# ABSOLUTE VALUES II: TOPOLOGIES, COMPLETIONS AND THE EXTENSION PROBLEM 

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## Contents

2.1. Introduction and Reorientation ..... 1
2.2. Reminders on metric spaces ..... 5
2.3. Ultrametric spaces ..... 7
2.4. Normed abelian groups ..... 8
2.5. The topology on a normed field ..... 10
2.6. Completion of a metric space ..... 10
2.7. Completions of normed abelian groups and normed fields ..... 12
2.8. Non-Archimedean Functional Analysis: page 1 ..... 14
2.9. Ostrowski's Theorem Revisited ..... 16
2.10. Proof of Theorem 3 Part I: Uniqueness ..... 16
2.11. Proof of Theorem 5 ..... 17
2.12. Proof of Theorem 3 Part II: Existence ..... 20

### 2.1. Introduction and Reorientation.

In this chapter we will study more explicitly the topology on a field induced by a norm. Especially interesting from this perspective are the (nontrivially) normed fields which are locally compact with respect to the norm topology.

But we have been studying normed fields for a little while now. Where are we going? What problems are we trying to solve?

Problem 1: Local/Global Compatibility. Arguably the most interesting results in Chapter 1 were the complete classification of all norms on a global field $K$, i.e., a finite extension of either $\mathbb{Q}$ (a number field) or $\mathbb{F}_{q}(t)$ for some prime power $q$ (a function field).

We interrupt for two remarks:
Remark 1: Often when dealing with function fields, we will say "Let $K / \mathbb{F}_{q}(t)$ be a finite separable field extension". It is not true that every finite field extension of $\mathbb{F}_{q}(t)$ is separable: e.g. $\mathbb{F}_{q}\left(t^{\frac{1}{q}}\right) / \mathbb{F}_{q}(t)$ is an inseparable field extension. However, the following is true: if $\iota: \mathbb{F}_{q}(t) \hookrightarrow K$ is a finite degree field homomorphism -

[^0]don't forget that this wordier description is the true state of affairs which is being elided when we speak of "a field extension $K / F$ " - then there is always another finite degree field homomorphism $\iota^{\prime}: \mathbb{F}_{q}(t) \hookrightarrow K$ which makes $K / \iota^{\prime}\left(\mathbb{F}_{q}(t)\right)$ into a separable field extension: e.g. [?, Cor. 16.18].

Remark 2: In the above passage we could of course have replaced $\mathbb{F}_{q}(t)$ by $\mathbb{F}_{p}(t)$. But the idea here is that for an arbitrary prime power $q$, the rational function field $\mathbb{F}_{q}(t)$ is still highly analogous to $\mathbb{Q}$ rather than to a more general number field. For instance, if $K$ is any number field, then at least one prime ramifies in the extension of Dedekind domains $\mathbb{Z}_{K} / \mathbb{Z}$. However, the extension $\mathbb{F}_{q}[t] / \mathbb{F}_{p}[t]$ is everywhere unramified. Moreover, $\mathbb{F}_{q}[t]$ is always a PID. ${ }^{1}$

For a global field $K$, we saw that there is always a Dedekind ring $R$ with $K$ as its fraction field with "sufficiently large spectrum" in the sense that all but finitely many valuations on $K$ are just the $\mathfrak{p}$-adic valuations associated to the nonzero prime ideals of $R$. This suggests - correctly!- that much of the arithmetic of $K$ and $R$ can be expressed in terms of the valuations on $K$.

A homomorphism of normed fields $\iota:(K,| |) \rightarrow(L,| |)$ is a field homomorphism $\iota$ such that for all $x \in K,|x|=|\iota(x)|$. We say that the norm on $L$ extends the norm on $K$. When the normed is non-Archimedean, this has an entirely equivalent expression in the language of valuations: a homomorphism of valued fields $\iota:(K, v) \rightarrow(L, w)$ is a field homomorphism $\iota: K \hookrightarrow L$ such that for all $x \in K, v(x)=w(\iota(x))$. We say that $w$ extends $v$ or that $\left.w\right|_{K}=v$. (Later we will abbreviate this further to $w \mid v$.)

Problem 2: The Extension Problem. Let $(K,| |)$ be a normed field, and let $L / K$ be a field extension. In how many ways does $v$ extend to a norm on $L$ ?

Theorem 1. Let $(K,| |)$ be a normed field and $L / K$ an extension field. If either of the following holds, then there is a norm on $L$ extending the given norm on $K$ : (i) $L / K$ is algebraic.
(ii) $(K,| |)$ is non-Archimedean.

Example: Let $K=\mathbb{Q},\left|\left|=| |_{2}\right.\right.$ and $L=\mathbb{R}$. Then there exists a norm on $\mathbb{R}$ which extends the 2-adic norm on $\mathbb{Q}$. This may seem like a bizarre and artifical example, but it isn't: this is the technical heart of the proof of a beautiful theorem of Paul Monsky [?]: it is not possible to dissect a square into an odd number of triangles such that all triangles have the same area. In fact, after 40 years of further work on this and similar problems, to the best of my knowledge no proof of Monsky's theorem is known which does not use this valuation-theoretic fact.

Exercise 2.1: Let $(K,| |)$ be an Archimedean norm.
a) Suppose that $L / K$ is algebraic. Show that $\|$ extends to a norm on $L$.
b) Give an example where $L / K$ is transcendental and the norm on $K$ does extend to a norm on $L$.
c) Give an example where $L / K$ is transcendental and the norm on $K$ does not

[^1]extend to a norm on $L$.
Hint for all three parts: use the Big Ostrowski Theorem.
In view of Exercise 2.1, we could restrict our attention to non-Archimedean norms and thus to valuations. Nevertheless it is interesting and useful to see that the coming results hold equally well in the Archimedean and non-Archimedean cases.

Theorem 1 addresses the existence of an extended norm but not the number of extensions. We have already seen examples to show that if $L / K$ is transcendental, the number of extensions of a norm on $K$ to $L$ may well be infinite. The same can happen for algebraic extensions of infinite degree: e.g., as we will see later, for any prime $p$, there are uncountably many extensions of the $p$-adic norm to $\overline{\mathbb{Q}}$.

