \cong K^n$.

Proof. a) It is easy to see that $|\ |_\infty$ is a norm on V. For instance, for all $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in K$ we have

$$||(\alpha_1 v_1 + \ldots + \alpha_n v_n) + (\beta_1 v_1 + \ldots + \beta_n v_n)||_\infty$$
$$= ||(\alpha_1 + \beta_1)v_1 + \ldots + (\alpha_n + \beta_n)v_n||_\infty$$
$$= \max_i |\alpha_i + \beta_i| \leq \max_i |\alpha_i| + \max_i |\beta_i| = ||\alpha_1 v_1 + \ldots + \alpha_n v_n||_\infty + ||\beta_1 v_1 + \ldots + \beta_n v_n||_\infty.$$

Any finite product of metric spaces (X_i, d_i) can be endowed with this "∞-metric" – i.e., $d(x, y) = \max_i d(x_i, y_i)$ – and the induced topology is indeed just the product topology. Moreover a finite product of complete metric spaces is complete. This completes the proof of part a).
If $n = 1$, the remaining statements of the theorem are obvious. We now proceed by induction on n. Let $||\ ||$ be any norm on the n-dimensional vector space V. We claim that there are positive constants A and B such that for all $v \in V$,

$$A||v||_\infty \leq ||v|| \leq B||v||_\infty.$$

Indeed, we may take $B = \sum_{i=1}^n ||v_i||$. Now consider the subspace $U_1 = \langle v_2, \ldots, v_n \rangle_K$. By induction, $(U, ||\ ||)$ is complete, hence closed in V. Translation by any vector gives a homeomorphism, so $v_1 + U$ is also closed in V. It follows that there exists a neighborhood $\mathcal{N}_1$ of 0 which contains no vector whose first coordinate with respect to the basis $(v_1, \ldots, v_n)$ is 1. Applying the same argument to the subspace U_i generated by all basis vectors but v_i, we get a neighborhood $\mathcal{N}_i$, and let $A > 0$ be such that the open disk $B_0(A)$ is contained in $\mathcal{N}_1 \cap \ldots \cap \mathcal{N}_n$. Now let $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ be nonzero, say $||v||_\infty = |\alpha_i|$. Then the ith coordinate of $= \frac{1}{\alpha_i} v_i$ is equal to 1, i.e., $||\frac{1}{\alpha_i} v|| \geq A$, $||v|| \geq A|\alpha_i| = A||v||_\infty$. Thus we have shown that an arbitrary norm $||\ ||$ on V is equivalent to the infinity norm. The rest follows immediately. $\square$

Theorem 15. *Let $(K, |\ |)$ be a complete normed field and $(V, ||\ ||)$ a normed K-linear space. Let W be a finite-dimensional K-subspace of V. Then W is closed.*

Exercise 2.36: Use Theorem 14 to prove Theorem 15.

When one thinks of "Archimedean functional analysis", Theorems 14 and 15 are probably not the first two which come to mind, perhaps because they are not very interesting! On the other hand, for our purpose these results are remarkably useful: they are just what we need to prove the uniqueness in Theorem 3, a topic to which we now turn.

2.9. **Big Ostrowski Revisited.**

The goal of this section is to prove the Big Ostrowski Theorem (Theorem 1.10). Our proof follows an approach taken by David Krumm (a student in the 2010 course) who was in turn following Neukirch's *Algebraic Number Theory*, but with some modifications. In fact we will prove the following result.

Theorem 16. *Let $(K, | \, |)$ be a complete Archimedean field with Artin constant 2 (equivalently, $|2| = 2$). Then $(K, | \, |)$ is isomorphic either to $\mathbb{R}$ with its standard absolute value or $\mathbb{C}$ with its standard absolute value.*

Let us first establish that Theorem 16 is *equivalent* to the Big Ostrowski Theorem. Indeed, first assume Theorem 1.10, and let $(K, | \, |_1)$ be an Archimedean normed field with Artin constant 2. Then we have an embeddding $\iota : K \hookrightarrow \mathbb{C}$ such that for all $x \in K$, $|x|_1 = |\iota(x)|$. There is an induced map $\hat{\iota} : (\hat{K}, | \, |_1) \to (\hat{\mathbb{C}}, | \, |) = (\mathbb{C}, | \, |)$, since the standard norm on $\mathbb{C}$ is complete. Thus $\iota(\hat{K})$ is an isometrically embedded complete subfield of $\mathbb{C}$. Since $\iota(\hat{K})$ contains $\mathbb{Q}$, it contains the closure of $\mathbb{Q}$ in $\mathbb{C}$, namely $\mathbb{R}$. Thus $\mathbb{R} \subset \iota(\hat{K}) \subset \mathbb{C}$. Since $[\mathbb{C} : \mathbb{R}] = 2$, we have little choice: either

$$(\hat{K}, | \, |) \cong (\iota(\hat{K}), | \, |_\infty) = (\mathbb{R}, | \, |_\infty)$$

or

$$(\hat{K}, | \, |) \cong (\iota(\hat{K}), | \, |_\infty) = (\mathbb{C}, | \, |_\infty).$$

Conversely, assume Theorem 16, and let $(K, | \, |)$ be an Archimedean normed field with Artin constant 2. Then $(K, | \, |)$ is a normed subfield of its completion $\hat{K}$, which is isomorphic to either $(\mathbb{R}, | \, |)_\infty)$ or $(\mathbb{C}, | \, |_\infty)$. Of course, $(\mathbb{R}, | \, |_\infty)$ is a normed subfield of $(\mathbb{C}, | \, |_\infty)$, so either way $(K, | \, |)$ can be isometrically embedded in $(\mathbb{C}, | \, |_\infty)$.

Now we turn to the proof of Theorem 16. First, since $(K, | \, |)$ is a complete Archimedean normed field, it has characteristic zero (Corollary 1.9) and thus contains $\mathbb{Q}$. By Ostrowski's Lemma (Lemma 1.10) and the computation of the Artin constant (Theorem 1.11), the restriction of $| \, |$ to $\mathbb{Q}$ must be the standard Archimedean absolute value $| \, |_\infty$. So $(\mathbb{Q}, | \, |)_\infty \hookrightarrow (K, | \, |)$ is an isometric embedding of normed fields. Taking completions, we get an isometric embedding $(\mathbb{R}, | \, |_\infty) \hookrightarrow (K, | \, |)$. The crux of the matter is the following claim.

Claim: The field extension $K/\mathbb{R}$ is algebraic.

Let us first argue for the sufficiency of the claim (which is easy) and then come back to prove the claim (which is somewhat tricky). Of course if $K/\mathbb{R}$ is algebraic then either $K = \mathbb{R}$ – so we are done in this case – or $[K : \mathbb{R}] = 2$ and K is isomorphic as an $\mathbb{R}$-algebra to the complex field. In this case there is something more to be shown, namely that the normed field $(K, | \, |)$ is isomorphic, as an extension of the normed field $(\mathbb{R}, | \, |_\infty)$, to $(\mathbb{C}, | \, |)$. Happily, the tools for this were developed in the previous section. Indeed, $(K, | \, |)$ is a finite dimensional normed $\mathbb{R}$-space (and $(\mathbb{R}, | \, |_\infty$ is complete!), so that the metric topology induced by the norm is the product topology on $\mathbb{R}^2$. We may use the $\mathbb{R}$-isomorphism of K with $\mathbb{C}$ to transport the norm $| \, |$ to $\mathbb{C}$. On $\mathbb{C}$ we also have the standard Archimedean norm $| \, |_\infty$. By the above remark, these two norms induce the same topology on $\mathbb{C}$ so are equivalent. Moreover, they both have Artin constant 2, so in fact they are equal. In other words, our isomorphism of $\mathbb{R}$-algebras $K \stackrel{\sim}{\to} \mathbb{C}$ is indeed an isomorphism of normed fields $(K, | \, |) \stackrel{\sim}{\to} (\mathbb{C}, | \, |_\infty)$.

Now we move on to prove the claim. Of course it is equivalent to show that every element of K is the root of a quadratic polynomial with $\mathbb{R}$-coefficients, and this is indeed how we will proceed.

A preliminary remark: the coming proof will use both the abstract norm $|\ |$ on K and the standard Archimidean norm $|\ |_\infty$ on $\mathbb{C}$. This is potentially confusing. In an attempt to lessen the confusion, we write complex numbers as (possibly subscripted) z's and w's and elements of K as Greek letters.

Let $\alpha \in K \setminus \mathbb{R}$. For $z \in \mathbb{C}$, put

$$P_z(t) = t^2 - (z + \overline{z})t + z\overline{z}.$$

Thus $P_z(t) \in \mathbb{R}$ is a quadratic polynomial whose roots in $\mathbb{C}$ are z and $\overline{z}$ (a double root, if $z \in \mathbb{R}$). Moreover, define a map $f : \mathbb{C} \to \mathbb{R}^{\geq 0}$ by

$$f(z) = |P_z(\alpha)|.$$

To say that α is quadratic over $\mathbb{R}$ is to say that there exists some $z \in \mathbb{C}$ such that $f(z) = 0$. We will prove this by a somewhat sneaky argument mixing algebra and topology. First, it is easy to see that f is continuous and that $f(z)$ tends to $+\infty$ with the modulus of z. Indeed, for $|z|_\infty$ sufficiently large, the constant term of $P_z(\alpha)$ dominates. Therefore f attains a minimum value $m \in \mathbb{R}^{\geq 0}$.

Seeking a contradiction, we assume that $m > 0$. Since f is continuous, the level set $Z = f^{-1}(m)$ is closed; since f tends to infinity Z is also bounded, i.e., compact. So there exists some $z_1 \in \mathbb{C}$ such that $f(z_1) = m$ and that $|z_1|_\infty$ is maximal among all $z \in Z$. If we can produce an element $w_1 \in \mathbb{C}$ such that $|w_1|_\infty > |z_1|_\infty$ but $f(w_1) \leq m$, then we will have attained our contradiction.

To do so, choose $\epsilon \in \mathbb{R}$ with $0 < \epsilon < m$, and let $w_1 \in \mathbb{C}$ be a root of the "perturbed polynomial" $P_{z_1}(t) + \epsilon$. The discriminant of $P_{z_1}(t) + \epsilon$ is strictly smaller than the discriminant of $P_{z_1}(t)$ (a quadratic polynomial which does not have distinct real roots) hence is negative; that is, $w \in \mathbb{C} \setminus \mathbb{R}$. Hence

$$P_{z_1}(t) + \epsilon = (t - z_1)(t - \overline{z_1}) + \epsilon = (t - w_1)(t - \overline{w_1}) = t^2 - (w_1 + \overline{w_1})t + w_1\overline{w_1}.$$

Comparing constant coefficients in these two expressions gives

$$|z_1|_\infty^2 = |w_1|_\infty^2 - \epsilon < |w_1|_\infty^2,$$

so $|z_1|_\infty < |w_1|_\infty$. Thus by our above setup we must have $f(w_1) > m$. But we claim that we also have $f(w_1) \leq m$. This is established as follows: let n be an odd positive integer, and define

$$g(t) = P_{z_1}(t)^n + \epsilon^n.$$

Factor $g(t)$ over $\mathbb{C}$ as

$$g(t) = \prod_{i=1}^{2n} (t - w_i).$$

Note that since n is odd, $P_{z_1}(t) + \epsilon$ divides $g(t)$ so that indeed w_1 is one of the roots of g: our notation is consistent. Also $g(t) \in \mathbb{R}[t]$ so we must also have

$$g(t) = \prod_{i=1}^{2n} (t - \overline{w_i}).$$

Thus

$$g(t)^2 = \prod_{i=1}^{2n} (t - w_i)(t - \overline{w_i}) = \prod_{i=1}^{2n} (t^2 - (w_i + \overline{w_i})t + w_i\overline{w_i}).$$

It folows that

$$|g(\alpha)|^2 = \prod_{i=1}^{2n} |\alpha^2 - (w_i + \overline{w_i})\alpha + w_i\overline{w_i}| = \prod_{i=1}^{2n} f(w_i) \geq f(w_1)m^{2n-1}.$$

On the other hand,

$$|g(\alpha)| \leq |P_{z_1}(\alpha)|^n + \epsilon^n = f(z_1)^n + \epsilon^n = m^n + \epsilon^n.$$

Therefore

$$f(w_1) \leq \frac{|g(\alpha)|^2}{m^{2n-1}} \leq \frac{(m^n+\epsilon^n)^2}{m^{2n-1}} = m\left(1 + (\frac{\epsilon}{m})^n\right)^2.$$

Since $0 < \epsilon < m$, sending n to infinity gives $f(w_1) \leq m$, contradiction! This completes the proof of Theorem 16.