Exercise $2.2 \mathrm{~T}^{2}$ : Let $K$ be a field and $\left\{K_{i}\right\}_{i \in I}$ be a family of subfields of $K$ such that: (i) for all $i, j \in I$ there exists $k \in I$ such that $K_{i} \cup K_{j} \subset K_{k}$ and (ii) $\bigcup_{i} K_{i}=K$. (Thus the family of subfields is a directed set under set inclusion, whose direct limit is simply $K$.) Suppose that for each $i$ we have a norm $\left|\left.\right|_{i}\right.$ on $K_{i}$, compatibly in the following sense: whenever $K_{i} \subset K_{j},| |_{j}$ extends $\left|\left.\right|_{i}\right.$. Show that there is a unique norm || on $K$ extending each norm $\left|\mid\right.$ on $K_{i}$.

Exercise 2.3: Let $(k,| |)$ be a non-Archimedean normed field. Let $R=k[t]$ and $K=k(t)$. For $P(t)=a_{n} t^{n}+\ldots+a_{1} t+a_{0} \in R$, define the Gauss norm $|P|=\max _{i}\left|a_{i}\right|$. Show that this is indeed a norm on $k[t]$ and thus induces a norm on the fraction field $K=k(t)$ extending the given norm on $k$. Otherwise put, this shows that every valuation on a field $k$ extends to a valuation on $k(t)$.

Exercise 2.4: Let $(K, v)$ be a valued field, and let $L / K$ be a purely transcendental extension, i.e., the fraction field of a polynomial ring over $K$ (in any number of indeterminates, possibly infinite or uncountable). Use the previous Exercise to show that $v$ extends to a valuation on $L$. (Suggestion: this is a case where a transfinite induction argument is very clean.)

Exercise 2.4 and basic field theory reduces Theorem 1 to the case of an algebraic extension $L / K$. As we will see, this can be further reduced to the case of finite extensions. Moreover, when $(K, v)$ is a valued field and $L / K$ is a finite extension, we wish not only to show that an extension $w$ of $v$ to $L$ exists but to classify (in particular, to count!) all such extensions. We saw in Chapter 1 that this recovers one of the core problems of algebraic number theory. Somewhat more generally, if $v$ is discrete, then the valuation ring $R$ is a DVR - in particular a Dedekind domain - and then its integral closure $S$ in $L$ is again a Dedekind domain, and we are asking how the unique nonzero prime ideal $\mathfrak{p}$ of $R$ splits in $S$ : i.e., $\mathfrak{p} S=\mathcal{P}_{1}^{e_{1}} \cdots \mathcal{P}_{r}^{e_{r}}$. With suitable separability hypotheses, we get the fundamental relation $\sum_{i=1}^{r} e_{i} f_{i}=[L: K]$.

The key idea that makes this bookkeeping automatic - and has many other virtues besides - is that of the completion $\hat{K}$ of a normed field $(K,| |)$. This is indeed a

[^2]special case of the completion of a metric space - a concept which we will review but bears further scrutiny in this case because we wish $\hat{K}$ to itself have the structure of a normed field. Here are some fundamental results:

Theorem 2. Let $(K,| |)$ be a normed field.
a) There is a complete normed field $(\hat{K},| |)$ and a homomorphism of normed fields $\iota:(K,| |) \rightarrow(\hat{K},| |)$ such that $\iota(K)$ is dense in $\hat{K}$.
b) The homomorphism $\iota$ is universal for norm-preserving homomorphisms of $K$ into complete normed fields.
c) In particular, $\hat{K}$ is unique up to canonical isomorphism.
d) It follows that any homomorphism of normed fields extends uniquely to a homomorphism on the completions.

Remark: In categorical language, these results amount to the following: completion is a functor from the category of normed fields to the category of complete normed fields which is left adjoint to the forgetful functor from the category of complete normed fields to the category of normed fields. We stress that, for our purposes here, it is absolutely not necessary to understand what the previous sentence means.

Theorem 3. Let $(K,| |)$ be a complete normed field and let $L / K$ be algebraic.
a) There exists a unique norm $\left|\left.\right|_{L}\right.$ on $L$ such that $(K,| |) \rightarrow\left(L,| |_{L}\right)$ is a homomorphism of normed fields.
b) If $L / K$ is finite, then $\left(L,| |_{L}\right)$ is again complete.

Corollary 4. If $(K,| |)$ is a normed field and $L / K$ is an algebraic extension, then there is at least one norm on $L$ extending the given norm on $K$.
Proof. We may as well assume that $L=\bar{K}$. The key step is to choose a field embedding $\Phi: \bar{K} \hookrightarrow \overline{\widehat{K}}$. This is always possible by basic field theory: any homomorphism from a field $K$ into an algebraically closed field $F$ can be extended to any algebraic extension $L / K$. Since this really is the point, we recall the proof. Consider the set of all embeddings $\iota_{i}: L_{i} \hookrightarrow F$, where $L_{i}$ is a subextension of $L / K$. This set is partially ordered by inclusion. Moreover the union of any chain of elements in this poset is another element in the poset, so by Zorn's Lemma we are entitled to a maximal embedding $\iota_{i}: L_{i} \hookrightarrow F$. If $L_{i}=L$, we're done. If not, there exists an element $\alpha \in L \backslash L_{i}$, but then we could extend $\iota_{i}$ to $L_{i}[\alpha]$ by sending $\alpha$ to any root of its $\iota_{i}\left(L_{i}\right)$-minimal polynomial in $F$. By Theorem 3, there is a unique norm on $\overline{\widehat{K}}$ extending the given norm on $K$. Therefore we may define a norm on $L$ by $x \mapsto|\Phi(x)|$.
Exercise 2.5: Use Corollary 4 and some previous exercises to prove Theorem 1.
Theorem 5. Let $(K,| |)$ be a normed field and $L / K$ a finite extension. Then there is a bijective correspondence between norms on $L$ extending the given norm on $K$ and prime ideals in the $\hat{K}$-algebra $L \otimes_{K} \hat{K}$.
There is a beautiful succinctness to the expression of the answer in terms of tensor products, but let us be sure that we understand what it means in more down-toearth terms. Suppose that there exists a primitive element $\alpha \in L$ i.e., such that $L=K(\alpha)$. Recall that this is always the case when $L / K$ is separable or $[L: K]$ is prime. In fact, the existence of primitive elements is often of mostly psychological usefulness: in the general case we can of course write $L=K\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and
decompose $L / K$ into a finite tower of extensions, each of which has a primitive element.