2.10. **Theorems of Mazur, Gelfand and Tornheim.**

In this section we will sketch a different approach to proving the Big Ostrowski Theorem – or rather, the equivalent Theorem 16. Let $(K, |\ |)$ be a field which is complete with respect to a norm with Artin constant 2. We want to show that $(K, |||)$ is isomorphic to $(\mathbb{R}, |\ |_\infty)$ or to $(\mathbb{C}, |\ |_\infty)$. As in the previous section, applying the Little Ostrowski Theorem we easily see that $(K, |\ |)$ has a normed subfield isomorphic to $(\mathbb{R}, |\ |)$. The new idea here is that this implies that $(K, |\ |)$ is a real Banach algebra: i.e., an $\mathbb{R}$-vector space A complete with respect to a norm $|\ |$ with $|1| = 1$ and which is (at least) *sub*-multiplicative: for all $x, y \in A$, $|xy| \leq |x||y|$.

We will prove the following result, which is a generalization of Theorem 16.

Theorem 17. *(Gelfand-Tornheim) Let $(K, ||\ ||)$ be a real Banach space which is also a field, and such that $||1|| = 1$ and for all $x, y \in K$, $||xy|| \leq ||x||||y||$. Then $(K, ||\ ||)$ is isomorphic to $(\mathbb{R}, |\ |_\infty)$ or to $(\mathbb{C}, |\ |_\infty)$.*

There is an evident corresponding notion of a *complex* Banach algebra, and in fact the theory of complex Banach algebras is much better developed than the theory of real Banach algebras, so our first step is to reduce to the complex case. This is handled as follows: if K contains a square root of -1, then indeed it contains a subfield isomorphic to $\mathbb{C}$ (even as a normed field, by the uniqueness up to equivalence of the norm on a finite extension of a complete field). So if K does not contain a square root of -1, we would like to replace K by $K(\sqrt{-1})$.

The natural result to try to prove is the following.

Theorem 18. *Let K be a field of characteristic different from 2 which is complete with respect to a norm $|\ |$. Let L/K be a quadratic extension. Then $x \in L \mapsto |N_{L/K}(x)|^{\frac{1}{2}}$ is a norm on L extending the given norm on K.*

Note that Theorem 18 is a very special case of Theorem 3. In particular, the Archimedean case follows from the Big Ostrowski theorem and the non-Archimedean case will be proved later by other methods. A direct proof of Theorem 18 is indeed possible, but somewhat lengthy and unpleasant. The reader who wants to see it may consult [BAII, §9.5].

However, in order to prove Theorem 17 we can get away with less than this:[8] it

[8]This simpler path is taken from [Bou, §VI.6.4].

is enough to endow $K(\sqrt{-1})$ with an $\mathbb{R}$-algebra norm which is *sub*multiplicative, i.e., for all $x, y \in K(\sqrt{-1})$, $||xy|| \leq ||x|| ||y||$. But this is easy: for $x, y \in K$ we put $||x + \sqrt{-1}y|| := ||x|| + ||y||$. Certainly this endows $K(\sqrt{-1})$ with the structure of a real Banach space. Moreover, for $z = x + \sqrt{-1}y$, $z' = x' + \sqrt{-1}y'$ in $K(\sqrt{-1})$, we calculate

$$\begin{aligned}||zz'|| = ||xx' - yy'|| + ||xy' + x'y|| &\leq ||xx'|| + ||yy'|| + ||xy'|| + ||x'y|| \\ &\leq ||x|| ||x'|| + ||y|| ||y'|| + ||x|| ||y'|| + ||x'|| ||y|| \\ &= (||x|| + ||y||)(||x'|| + ||y'||) = ||z|| ||z'||.\end{aligned}$$

Putting $L = K(\sqrt{-1})$, we have endowed L with the structure of a complex Banach algebra. To complete the proof of Theorem 17, it is enough to show that the only field which is a complex Banach algebra is $\mathbb{C}$ itself. Again we shall prove rather more than this. First a few preliminaries.

Lemma 19. *(Neumann) Let $(A, || \, ||)$ be a complex Banach algebra, and let $x \in A$ be such that $||x|| < 1$. Then $1 - x \in A^\times$; explicitly,*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Proof. Since $||x|| < 1$, we have $\sum_{n=0}^\infty ||a^n|| \leq \sum_{n=0}^\infty ||a||^n < \infty$. That is, the series $\sum_{n=0}^\infty x^n$ is absolutely convergent and thus, by completeness, convergent; denote the sum by b. It is then easily seen that $(1-a)b = b(1-a) = 1$. $\square$

Let $(A, || \, ||)$ be a complex Banach algebra and let $x \in A$. We make a key definition: the **spectrum** $\sigma(x)$ of x is the set of all complex numbers z such that $z - x \in A \setminus A^\times$, i.e., such that $z - x$ is not invertible. The complement $\mathbb{C} \setminus \sigma(x)$ is called the **resolvent set** of x. From Lemma 19 it follows that $A^\times$ contains an open neighborhood of 1 and thus is open. It follows easily that the resolvent set of x is open and thus the spectrum $\sigma(x)$ is closed. Moreover, if $z \in \mathbb{C}$ is such that $|z| > ||x|$, then $||z^{-1}x|| < 1$ and so $1 - z^{-1}x \in A^\times$; since also $z \in A^\times$, we find $z - x \in A^\times$. Thus the spectrum is also bounded, so the resolvent set is in particular nonempty.

Lemma 20. *Let $(A, || \, ||)$ be a complex Banach algebra, and let $x \in A$. Let $U(x)$ be the resolvent set of x. Let $\varphi : A \to \mathbb{C}$ be any bounded (equivalently, continuous) linear functional. The map $f : U \to \mathbb{C}$ by $z \mapsto \varphi(\frac{1}{z-x})$ is holomorphic.*

Proof. Fix $z \in U(x)$. Then

$$\begin{aligned}\lim_{h \to 0} \frac{1}{h}(\varphi(\frac{1}{z+h-x}) - \varphi(\frac{1}{z-x})) &= \lim_{h\to 0} \frac{1}{h}\varphi(\frac{1}{z+h-x} - \frac{1}{z-x}) \\ = \lim_{h \to 0} \frac{1}{h}\varphi(\frac{h}{(z+h-x)(z-x)}) &= \lim_{h\to 0} \varphi(\frac{1}{(z+h-x)(z-x)}) \\ &= \varphi(\lim_{h\to 0} \frac{1}{(z+h-x)(z-x)}).\end{aligned}$$

Lemma 19 implies that $x \mapsto \frac{1}{z-x}$ is continuous on $U(x)$, so this last limit exists. $\square$

Now we can prove the following celebrated result, a complete classification of all complex Banach division algebras, commutative or otherwise.

Theorem 21. *(Gelfand-Mazur) Let $(A, || \ ||)$ be a complex Banach algebra such that $A \setminus \{0\} = A^\times$. Then $A \cong \mathbb{C}$.*

Proof. We claim that for all $x \in A$, the spectrum $\sigma(x)$ is non-empty.

Indeed, this immediately implies the Gelfand-Mazur theorem: let A be a complex Banach algebra such that every nonzero element is invertible. If $A \neq \mathbb{C}$ then there exists $x \in A \setminus \mathbb{C}$. But then for every $z \in \mathbb{C}$, $z - x$ is not in $\mathbb{C}$, so certainly it is not zero, so it is invertible: $\sigma(x) = \emptyset$.

Now we prove the nonemptiness of $\sigma(x)$: suppose for a contradiction that $\sigma(x)$ is empty, so the resolvent set $U(x) = \mathbb{C}$. Let $\varphi : A \to \mathbb{C}$ be a bounded linear functional such that $\varphi(-\frac{1}{x}) \neq 0$. (To see that such a thing exists, choose a $\mathbb{C}$-basis for A in which $\frac{-1}{x}$ is the first element, and define φ by $\varphi(\frac{-1}{x}) = 1$ and for every other basis element e_i, $\varphi(e_i) = 0$. Note also that this is a special case of the Hahn-Banach Theorem.) Let $f : \mathbb{C} \to \mathbb{C}$ be the function $z \mapsto \varphi(\frac{1}{z-x})$. By Lemma 20, f is holomorphic on all of $\mathbb{C}$, i.e., entire. Let $C = ||\varphi||$, i.e., the least number such that for all $a \in A$, $|\varphi(a)| \leq C||a||$. For $|z| > 2||x||$, we have

$$|f(z)| = |\varphi(\frac{1}{z-x})| = \frac{1}{|z|}|\varphi(\frac{1}{1 - z^{-1}x})|$$

$$= \frac{1}{|z|}|\varphi\left(\sum_{n=0}^{\infty}(z^{-1}x)^n\right)| \leq \frac{1}{|z|}\sum_{n=0}^{\infty}|\varphi((z^{-1}x)^n)|$$

$$\leq \frac{C}{|z|}\sum_{n=0}^{\infty}||z^{-1}x||^n < \frac{C}{|z|}\sum_{n=0}^{\infty}(\frac{1}{2})^n = \frac{2C}{|z|}.$$

Thus the entire function f is bounded, hence constant. Moreover, $\lim_{|z|\to\infty} f(z) = 0$, so f must be identically zero. But $f(0) = \varphi(\frac{-1}{x}) = 1$: contradiction! □

Finally, we remark that the only other division Banach algebra over $\mathbb{R}$ is $\mathbb{H}$, the quaternions. For a proof of this, see e.g. [Bou, §VI.6.4].

2.11. **Proof of Theorem 3 Part I: Uniqueness.**

Theorem 22. *Let $(K, | \ |)$ be a complete normed field, let L/K be a field extension of finite degree d, and let $| \ |$ be a norm on L extending the given norm on K. Then we* must have *that for all $x \in L$,*

$$|x| = |N_{L/K}(x)|^{\frac{1}{d}} \tag{1}$$

Proof. Step 1: We may assume without loss of generality that L/K is normal. This reduction is left as an exercise.
Step 2: Suppose first that L/K is separable, so WLOG L/K is Galois and $N_{L/K}(x) = \prod_{\sigma\in\mathrm{Aut}(L/K)}\sigma(x)$. Then by the preceding theorem we have

$$|N_{L/K}(x)| = |\prod_{\sigma\in\mathrm{Aut}(L/K)}\sigma(x)| = \prod_{\sigma\in\mathrm{Aut}(L/K)}|x| = |x|^d.$$

Step 3: In the general case, let d_s be the number of distinct K-embeddings of L into an algebraic closure $\overline{K}$ of K (the "separable degree") and let $d_i = \frac{d}{d_s}$ (the "inseparable degree"). As a piece of basic field theory, we have that – under the assumption that L/K is normal – $N_{L/K}(x) = (\prod_{\sigma\in\mathrm{Aut}(L/K)}\sigma(x))^{d_i}$. The proof now proceeds as in Step 2 above. □

Exercise 2.37: Work out the details of Step 1 of the proof of Theorem 3.

Corollary 23. *Let $(K, | \ |)$ be a complete normed field, and let L/K an algebraic extension.*
a) There exists at most one norm on L extending $| \ |$ on K.
b) Suppose that for every finite subextension M of L/K, the mapping $x \in M \mapsto |N_{M/K}(x)|^{\frac{1}{[M:K]}}$ of (1) is indeed a norm on M. Then the map

$$x \in L \mapsto |N_{K[x]/K}(x)|^{\frac{1}{[K[x]:K]}} \tag{2}$$

is a norm on L.

Exercise 2.38: Prove Corollary 23. (Hint: use Exercise 2.2.)

Exercise 2.39: Suppose that $(K, | \ |)$ is Archimedean and L/K is an algebraic field extension of K. Use Theorem 3 to show that (2) is the unique norm on L extending $| \ |$ on K (i.e., reprove uniqueness and verify existence!).

2.12. **Proof of Theorem 5.**

We come now to the most technically complicated of the basic extension theorems, Theorem 5. The reader will surely have noticed that we have taken some time to build up suitable tools and basic facts. Now our hard work comes to fruition: given what we already know, the proof of Theorem 5 (modulo the existence part of Theorem 3 in the non-Archimedean case, which we will treat last) is rather straightforward and elegant.

Let us begin by recalling the setup and what we already know. Let $(K, | \ |)$ be a normed field (note that we are certainly interested in the Archimedean case, and even the case of the standard Archimedean norm on $K = \mathbb{Q}$!). Let L/K be a degree n extension. Let $\overline{\hat{K}}$ be the algebraic closure of the completion of $(K, | \ |)$. We know:

- There is a unique norm on $\overline{\hat{K}}$ extending the given norm on K.
- Every norm on L extending $| \ |$ comes from an embedding $\iota : L \hookrightarrow \overline{\hat{K}}$.