Now let $P(t) \in K[t]$ be the minimal polynomial of $\alpha$ over $K$, so $P(t)$ is irreducible and $L \cong K[t] /(P(t))$. In this case, for any field extension $F / K$, we have isomorphisms

$$
L \otimes_{K} F \cong K[t] /(P(t)) \otimes_{K} F \cong F[t] /(P(t))
$$

Thus, $L \otimes_{K} F$ is an $F$-algebra of dimension $d=\operatorname{deg} P=[L: K]$. It need not be a field, but its structure is easy to analyze using the Chinese Remainder Theorem in the Dedekind ring $F[t]$. Namely, we factor $P(t)$ into irreducibles: say $P(t)=$ $P_{1}^{e_{1}} \cdots P_{r}^{e_{r}}$. Then CRT gives an isomorphism

$$
L \otimes_{K} F \cong F[t] /(P(t)) \cong \bigoplus_{i=1}^{r} F[t] /\left(P_{i}^{e_{i}}\right)
$$

Let us put $A_{i}=F[t] /\left(P_{i}^{e_{i}}\right)$. This is a local Artinian $F$-algebra with unique prime ideal $P_{i} / P_{i}^{e_{i}}$. Thus the number of prime ideals in $L \otimes_{K} F$ is $r$, the number of distinct irreducible factors of $F$. Moreover, suppose that $L / K$ is separable. Then $P(t)$ splits into distinct linear factors in the algebraic closure of $K$, which implies that when factored over the extension field $F$ (algebraic or otherwise), it will have no multiple factors. In particular, if $L / K$ is separable (which it most often will be for us, in fact, but there seems to be no harm in briefly entertaining the general case), then all the $e_{i}$ 's are equal to 1 and $A_{i}=F[t] /\left(P_{i}\right)$ is a finite, separable field extension of $F$.

Example: We apply this in the case $(K,| |)$ is the rational numbers equipped with the standard Archimedean norm. Then the number of extensions of $|\mid$ to $L \cong K[t] /(P(t))$ is equal to the number of (necessarily distinct) irreducible factors of $P(t)$ in $\mathbb{R}=\hat{\mathbb{Q}}$. How does a polynomial factor over the real numbers? Every irreducible factor has degree either 1 - corresponding to a real root - or 2 - corresponding to a conjugate pair of complex roots. Thus $L \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_{1}} \oplus \mathbb{C}^{r_{2}}$ and the number of extensions is $r_{1}+r_{2}$, as advertised - but not proved! - in the Remark following Theorem 1.16.

It remains to prove Theorems 2, 3 and 5. Before proving Theorem 1, we give several short sections of "review" on topics which are probably somewhat familiar from previous courses but are important enough to revisit from a slightly more sophisticated perspective. In $\S 2.7$ we give the proof of Theorem 2.

### 2.2. Reminders on metric spaces.

Let $X$ be a set. A function $\rho: X \times X \rightarrow \mathbb{R}^{\geq 0}$ is a metric on $X$ if it satisfies all of the following:
(M1) (positive definiteness) $\forall x, y \in X, \rho(x, y)=0 \Longleftrightarrow x=y$.
(M2) (symmetry) $\forall x, y \in X, \rho(x, y)=\rho(y, x)$.
(M3) (triangle inequality) $\forall x, y, z \in X, \rho(x, z) \leq \rho(x, y)+\rho(y, z)$.
A metric space is a pair $(X, d)$ where $d$ is a metric on $X$.

For $x$ an element of a metric space $X$ and $r \in \mathbb{R}^{>0}$, we define the open ball

$$
B(x, r)=\{y \in X \mid \rho(y, x)<r\} .
$$

The open balls form the base for a topology on $X$, the metric topology. With your indulgence, let's check this. What we must show is that if $z \in B\left(x, r_{1}\right) \cap B\left(y, r_{2}\right)$, then there exists $r_{3}>0$ such that $B\left(z, r_{2}\right) \subset B\left(x, r_{1}\right) \cap B\left(y, r_{2}\right)$. Let $r_{3}=$ $\min \left(r_{1}-\rho(x, z), r_{2}-\rho(y, z)\right)$, and let $w \in B\left(z, r_{3}\right)$. Then by the triangle inequality $\rho(x, w) \leq \rho(x, z)+\rho(z, w)<\rho(x, z)+\left(r_{1}-\rho(x, z)\right)=r_{1}$, and similarly $\rho(y, w)<r_{2}$.

Given a finite collection of metric spaces $\left\{\left(X_{i}, \rho_{i}\right)\right\}_{1 \leq i \leq n}$, we define the product metric on $X=\prod_{i=1}^{n} X_{i}$ to be $\rho(x, y)=\max _{i} \rho_{i}\left(x_{i}, y_{i}\right) .{ }^{3}$

Remark: As is typical, instead of referring to "the metric space $(X, \rho)$ ", we will often say instead "the metric space $X$ ", i.e., we allow $X$ to stand both for the set and for the pair $(X, \rho)$.

Exercise 2.6 (pseudometric spaces): Let $X$ be a set. A function $\rho: X \times X \rightarrow \mathbb{R}^{\geq 0}$ satisfying (M2) and (M3) is called a pseudometric, and a set $X$ endowed with a pseudometric is called a pseudometric space.
a) Show that all of the above holds for pseudometric spaces - in particular, the open balls form the base for a topology on $X$, the pseudometric topology.
b) Show that for a pseudometric space $(X, \rho)$, the following are equivalent:
(i) $\rho$ is a metric.
(ii) The topological space $X$ is Hausdorff.
(iii) The topological space $X$ is separated (i.e., $T_{1}$ : points are closed).
(iv) The topological space $X$ is Kolmogorov (i.e., $T_{0}$ : no two distinct points have exactly the same open neighborhoods).
c) Define an equivalence relation $\sim$ on $X$ by $x \sim y \Longleftrightarrow \rho(x, y)=0$. Let $\bar{X}=X / \sim$ be the set of equivalence classes. Show that $\rho$ factors through a function $\bar{\rho}: \bar{X} \times \bar{X} \rightarrow \mathbb{R}^{\geq 0}$ and that $\bar{\rho}$ is a metric on $\bar{X}$. Show that the map $q: X \rightarrow \bar{X}$ is the Kolmogorov completion of the topological space $X$, i.e., it is the universal continuous map from $X$ into a $T_{0}$-space.

A Cauchy sequence in a metric space $(X, \rho)$ is a sequence $\left\{x_{n}\right\}$ in $X$ such that for all $\epsilon>0$, there exists $N \in \mathbb{Z}^{+}$such that $m, n \geq N \Longrightarrow \rho\left(x_{m}, x_{n}\right)<\epsilon$. Every convergent sequence is convergent. Conversely, we say that a metric space $X$ is complete if every Cauchy sequence converges.