Since L/K is algebraic and $\overline{\hat{K}}$ is an algebraically closed field containing K, certainly there exists at least one K-algebra embedding $\iota : L \hookrightarrow \overline{\hat{K}}$, thus at least one extended norm on L. Since $[L : K] = n < \infty$, the number of such embeddings ι is at most n, in particular it is finite. Therefore, let g be the number of norms on L extending K. We have:

$$1 \leq g \leq n = [L : K]$$

and the problem is to compute g exactly in terms of L, K and $| \ |$.

For $1 \leq i \leq g$, let $| \ |_i$ be the norms on L extending $| \ |$ on K. Then $(L, | \ |_i)$ is a normed field and we may take the completion, say $\hat{L}_i$.

Now I claim that there is a canonical ring homomorphism

$$\Phi : L \otimes_K \hat{K} \to \prod_{i=1}^{g} \hat{L}_i.$$

Indeed, to define it, we will use the universal properties of the direct product and the tensor product to reduce to a situation where we can easily guess what the definition should be. First, just by writing out Φ in coordinates we have $\Phi = (\Phi_i)_{i=1}^g$ where $\Phi_i : L \otimes_K \hat{K} \to \hat{L}_i$. In other words, to define Φ, it is necessary and sufficient to define each Φ_i. Moreover, by the universal property of the tensor product, to define Φ_i what we need is precisely a K-bilinear map $\varphi_i : L \times \hat{K} \to \hat{L}_i$. What is the "obvious" map here? Well, **observe** that $\iota_i(L)$ and $\hat{K}$ are both subfields of $\hat{L}_i$, so given an element $x \in L$ and $y \in \hat{K}$, we may use ι_i to map L into $\hat{L}_i$ and then multiply them in $\hat{L}_i$. Explicitly,

$$\varphi_i(x, y) := \iota_i(x) \cdot y,$$

and thus Φ is defined (on "simple tensors" $x \otimes y$, and then uniquely extended by linearity) as $\Phi(x \otimes y) = (\iota_i(x)y)_{i=1}^g$.

Let us stop and note that Φ is a map between two objects each with a lot of structure. Both the source and target of Φ are finite dimensional $\hat{K}$-algebras, and Φ is a $\hat{K}$-linear map. Indeed, another perspective on the definition of Φ is to define the diagonal map $\Delta : L \hookrightarrow \prod_{i=1}^g \hat{L}_i$, $x \mapsto (\iota_i(x))$, note that Δ is K-linear and that $\prod_{i=1}^g \hat{L}_i$ is also a $\hat{K}$-algebra, so that Φ is the unique map corresponding to Δ under the canonical "adjunction" isomorphism

$$\mathrm{Hom}_K(L, \prod_i \hat{L}_i) = \mathrm{Hom}_{\hat{K}}(L \otimes_K \hat{K}, \prod_i \hat{L}_i).$$

Moreover, like any two finite-dimensional $\hat{K}$-vector spaces, $L \otimes_K \hat{K}$ and $\prod_{i=1}^g \hat{L}_i$ each come with a canonical topology, such that any $\hat{K}$-linear map between them (for instance, Φ!) is necessarily continuous.

The precise result we want to prove is the following:

Theorem 24. *Let $\Phi : L \otimes_K \hat{K} \to \prod_{i=1}^g \hat{L}_i$ be the homomorphism defined above.*
a) Φ is surjective.
b) The kernel of (Φ) is the Jacobson radical of the Artinian ring $L \otimes_K \hat{K}$, i.e., the intersection of all the maximal ideals. More precisely, there are precisely g maximal ideals in $L \otimes_K \hat{K}$; suitably labelled as $\mathfrak{m}_1, \ldots, \mathfrak{m}_g$, the map Φ can be identified as the Chinese Remainder Theorem homomorphism

$$L \otimes_K \hat{K} \to \prod_{i=1}^g (L \otimes_K \hat{K})/\mathfrak{m}_i.$$

d) It follows that if L/K is separable, Φ is an isomorphism.

Proof. For brevity, we put $A = L \otimes_K \hat{K}$.
a) Let $W = \Phi(A)$. We wish to show that $W = \prod_{i=1}^g \hat{L}_i$. Since Φ is $\hat{K}$-linear, W is a $\hat{K}$-subspace of the finite-dimensional $\hat{K}$-subspace $\prod_{i=1}^g \hat{L}_i$. By Theorem 15, W is closed. On the other hand, by Artin-Whaples, the image of of L iunder $\Delta : L \hookrightarrow \prod_{i=1}^g L_i$ is dense, and by definition of completion (and an easy verification involving the topology on a finite product of metric spaces) $\prod_{i=1}^g L_i$ is dense in $\prod_{i=1}^g \hat{L}_i$. Certainly "is dense in" is a transitive relation among subspaces of a topological space, so $\Phi(L) = \Phi(L \otimes 1)$ is dense in $\prod_{i=1}^g \hat{L}_i$. Thus Φ is surjective.

b) Since A is a finite-dimensional $\hat{K}$-algebra, it is an Artinian ring and therefore has finitely many maximal (= prime, here) ideals, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_N$. So the Jacobson (= nil, here) radical $J = \bigcap_{i=1}^N \mathfrak{m}_i$. We have a finite set of pairwise comaximal ideals in a commutative ring, so the Chinese Remainder Theorem gives an isomorphism

$$\Psi : A/J \xrightarrow{\sim} \prod_{j=1}^N A/\mathfrak{m}_j.$$

For each i, $A/\mathfrak{m}_i$ is a finitely generated $\hat{K}$-module and also (since we modded out by a maximal ideal) a field, hence is a finite degree field extension of $\hat{K}$, say $L(\mathfrak{m}_j)$. Now our map Φ factors through Ψ and we get

$$A \to A/J \xrightarrow{\sim} \prod_{j=1}^N L(\mathfrak{m}_j) \xrightarrow{q} \prod_{i=1}^g \hat{L}_i.$$

Thus by part a) we have one finite product of finite field extensions of $\hat{K}$ surjecting onto another finite product of finite field extensions of $\hat{K}$. A little thought shows that the surjectivity means that we must have $N \geq g$ and can relabel the j's such that for all $1 \leq j \leq g$, $\mathfrak{m}_j = \mathrm{Ker}(\Phi_j)$ and thus

$$q : \prod_{j=1}^g \hat{L}_j \oplus \prod_{j>g} L(\mathfrak{m}_j) \to \prod_{j=1}^g \hat{L}_j$$

is projection onto the first factor. In other words, what we wish to show is that we have put enough factors on the right hand side: $g = N$.