Let $X$ and $Y$ be metric spaces. A function $f: X \rightarrow Y$ is uniformly continuous if for all $\epsilon>0$, there exists $\delta>0$ such that $\forall x, y \in X, \rho_{X}(x, y)<\delta \Longrightarrow$ $\rho_{Y}(f(x), f(y))<\epsilon$.

Exercise 2.7: Let $(X, \rho)$ be a metric space. Show that $\rho: X \times X \rightarrow \mathbb{R}$ is a uniformly continuous function: here $\mathbb{R}$ is endowed with the standard Euclidean metric $\rho(x, y)=|x-y|$.

[^3]Exercise 2.8: Let $X$ and $Y$ be metric spaces, and let $f: X \rightarrow Y$ be a continuous function.
a) If $f$ is uniformly continuous and $\left\{x_{n}\right\}$ is a Cauchy sequence in $X$, show that $\left\{f\left(x_{n}\right)\right\}$ is a Cauchy sequence in $Y$.
b) Give an example to show that a merely continuous function need not map Cauchy sequences to Cauchy sequences.

A Hausdorff topological space is compact if every open covering has a finite subcovering. ${ }^{4}$ A Hausdorff topological space is locally compact if every point admits a compact neighborhood. This is equivalent (thanks to the Hausdorff condition!) to the apparently stronger condition that every point has a local base of compact neighborhoods.

A metric space $(X, \rho)$ is ball compact ${ }^{5}$ if every closed bounded ball is compact.
Exercise 2.9: Consider the following properties of a metric space $(X, \rho)$ :
(i) $X$ is compact.
(ii) $X$ is ball compact.
(iii) $X$ is locally compact.
(iv) $X$ is complete.

Show that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iv), but none of the other implications hold.

### 2.3. Ultrametric spaces.

An ultrametric space is a metric space $(X, \rho)$ in which the following stronger version of the triangle inequality holds:

$$
\forall x, y, z \in X, \rho(x, z) \leq \max (\rho(x, y), \rho(y, z))
$$

Exercise 2.10: a) Suppose that $x, y, z$ are points in an ultrametric space such that $\rho(x, y) \neq \rho(y, z)$. Show that $\rho(x, z)=\max (\rho(x, y), \rho(y, z))$.
b) In particular, every triangle in an ultrametric space is isosceles.
c) Let $B=B(x, r)$ be an open ball in an ultrametric space $(X, \rho)$ and let $y \in B(x, r)$. Show that $y$ is also a center for $B: B=B(y, r)$. Does the same hold for closed balls?

Exercise 2.11: Let $B_{1}, B_{2}$ be two balls (each may be either open or closed) in an ultrametric space $(X, \rho)$. Show that $B_{1}$ and $B_{2}$ are either disjoint or concentric: i.e., there exists $x \in X$ and $r_{1}, r_{2} \in(0, \infty)$ such that $B_{i}=B\left(x, r_{i}\right)$ or $B_{c}\left(x, r_{i}\right)$.

Exercise 2.12: Let $(X, \rho)$ be an ultrametric space.
a) Let $r \in(0, \infty)$. Show that the set of open (resp. closed) balls with radius $r$ forms a partition of $X$.
b) Deduce from part a) that every open ball is also a closed subset of $X$ and that

[^4]every closed ball of positive radius is also an open subset of $X$.
c) A topological space is zero-dimensional if there exists a base for the topology consisting of clopen (= closed and open) sets. Thus part b) shows that an ultrametric space is zero-dimensional. Show that a zero-dimensional Hausdorff space is totally disconnected. In particular, an ultrametric space is totally disconnected.

Exercise 2.13: Prove or disprove: it is possible for the same topological space $(X, \tau)$ to have two compatible metrics $\rho_{1}$ and $\rho_{2}$ (i.e., each inducing the given topology $\tau$ on $X$ ) such that $\rho_{1}$ is an ultrametric and $\rho_{2}$ is not.

Exercise 2.14: Let $\Omega$ be a nonempty set, and let $\mathcal{S}=\prod_{i=1}^{\infty} \Omega$, i.e., the space of infinite sequences of elements in $\Omega$, endowed with the metric $\rho(x, y)=2^{-N}$ if $x_{n}=y_{n}$ for all $n<N$ and $x_{N} \neq y_{N}$. (If $x_{n}=y_{n}$ for all $n$, then we take $N=\infty$.)
a) Show that $(\mathcal{S}, \rho)$ is an ultrametric space, and that the induced topology coincides with the product topology on $\mathcal{S}$, each copy of $\Omega$ being given the discrete topology.
b) Show that $\mathcal{S}$ is a complete ${ }^{6}$ metric space without isolated points.
c) Without using Tychonoff's theorem, show that $\mathcal{S}$ is compact iff $\Omega$ is finite. (Hint: since $\mathcal{S}$ is metrizable, compact is equivalent to sequentially compact. Show this via a diagonalization argument.)
d) Suppose $\Omega_{1}$ and $\Omega_{2}$ are two finite sets, each containing more than one element. Show that the spaces $\mathcal{S}\left(\Omega_{1}\right)$ and $\mathcal{S}\left(\Omega_{2}\right)$ are homeomorphic.

### 2.4. Normed abelian groups.

Let $G$ be an abelian group, written additively. By a norm on $G$ we mean a map ||:G $\rightarrow \mathbb{R}^{\geq 0}$ such that:
(NAG1) $|g|=0 \Longleftrightarrow g=0$.
(NAG2) $\forall g \in G,|-g|=|g|$.
(NAG3) $\forall g, h \in G,|g+h| \leq|g|+|h|$.
For example, an absolute value on a field $k$ is (in particular) a norm on ( $k,+$ ). By analogy to the case of fields, we will say that a norm is non-Archimedean if $\forall g, h \in G,|g+h| \leq|g|+|h|$.

For a normed abelian group $(G,| |)$, define $\rho: G^{2} \rightarrow \mathbb{R}^{\geq 0}$ by $\rho(x, y)=|x-y|$.
Exercise 2.15: Show that $\rho$ defines a metric on $G$. Show that the norm is nonArchimedean iff $\rho$ is an ultrametric.

Exercise 2.16: Show that the norm $\|: X \rightarrow \mathbb{R}$ is uniformly continuous.
The metric topology on $X$ is Hausdorff and first countable, so convergence can be described in terms of sequences: a sequence $\left\{x_{n}\right\}$ in $X$ converges to $x \in X$ if for all $\epsilon>0$, there exists $N=N(\epsilon)$ such that for all $n \geq N, \rho\left(x_{n}, x\right)<\epsilon$. A sequence is said to be convergent if it converges to some $x$. Since $X$ is Hausdorff, a sequence converges to at most one point.