So let's try. What we have put on the right hand side is, precisely, one factor for each inequivalent norm on L extending $|\ |$ of K. Each $L(\mathfrak{m}_j)$ is a finite degree extension of the complete field $\hat{K}$ so has a unique norm, say $||_j$, which restricts to a norm on L. So if $N > g$ there exists $j_1 \leq g$ and $j_2 > g$ such that $|\ |_{j_1} = |\ |_{j_2}$ as norms on L. Now consider the projection of Ψ onto just these two factors, i.e.,

$$\Psi_{j_1,j_2} : L \otimes_K \hat{K} \to \widehat{(L, |\ |_{j_1})} \times \widehat{(L, |\ |_{j_2})}.$$

We claim that Ψ_{j_1,j_2} is not in fact surjective, thus we have a contradiction. But this map[9] is not so mysterious: it is determined by the images of L and of $\hat{K}$. In particular, consider Ψ_{j_1,j_2} restricted to L: this is just the diagonal map; since the norms are equivalent, the topologies are the same, and thus the image of L is **closed** in $(L, |\ |_{j_1}) \times (L, |\ |_{j_2})$. Moreover, tensoring this diagonal map with $\hat{K}$ has the effect of completing these normed spaces (to see this, all we have to check is that after tensoring with $\hat{K}$ we have complete spaces and that the image is dense in the tensorization with $\hat{K}$). We have the same topology on both factors, so the closure of the diagonal is the diagonal of the closure, and thus the image of A under Ψ_{j_1,j_2} has $\hat{K}$-dimension $\dim L(\mathfrak{m}_{j_1}) = \dim L(\mathfrak{m}_{j_2})$ and hence not equal to $\dim L(\mathfrak{m}_{j_1}) \times L(\mathfrak{m}_{j_2}) = 2 \dim L(\mathfrak{m}_{j_1})$, contradiction.

c) If L/K is a separable field extension, then $A = L \otimes_K \hat{K}$ is a separable K-algebra, i.e., a product of finite separable field extensions. To see this, write $L = K[t]/(P(t))$ with $P(t)$ an irreducible separable polynomial (this is possible by the primitive element theorem). Being a separable polynomial is unaffected by extending the field:

[9]Despite its complication notation!

if M is any extension of K, however large, then $P \in M[t]$ factors into distinct irreducible polynomials. (To see this, use e.g. the derivative criterion for separability: P is separable iff $\gcd(P, P') = 1$.) Applying this remark with $M = \hat{K}$, we get $P = P_1 \cdots P_g \in \hat{K}[t]$ with the P_i's distinct irreducible polynomials; thus the set of ideals (P_i) are pairwise comaximal, and the CRT isomorphism is

$$A = \hat{K}[t]/(P) = \hat{K}[t]/(P_1 \cdots P_g) \cong \prod_{i=1}^{g} \hat{K}[t]/(P_i) \cong \prod_{i=1}^{g} \hat{L}_i.$$

□

2.13. **Proof of Theorem 3 Part II: Existence Modulo Hensel-Kürschák.**

We return to the situation of a complete normed field $(K, | \ |)$ and a degree $n < \infty$ field extension L/K. We have seen that if there exists an extension of $| \ |$ to a norm on L, it must be the map

$$x \in L \mapsto |N_{L/K}(x)|^{\frac{1}{[L:K]}}.$$

In the Archimedean case, the Ostrowski theorem reduces us to checking that this is indeed the correct recipe for the standard norm on $\mathbb{C}$ as a quadratic extension of $\mathbb{R}$. Thus we are left to deal with the non-Archimedean case. As mentioned above, we really need to check the ultrametric triangle inequality.

From our study of Artin valuations in §1, we know that we do not change whether a mapping is a non-Archimedean norm by raising it to any power, so we might we well look at the mapping $x \mapsto |N_{L/K}(x)|$ instead. Moreover, we also know that the non-Archimedean triangle inequality is equivalent to: for all $x \in L$, $|x| \leq 1 \implies |x+1| \leq 1$.

This is what we will check. In fact, as came out in the lecture, it is convenient to make a further reduction: since $N_{L/K}(x) = (N_{K[x]/K}(x))^{[L:K[x]]}$, we may as well assume that $L = K[x]$.

Lemma 25. *(Hensel-Kürschák) Let $(K, | \ |)$ be a complete, non-Archimedean normed field. Suppose that $P(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0 \in K[t]$ is irreducible and such that $|a_0| \leq 1$. Then $|a_i| \leq 1$ for all $0 < i < n$.*

Let us postpone the proof of Lemma 25 and see why it is useful for us. For $\alpha \in L$, let $P(t)$ be the minimal polynomial of α over K, so $P(t)$ is a monic irreducible polynomial of degree $m = [K[\alpha] : K]$ and has constant coefficient $a_0 = (-1)^m N_{K[\alpha]:K}(x)$. By assumption,

$$1 \geq |N_{L/K}(\alpha)| = |N_{K[\alpha]/K}(\alpha)^{n/m}| = |a_0|^{n/m},$$

so $|a_0| \leq 1$. Now the minimal polynomial for $\alpha + 1$ is $P(t-1)$ (note that $K[\alpha] = K[\alpha+1]$). Plugging in $t = 0$, we get

$$(-1)^m N_{K[\alpha]/K}(\alpha+1) = P(-1) = (-1)^m + a_{m-1}(-1)^{m-1} + \ldots + (-1)a_1 + a_0.$$

By Lemma 25 we have $|a_i| \leq 1$ for all i, and by the non-Archimedean triangle inequality in K we conclude that

$$|N_{L/K}(\alpha+1) = |N_{K[\alpha]/K}(\alpha+1)|^{n/m} \leq 1^{n/m} \leq 1.$$

Now we should discuss the proof of the Hensel-Kürshák Lemma. And we will – but in the next chapter of the notes, along with other equivalent forms of the all-important **Hensel's Lemma**. In the next (optional) section, we discuss a Hensel-less proof of the existence of the extended valuation, which employs the somewhat more general concept of a **Krull valuation** that we have been trying to keep out of our main exposition.

2.14. **Proof of Theorem 3 Part III: Krull Valuations.**

In this optional section we give a proof of the existence statement in Theorem 3 using the concept of a Krull valuation and an important (but not terribly difficult) result from commutative algebra.