[^5]Exercise (semi-normed group): A semi-norm on an abelian group is a map ||: $G \rightarrow \mathbb{R}^{\geq 0}$ which satisfies (NAG2) and (NAG3). Show that a semi-norm induces a pseudometric on $G$.

Exercise 2.17: Suppose $G$ is an arbitrary (i.e., not necessarily abelian) group with identity element $e$ and group law written multiplicatively - endowed with a function $\left|\mid: G \rightarrow \mathbb{R}^{\geq 0}\right.$ satisfying:
(NG1) $|g|=0 \Longleftrightarrow g=e$.
(NAG2) $\forall g \in G,\left|g^{-1}\right|=|g|$.
(NAG3) $\forall g, h \in G,|g h| \leq|g|+|h|$.
a) Show that $d: G \times G \rightarrow \mathbb{R},(g, h) \mapsto\left|g h^{-1}\right|$ defines a metric on $G$.
b) If \| is a norm on $G$ and $C \in R^{>0}$, show that $C|\mid$ is again a norm on $G$. Let us write $\left.\left|\left.\right|_{1} \approx\right|\right|_{2}$ for two norms which differ by a constant in this way.
c) Define on any group $G$ a trivial norm; show that it induces the discrete metric.

In any topological abelian group, it makes sense to discuss the convergence of infinite series $\sum_{n=1}^{\infty} a_{n}$ in $G$ : as usual, we say $\sum_{n=1}^{\infty} a_{n}=S$ if the sequence $\left\{\sum_{k=1}^{n} a_{k}\right\}$ of partial sums converges to $S$.

A series $\sum_{n=1}^{\infty} a_{n}$ is unconditionally convergent if there exists $S \in G$ such that for every permutation $\sigma$ of the positive integers, the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges to $S$.

In a normed abelian group $G$ we may speak of absolute convergence: we say that $\sum_{n=1}^{\infty} a_{n}$ is absolutely convergent if the real series $\sum_{n=1}^{\infty}\left|a_{n}\right|$ converges.

Exercise 2.18: For a normed group $G$, show that TFAE:
(i) Every absolutely convergent series is unconditionally convergent.
(ii) $G$ is complete.

Whether unconditional convergence implies absolute convergence is more delicate. If $G=\mathbb{R}^{n}$ with the standard Euclidean norm, then it follows from the Riemann Rearrangement Theorem that unconditional convergence implies absolute convergence. On the other hand, it is a famous theorem of Dvoretsky-Rogers that in any infinite dimensional real Banach space (i.e., a complete, normed $\mathbb{R}$-vector space) there exists a series which is unconditionally convergent but not absolutely convergent.

The theory of convergence in complete non-Archimedean normed groups is in fact much simpler:
Proposition 6. Let $G$ be a complete, non-Archimedean normed group, and let $\left\{a_{n}\right\}_{n=1}^{\infty}$ be a sequence in G. TFAE:
(i) The series $\sum_{n=1}^{\infty} a_{n}$ is unconditionally convergent.
(ii) The series $\sum_{n=1}^{\infty} a_{n}$ is convergent.
(iii) $\lim _{n \rightarrow \infty} a_{n}=0$.

Exercise 2.19: Prove Proposition 6.

Exercise 2.20: Use Proposition 6 to give an explicit example of a series in $\mathbb{Q}_{p}$ which is unconditionally convergent but not absolutely convergent.

Exercise 2.21: Let $(G,| |)$ be a normed abelian group. Suppose that $G$ is locally compact in the norm topology.
a) Show that $G$ is complete.
b) Must $G$ be ball compact?

### 2.5. The topology on a normed field.

Let $k$ be a field and $|\mid$ an Artin absolute value on $k$. We claim that there is a unique metrizable topology on $k$ such that a sequence $\left\{x_{n}\right\}$ in $k$ converges to $x \in k$ iff $\left|x_{n}-x\right| \rightarrow 0$. To see this, first note that the condition $\left|x_{n}-x\right| \rightarrow 0$ depends only on the equivalence class of the Artin absolute value, since certainly $\left|x_{n}-x\right| \rightarrow 0 \Longleftrightarrow\left|x_{n}-x\right|^{\alpha} \rightarrow 0$ for any positive real number $\alpha$. So without changing the convergence of any sequence, we may adjust || in its equivalence class to get an absolute value (i.e., with Artin constant $C \leq 2$ ) and then we define the topology to be the metric topology with respect to $\rho(x, y)=|x-y|$ as above. Of course this recovers the given notion of convergence of sequences. Finally, we recall that a metrizable topological space is first countable and that there exists at most one first countable topology on a set with a given set of convergent sequences. We call this topology the valuation topology.

Exercise 2.22: Show that the trivial valuation induces the discrete topology.
Exercise 2.23: Let $(k,| |)$ be a valued field, and let $\left\{x_{n}\right\}$ be a sequence in $k$. Show that $x_{n} \rightarrow 0$ iff $\left|x_{n}\right| \rightarrow 0$.

Proposition 7. Let $\left.\left|\left.\right|_{1}\right.$ and $|\right|_{2}$ be norms on a field $k$. TFAE:
(i) $\left.\left|\left.\right|_{1} \sim\right|\right|_{2}$ in the sense of Theorem 1.4.
(ii) The topologies induced by $\left.\left|\left.\right|_{1}\right.$ and $|\right|_{2}$ coincide.

Proof. The direction (i) $\Longrightarrow$ (ii) follows from the discussion given above. Assume (ii). Let $x \in k$. Then $|x|_{1}<1 \Longleftrightarrow x^{n} \rightarrow 0$ in the $\left|\left.\right|_{1}\right.$-topology iff $x^{n} \rightarrow 0$ in the $\left.\left|\left.\right|_{2}\right.$-metric topology $\left.\Longleftrightarrow\right| x\right|_{2}<1 \Longleftrightarrow| |_{1} \sim| |_{2}$.
An equivalent topological statement of Artin-Whaples approximation is:
Theorem 8. (Artin-Whaples Restated) Let $k$ be a field and, for $1 \leq i \leq n$, let $\left|\left.\right|_{i}\right.$ be inequivalent nontrivial norms on $k$. Let $\left(k, \tau_{i}\right)$ denote $k$ endowed with the $\left|\left.\right|_{i}\right.$-norm topology, and let $k^{n}=\prod_{i=1}^{n}\left(k, \tau_{i}\right)$. Then the diagonal map $\Delta: k \hookrightarrow$ $k^{n}, x \mapsto(x, \ldots, x)$ has dense image.
Exercise 2.24: Convince yourself that this is equivalent to Theorem 1.5.
Exercise 2.25: Show that any two closed balls of finite radius in a normed field are homeomorphic. Deduce that a locally compact normed field is ball compact. (In particular, it is complete, although we knew that already by Exercise X.X.)

### 2.6. Completion of a metric space.

Lemma 9. Let $\left(X, \rho_{X}\right)$ be a metric space, $\left(Y, \rho_{Y}\right)$ be a complete metric space, $Z \subset X$ a dense subset and $f: Z \rightarrow Y$ a continuous function.
a) There exists at most one extension of $f$ to a continuous function $F: X \rightarrow Y$. (N.B.: This holds for for any topological space $X$ and any Hausdorff space $Y$.)
b) $f$ is uniformly continuous $\Longrightarrow f$ extends to a uniformly continuous $F: X \rightarrow Y$.
c) If $f$ is an isometric embedding, then its extension $F$ is an isometric embedding.

Exercise 2.26: Prove Lemma 9.
Let us say that a map $f: X \rightarrow Y$ of topological spaces is dense if $f(X)$ is dense in $Y$. An isometric embedding is a map $f:\left(X, \rho_{X}\right) \rightarrow\left(Y, \rho_{Y}\right)$ such that for all $x_{1}, x_{2} \in X, \rho_{Y}\left(f\left(x_{1}\right), f\left(x_{2}\right)\right)=\rho_{X}\left(x_{1}, x_{2}\right)$. An isometry is a surjective isometric embedding.

Exercise 2.27: Let $f$ be an isometric embedding of metric spaces.
a) Show that $f$ is uniformly continuous with $\delta=\epsilon$.
b) Show that $f$ is injective. Therefore an isometry is bijective. Show that if $f$ is an isometry, then $f^{-1}$ is also an isometry.

Theorem 10. let $(X, \rho)$ be a metric space.
a) There is a complete metric space $\hat{X}$ and a dense isometric embedding $\iota: X \rightarrow \hat{X}$.
b) The completion $\iota$ satisfies the following universal mapping property: if $(Y, \rho)$ is a complete metric space and $f: X \rightarrow Y$ is a uniformly continuous map, then there exists a unique uniformly continuous map $F: \hat{X} \rightarrow Y$ such that $f=F \circ \iota$.
c) If $\iota^{\prime}: X \hookrightarrow \hat{X}^{\prime}$ is another isometric embedding into a complete metric space with dense image, then there exists a unique isometry $\Phi: \hat{X} \rightarrow \hat{X}^{\prime}$ such that $\iota^{\prime}=\Phi \circ \iota$.

Proof. a) Let $X^{\infty}=\prod_{i=1}^{\infty} X$ be the set of all sequences in $X$. Inside $X$, we define $\mathcal{X}$ to be the set of all Cauchy sequences. We introduce an equivalence relation on $\mathcal{X}$ by $x_{\bullet} \sim y_{\bullet}$ if $\rho\left(x_{n}, y_{n}\right) \rightarrow 0$. Put $\hat{X}=\mathcal{X} / \sim$. For any $x \in X$, define $\iota(x)=(x, x, \ldots)$, the constant sequence based on $x$. This of course converges to $x$, so is Cauchy and hence lies in $\mathcal{X}$. The composite map $X \xrightarrow{\iota} \mathcal{X} \xrightarrow{\sim} \hat{X}$ (which we continue to denote by $\iota)$ is injective, since $\rho\left(x_{n}, y_{n}\right)=\rho(x, y)$ does not approach zero. We define a map $\hat{\rho}: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ by

$$
\hat{\rho}\left(x_{\bullet}, y_{\bullet}\right)=\lim _{n \rightarrow \infty} \rho\left(x_{n}, y_{n}\right) .
$$

To see that this limit exists, we may reason (for instance) as follows: the sequence $x_{\bullet} \times y_{\bullet}$ is Cauchy in $X \times X$, hence its image under the uniformly continuous function $\rho$ is Cauchy in the complete metric space $\mathbb{R}$, so it is convergent. It is easy to see that $\hat{\rho}$ factors through to a map $\hat{\rho}: \hat{X} \rightarrow \hat{X} \rightarrow \mathbb{R}$. The verification that $\hat{\rho}$ is a metric on $\hat{X}$ and that $\iota: X \rightarrow \hat{X}$ is an isometric embedding is straightforward and left to the reader. Moreover, if $x_{\bullet}=\left\{x_{n}\right\}$ is a Cauchy sequence in $X$, then the sequence of constant sequences $\left\{\iota\left(x_{n}\right)\right\}$ is easily seen to converge to $x_{\bullet}$ in $\hat{X}$.
b) Let $x_{\bullet} \in \mathcal{X}$ be a Cauchy sequence in $X$. As above, since $f$ is uniformly continuous and $Y$ is complete, $f\left(x_{\bullet}\right)$ is convergent in $Y$ to a unique point, say $y$, and we put $y=F\left(x_{\bullet}\right)$. Since $X$ is dense in $\hat{X}$ this is the only possible choice, and by Lemma 9 it does indeed give a well-defined uniformly continuous function $F: X \rightarrow Y$.
c) Isometric embeddings are uniformly continuous, so we may apply the universal mapping property of part b) to the map $\iota^{\prime}: X \hookrightarrow \hat{X}^{\prime}$ to get a map $\Phi: \hat{X} \rightarrow \hat{X}^{\prime}$. Similarly, we get a map $\Phi^{\prime}: \hat{X}^{\prime} \rightarrow \hat{X}$. The compositions $\Phi^{\prime} \circ \Phi$ and $\Phi^{\prime} \circ \Phi$ are uniformly continuous maps which restrict to the identity on the dense subspace $X$,
so they must each by the identity map, i.e., $\Phi$ and $\Phi^{\prime}$ are mutually inverse bijections. By Lemma 9c), $\Phi$ is an isometric embedding, and therefore it is an isometry.
We refer to $\hat{X}$ as the completion of $X .^{7}$
Corollary 11. (Functoriality of completion) a) Let $f: X \rightarrow Y$ be a uniformly continuous map between metric spaces. Then there exists a unique map $F: \hat{X} \rightarrow \hat{Y}$ making the following diagram commute:

$$
\begin{aligned}
& X \xrightarrow{f} Y \\
& \hat{X} \xrightarrow{F} \hat{Y} .
\end{aligned}
$$

b) If $f$ is an isometric embedding, so is $F$.
c) If $f$ is an isometry, so is $F$.