Recall that a valuation ring is an integral domain R such that for every nonzero x in the fraction field K, at least one of x, x^{-1} lies in R. A valuation ring is necessarily local, say with maximal ideal $\mathfrak{m}$. Moreover:

Lemma 26. *A valuation ring is integrally closed.*

Proof. Let R be a valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Let $a_0, \ldots, a_{n-1} \in R$ and let $x \in K$ such that $x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$. By definition of a valuation ring, if $x \notin R$, then $x^{-1} \in \mathfrak{m}$, so $1 = -(a_{n-1}x^{-1} + \ldots + a_0 x^{-n})) \in \mathfrak{m}$, contradiction. $\square$

Remark: An integral domain R is called a **Bézout domain** if every finitely generated ideal is principal. In a valuation domain, every finitely generated ideal is generated by any element of minimal valuation, so valuation domains are Bézout. For a non-Noetherian domain, Bézout domains are very nice: see e.g. [Cla-CA, §12.4]. (In particular, all Bézout domains are integrally closed.)

Valuation rings have naturally arisen in our study of normed fields: If $(K, |\ |)$ is a normed field, then the set $R = \{x \in K \mid |x| \leq 1\}$ is a valuation ring. Equivalently, if $v = -\log |\ | : K^\times \to \mathbb{R}$ is an associated valuation, then $R = \{x \in K \mid v(x) \geq 0\}$. However, not every valuation ring comes from a normed field, or a valuation $v : K^\times \to \mathbb{R}$, in this way. We can however get a bijective correspondence by generalizing our concept of valuation to a map $v : K^\times \to \Gamma$, where Γ is an ordered abelian group.

This may sound abstruse, but is easily motivated, as follows: let R be a domain with fraction field K. Consider the relation $\leq$ on $K^\times$ of R-divisibility: that is $x \leq y \iff \frac{y}{x} \in R$. The relation of R-divisibility is immediately seen to be reflexive and transitive. Even when $R = \mathbb{Z}$, it is not anti-symmetric: an integer and its additive inverse divide each other. However, any relation $\leq$ on a set X which is reflexive and transitive induces a partial ordering $\leq$ on the quotient $X/\sim$, where we decree $x \sim y$ iff $x \leq y$ and $y \leq x$. In the case of $R = \mathbb{Z}$, this amounts essentially to restricting to positive integers. For R-divisibility in general, it means that we are identifying associate elements, so the quotient is precisely the group $\Gamma = K^\times / R^\times$ of principal fractional R-ideals of K.

Proposition 27. *For an integral domain R with fraction field K and $\Gamma = K^\times/R^\times$, TFAE:*

(i) The induced partial ordering $\leq$ on Γ is a total ordering.
(ii) R is a valuation ring.

Proof. If $\leq$ is a total ordering, then for any $x \in K^\times$, take $y = 1$: then either $\frac{y}{x} = x^{-1}$ or $\frac{x}{y} = x$ lies in R, so that R is a valuation ring. Conversely, if R is a valuation ring, let $x, y \in K^\times$. Then either $\frac{x}{y} \in R$ – i.e., $y \leq x$ – or $\frac{y}{x} \in R$ – i.e., $x \leq y$. □

This motivates the following definition.

Let $(\Gamma, \leq)$ be an ordered abelian group. Then a Γ-valued valuation on a field K is a map $v : K^\times \to \Gamma$ such that for all $x, y \in K$, $v(xy) = v(x) + v(y)$ and $v(x+y) \geq \min v(x), v(y)$. As usual, we may formally extend v to 0 by $v(0) = +\infty$. A **Krull valuation** on K is a Γ-valued valuation for some ordered abelian group Γ.

Proposition 28. *Let $(\Gamma, \leq)$ be an ordered abelian group and $v : K \to \Gamma$ a Krull valuation. Then $R_v = \{x \in K \mid v(x) \geq 0\}$ is a valuation ring.*

Thus to a valuation ring R we can associated the Krull valuation $v : K^\times \to K^\times/R^\times$ and conversely to a Krull valuation we can associate a valuation ring. These are essentially inverse constructions. To be more precise, let $(K, v : K \to \Gamma_1)$ and $(L, w : L \to \Gamma_2)$ be two fields endowed with Krull valuations.

Lemma 29. *Let $(\Gamma_1, \leq)$ and $(\Gamma_2, \leq)$ be ordered abelian groups, and let $g : \Gamma_1 \to \Gamma_2$ be a homomorphism of abelian groups.*
a) The following conditions on γ are equivalent:
(i) $x_1 < x_2 \implies g(x_1) < g(x_2)$.
(ii) $x_1 \leq x_2 \iff g(x_1) \leq g(x_2)$.
b) If the equivalent conditions of part a) hold, then g is injective.
A homomorphism satisfying the equivalent conditions of part a) is said to be a ***homomorphism of ordered abelian groups****.*

Proof. a) Suppose g satisfies (i). Certainly $x_1 = x_2 \implies g(x_1) = g(x_2)$, so we have $x_1 \leq x_2 \implies g(x_1) \leq g(x_2)$. Now suppose that $g(x_1) \leq g(x_2)$ and that we do not have $x_1 \leq x_2$. Since the ordering is total, we then have $x_1 > x_2$, and then our assumption gives $g(x_1) > g(x_2)$, contradiction. Now suppose g satisfies (ii), and let $x_1 < x_2$. If $g(x_1) = g(x_2)$, then $g(x_1) \leq g(x_2)$ and $g(x_2) \leq g(x_1)$, so (ii) implies that $x_1 = x_2$, contradiction. Similarly, we cannot have $g(x_1) \geq g(x_2)$, so we must have $g(x_1) < g(x_2)$.
b) Assume (i) and let $x \in \Gamma_1$ be such that $g(x) = 0$. If $0 < x$, then $0 = g(0) < g(x) = 0$, contradiction. Similarly, if $x < 0$, then $0 = g(x) < g(0) = 0$, contradiction. So $x = 0$ and thus g is injective. □

A homomorphism of Krull-valued fields is a pair (ι, g), where $\iota : K \hookrightarrow L$ is a homomorphism of fields and $g : \Gamma_1 \hookrightarrow \Gamma_2$ is an injective homomorphism of ordered abelian groups such that for all $x \in K$, $w(\iota(x)) = g(v(x))$. Then the correspondences described above take isomorphic valuation rings to isomorphic Krull-valued fields and isomorphic Krull-valued fields to isomorphic valuation rings.