Proof. a) The map $f^{\prime}: X \rightarrow Y \hookrightarrow \hat{Y}$, being a composition of uniformly continuous maps, is uniformly continuous (check this if you haven't seen it before!). Applying the universal property of completion to $f^{\prime}$ gives a unique extension $\hat{X} \rightarrow \hat{Y}$.
Part b) follows immediate from Lemma 9b). As for part c), if $f$ is an isometry, so is its inverse $f^{-1}$. The extension of $f^{-1}$ to a mapping from $\hat{Y}$ to $\hat{X}$ is easily seen to be the inverse function of $F$.
Exercise 2.28: For a metric space ( $X, \rho$ ), define the distance set $\mathcal{D}(X)=\rho(X \times X)$, i.e., the set real numbers which arise distances between points in $X$.
a) Prove or disprove: if $\mathcal{D}$ is a discrete subset of $\mathbb{R}$, then $\rho$ is ultrametric.
b) Prove or disprove: if $\rho$ is an ultrametric, then $\mathcal{D}$ is discrete.
c) Let $\tilde{X}$ be the completion of $X$. Show that $\mathcal{D}(\tilde{X})=\overline{\mathcal{D}(X)}$ (closure in $\mathbb{R})$.
d)(U) Determine which subsets of $\mathbb{R}^{\geq 0}$ arise as distance sets of some metric space.

Exercise 2.29: The notion of a metric space and a completion seems to presuppose knowledge of $\mathbb{R}$, the set of real numbers. In particular, it is a priori logically unacceptable to define $\mathbb{R}$ to be the completion of $\mathbb{Q}$ with respect to the Archimedean norm $\left|\left.\right|_{\infty}\right.$. (Apparently for such reasons, Bourbaki's influential text General Topology avoids mention of the real numbers until page 329, long after a general discussion of uniform spaces and topological groups.) Show that this is in fact not necessary and that the completion of a metric space can be used to construct the real numbers. (Hint: first define a $\mathbb{Q}$-valued metric and its completion.)

### 2.7. Completions of normed abelian groups and normed fields.

When $G$ is a normed abelian group (or a field with an absolute value) we wish to show that the completion $\tilde{G}$ is, in a natural way, again a normed abelian group (or a field with an absolute value). This follows readily from the results in the previous section, but we take the opportunity to point out a simplification in the construction of $\hat{G}$ in this case.

As above, we put $G^{\infty}=\prod_{i=1}^{\infty}$ and $\mathcal{G}$ the subset of Cauchy sequences. But this time $G^{\infty}$ is an abelian group and $\mathcal{G}$ is a subgroup of $G^{\infty}$ (easy exercise). Furthermore, we may define $\mathfrak{g}$ to be the set of sequences converging to 0 , and then $\mathfrak{g}$ is a subgroup

[^6]of $\mathcal{G}$. Thus in this case we may define $\hat{G}$ simply to be the quotient group $\mathcal{G} / \mathfrak{g}$, so by its provenance it has the structure of an abelian group. Moreover, if $x_{\bullet}$ is a Cauchy sequence in $G$, then by Exercise X.X $\left|x_{\bullet}\right|$ is a Cauchy sequence in $\mathbb{R}$, hence convergent, and we may define
$$
\left|x_{\bullet}\right|=\lim _{n \rightarrow \infty}\left|x_{n}\right| .
$$

We leave it to the reader to carry through the verifications that this factors to give a norm on $\hat{G}$ whose associated metric is the same one that we constructed in the proof of Theorem XX.

Now suppose that $(k,| |)$ is a normed field. Then the additive group $(k,+)$ is a normed abelian group, so the completion $\hat{k}$ exists at least as a normed abelian group. Again though we want more, namely we want to define a multiplication on $\hat{k}$ in such a way that it becomes a field and that the norm satisfies $|x y|=|x||y|$. Again the porduct map on $k$ is uniformly continuous, so that it extends to $\hat{k}$, but to see that $\hat{k}$ is a field the algebraic construction is more useful. Indeed, it is not hard to show that $k^{\infty}$ is a ring, the Cauchy sequences $\mathcal{K}$ form a subring. But more is true:

Lemma 12. The set $\mathfrak{k}$ of sequences converging to 0 is a maximal ideal of the ring $\mathcal{K}$ of Cauchy sequences. Therefore the quotient $\mathcal{K} / \mathfrak{k}=\hat{k}$ is a field.
Proof. Since a Cauchy sequence is bounded, and a sequence which converges to 0 multiplied by a bounded sequence again converges to 0 , it follows that $\mathfrak{k}$ is an ideal of $\mathcal{K}$. To show that the quotient is a field, let $x_{\bullet}$ be a Cauchy sequence which does not converge to 0 . Then we need to show that $x_{\bullet}$ differs by a sequence converging to 0 from a unit in $\mathcal{K}$. But since $x_{\bullet}$ is Cauchy and not convergent to 0 , then (e.g. since it converges to a nonzero element in the abelian group $\hat{k}$ ) we have $x_{n} \neq 0$ for all sufficiently large $n$. Since changing any finite number of coordinates of $x_{\bullet}$ amounts to adding a sequence which is ultimately zero hence convergent to 0 , this is permissible as above, so after adding an element of $\mathfrak{k}$ we may assume that for all $n \in \mathbb{Z}^{+}, x_{n} \neq 0$, and then the inverse of $x_{\bullet}$ in $\mathcal{K}$ is simply $\left\{\frac{1}{x_{n}}\right\}$.
Exercise 2.30*: Find all maximal ideals in the ring $\mathcal{K}$.
Let $(k,| |)$ be a normed field, and let $\sigma: k \rightarrow k$ be a field automorphism. We say that $\sigma$ is an automorphism of the normed field $(k,| |)$ if $\sigma^{*}| |=| |$.

Exercise 2.31: Let $(k,| |)$ be a normed field and $\sigma$ an automorphism of $k$. Show that $\sigma$ is an automorphism of $(k,| |)$ iff it is continuous in the norm topology on $k$.

Exercise 2.32: Let $(k,| |)$ be a complete normed field, and let $\sigma$ be an automorphism of $k$. Put $\left|\left.\right|^{\prime}=\sigma^{*}\right| \mid$. Show that $k$ is also complete with respect to $\left|\left.\right|^{\prime}\right.$.