The **value group** of a Krull valuation is the image $v(K^\times)$. A Krull valuation is said to be trivial if its value group is the trivial abelian group (which has a unique ordering).

In this more general context, a nontrivial Krull valuation $v : K \to \Gamma$ is said to be **rank one** if there exists a homomorphism of ordered abelian groups $g : \Gamma \to \mathbb{R}$.[10]

Exercise 2.40: Show that any ordered abelian group Γ can serve as the value group of a Krull-valued field.
Suggestion: let k be any field, and let R be the group ring $k[\Gamma]$, i.e., the set of formal sums $x = \sum_{\gamma \in \Gamma} x_\gamma[\gamma]$ where $x_\gamma \in k$ for all γ and for a fixed x, all but finitely many x_γ's are zero. Define $v : R \setminus \{0\} \to \Gamma$ by letting $v(x)$ be the least γ such that $x_\gamma \neq 0$. Show that R is an integral domain, that v extends uniquely to its fraction field K and defines a valuation on K with value group Γ.

Exercise 2.41: Let Γ be a nontrivial ordered abelian group. TFAE:
(i) There exists a homomorphism of ordered abelian groups $g : \Gamma \to \mathbb{R}$ (Γ has rank one).
(ii) For all positive elements $x, y \in \Gamma$, there exists $n \in \mathbb{Z}^+$ such that $nx > y$ (Γ is Archimedean).

Combining the previous two exercises one gets many examples of Krull valuations which are not of rank one, e.g. $\Gamma = \mathbb{Z} \times \mathbb{Z}$ ordered **lexicographically**.

Lemma 30. *Let Γ be an ordered abelian group and $H \subset \Gamma$ be a finite index divisible subgroup. Then $H = \Gamma$.*

Proof. Suppose not, i.e., $[\Gamma : H] = n < \infty$. Let $x \in \Gamma$. Then $nx = h \in H$. By definition of divisibility, there exists $y \in H$ such that $ny = h$. Therefore $0 = h - h = n(x - y)$, i.e., $x - y \in \Gamma[n]$. But an ordered abelian group must be torsionfree, so $x = y \in H$. □

Lemma 31. *Let L be a field, $v : L^\times \to \Gamma$ a Krull valuation on L, and let K be a subfield of L with $[L : K] = n < \infty$. Then Γ is order isomorphic to a subgroup of $\Gamma_K = v(K^\times)$.*

Proof. ([BAII, p. 582]) For any nonzero $x \in L$, we have a relation of the form $\sum_{i=1}^k \alpha_i x^{n_i}$, where $\alpha_i \in K$ and the n_i are integers such that $[L : K] = n \geq n_1 > n_2 > \ldots > n_k \geq 0$. If there existed any index j such that for all $i \neq j$ we had $v(\alpha_i x^{n_i}) > v(\alpha_j x^{n_j})$, then $\infty = v(\sum_{i=1}^k \alpha_i x^{n_i}) = v(\alpha_j x^{x_j})$ and thus $\alpha_j x^{n_j} = 0$, a contradiction. Thus there exist $i > j$ such that $v(\alpha_i x^{n_i}) = v(\alpha_j x^{n_j})$, so

$$v(x)^{n_i - n_j} = v(\alpha_j \alpha_i^{-1}) \in \Gamma_K.$$

Thus, for any $x \in L^\times$, $(n!)v(x) \in \Gamma_K$. But since Γ is torsionfree, the endomorphism $[n!] : \Gamma \to \Gamma$ (i.e., multiplication by $n!$) is injective, and therefore $[n!] : \Gamma \hookrightarrow [n!]\Gamma \subset \Gamma_K$, qed. □

Corollary 32. *Let (K, v) be a rank one valued field, let L/K be a finite degree field extension, and let w be any Krull valuation on L such that there exists a homomorphsim of Krull-valued fields $(\iota, g) : (K, v) \to (L, w)$. Then w also has rank one.*

[10]There is a definition of the rank of an ordered abelian group, but we will not need it here.

Proof. This follows immediately from Lemma 31 and the definition of a rank one valuation as one whose value group is order isomorphic to a subgroup of $(\mathbb{R}, +)$. $\square$

But why would one want to use Krull valuations? One might equally well ask what is the use of general valuation rings, and the latter has a very satisfying answer:

Theorem 33. *Let R be an integral domain which is not a field, and let L be a field such that $R \subset L$. Let S be the integral closure of R in L. Then S is equal to the intersection of all nontrivial valuation rings of L containing R.*

Now let $(K, | \, |)$ be a nontrivial non-Archimedean valued field and L/K a finite field extension. We know that there is at most one norm on L which extends $| \, |$ on K. We will now give a proof of the **existence** of this extended norm which is independent of the as yet unproved Lemma 25. Namely, let v be any rank one valuation corresponding to $| \, |$, and let R be the valuation ring of K. Let S be the integral closure of R in K. It suffices to show that S is itself a valuation ring and the corresponding valuation has rank one.

Let $\mathcal{S} = \{R_w\}$ be the set of all nontrivial valuation rings of L which contain R. By Theorem 33, we have $S = \bigcap_{R_w \in \mathcal{S}} R_w$. For any valuation ring $R_w \in \mathcal{S}$, let $w : L \rightarrow \Gamma$ be the corresponding Krull valuation. By Corollary X.X, w is a rank one valuation, hence corresponds to an non-Archimedean norm on L which (certainly after rescaling in its equivalence class) restricts to $| \, |$ on K. By the uniqueness of extended norms, it follows that $\#\mathcal{S} = 1$, so that $S = R_w$ is a rank one valuation ring.

References

[Bak] M. Baker, *Algebraic Number Theory Course Notes (2006)*, available at http://people.math.gatech.edu/∼mbaker/pdf/ANTBook.pdf

[Bou] N. Bourbaki, *Commutative algebra. Chapters 1–7.* Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998.

[Cla-CA] P.L. Clark, *Commutative Algebra*, available at http://math.uga.edu/∼pete/integral.pdf

[Eis] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[BAII] N. Jacobson, *Basic algebra. II.* Second edition. W. H. Freeman and Company, New York, 1989.

[Mon] P. Monsky, *On dividing a square into triangles.* Amer. Math. Monthly 77 1970 161–164

[Wei] E. Weiss, *Algebraic number theory.* Reprint of the 1963 original. Dover Publications, Inc., Mineola, NY, 1998.