Exercise 2.33: Let $k$ be either $\mathbb{R}$ or $\mathbb{Q}_{p}$ for some prime $p$. We will show that $k$ is rigid, i.e., has no automorphisms other than the identity. Let $\sigma: k \rightarrow k$ be a field automorphism.
a) Suppose that $\sigma$ is continuous. Show that $\sigma=1_{k}$.
b) Show that any automorphism $\sigma$ of $k$ is continuous with respect to the norm topology. (Hint: Ostrowski's Theorem.)

Exercise 2.34: Let $k$ be a field complete with respect to a discrete, nontrivial valuation. Let $R$ be its valuation ring.
a) Show that $k$ is homeomorphic to the infinite disjoint union $\coprod_{i=1}^{\infty} R$.
b) Let $k_{1}, k_{2}$ be two fields complete with respect to discrete, nontrivial valuations, with valuation rings $R_{1}, R_{2}$. Suppose that $R_{1}$ and $R_{2}$ are compact. Show that $k_{1}$ and $k_{2}$ are homeomorphic, locally compact topological spaces.

We now give an alternate, more algebraic construction of the completion in the special case of a discretely valued, non-Archimedean norm on $k$. Namely, the norm is equivalent to a $\mathbb{Z}$-valued valuation $v$, with valuation ring

$$
R=\{x \in k \mid v(x) \geq 0\}
$$

and maximal ideal

$$
\mathfrak{m}=\{x \in k \mid v(x)>0\}=\{x \in k \mid v(x) \geq 1\} .
$$

Lemma 13. With notation above, suppose that $k$ is moreover complete. Then the ring $R$ is $\mathfrak{m}$-adically complete. Explicitly, this means that the natural map

$$
R \rightarrow \lim _{n} R / \mathfrak{m}^{n}
$$

is an isomorphism of rings.
Proof. This is straightforward once we unpack the definitions.
Injectivity: this amounts to the claim that $\bigcap_{n \in \mathbb{Z}^{+}} \mathfrak{m}^{n}=0$. In fact this holds for any nontrivial ideal in a Noetherian domain (Krull Intersection Theorem), but it is obvious here, because $\mathfrak{m}^{n}=\left(\pi^{n}\right)=\{x \in R \mid v(x) \geq n$, and the only element of $R$ which has valuation at least $n$ for all positive integers $n$ is 0 .
Surjectivity: Take any element $\mathbf{x}$ of the inverse limit, and lift each coordinate arbitarily to an element $x_{n} \in R$. It is easy to see that $\left\{x_{n}\right\}$ is a Cauchy sequence, hence convergent in $R$-since $k$ is assumed to be complete and $R$ is closed in $k, R$ is complete). Let $x$ be the limit of the sequence $x_{n}$. Then $x \mapsto \mathbf{x}$.

Exericse 2.35: Suppose now that $v$ is a discrete valuation on a field $k$. Let $\hat{R}=$ $\lim _{n} R / \mathfrak{m}^{n}$.
a) Show that $\hat{R}$ is again a discrete valuation ring - say with valuation $\hat{v}$ - whose maximal ideal $\hat{m}$ is generated by any uniformizer $\pi$ of $R$.
b) Let $\mathbb{K}$ be the fraction field of $\hat{R}$. Show that $\mathbb{K}$ is canonically isomorphic to $\hat{k}$, the completion of $k$ in the above topological sense.
c) Let $n \in \mathbb{Z}^{+}$. Explain why the natural topology on the quotient $R / \mathfrak{m}^{n}$ is the discrete topology.
d) Show that the following topologies on $\hat{R}$ all coincide: (i) the topology induced from the valuation $\hat{v}$; (ii) the topology $\hat{R}$ gets as a subset of $\prod_{n} R / \mathfrak{m}^{n}$ (the product of discrete topological spaces); (iii) the topology it inherits as a subset of $\hat{k}$ under the isomorphism of part b).

### 2.8. Non-Archimedean Functional Analysis: page 1.

$K$-Banach spaces: Let $(K,| |)$ be a complete normed field. In this context we can define the notion of a normed linear space in a way which directly generalizes the more familiar cases $K=\mathbb{R}, K=\mathbb{C}$. Namely:

A normed $K$-linear space is is a $K$-vector space $V$ and a map $|\mid: V \rightarrow \mathbb{R} \geq 0$ such that:
(NLS1) $\forall x \in V, x=0 \Longleftrightarrow|x|=0$.
(NLS2) $\forall \alpha \in K, x \in V,|\alpha x|=|\alpha||x|$.
(NLS3) $\forall x, y \in V,|x+y| \leq|x|+|y|$.
If $K$ is non-Archimedean, we require the stronger inequality $|x+y| \leq \max (|x|,|y|)$ in (NLS3).

Remark: Weakening (NLS1) to $\Longrightarrow$, we get the notion of a seminormed space.
Note that a normed linear space is a normed abelian group under addition. In particular it has a metric. A K-Banach space is a complete normed linear space over $K$.

The study of $K$-Banach spaces (and more general topological vector spaces) for a non-Archimedean field $K$ is called non-Archimedean functional analysis. This exists as a mathematical field which has real applications, e.g., to modern number theory (via spaces of $p$-adic modular forms). The theory is similar but not identical to that of functional analysis over $\mathbb{R}$ or $\mathbb{C}$. (Explain that the weak Hahn-Banach theorem only holds for spherically complete fields...)

Recall that two norms $\left|\left.\right|_{1},| |_{2}\right.$ on a $K$-vector space $V$ are equivalent if there exists $\alpha \in \mathbb{R}^{>0}$ such that for all $v \in V$,

$$
\frac{1}{\alpha}|v|_{1} \leq|v|_{2} \leq \alpha|v|_{1}
$$

Equivalent norms induce the same topology.
Theorem 14. Let $V$ be a finite dimensional $K$-vector space.
a) Choose a basis $v_{1}, \ldots, v_{n}$ of $V$, and define a map $\left|\left.\right|_{\infty}: V \rightarrow \mathbb{R} \geq 0\right.$ by $| \alpha_{1} v_{1}+$ $\ldots+\left.\alpha_{n} v_{n}\right|_{\infty}=\max _{i}\left|\alpha_{i}\right|$. Then $\left|\left.\right|_{\infty}\right.$ is a norm on $V$. The metric topology on $V$ is the one induced by pulling back the product topology on $K^{n}$ via the isomorphism $V \cong K^{n}$.
b) Any two norms on $V$ are equivalent.
c) It follows that for any norm $|\mid$ on $V,| |$ is complete and the induced topology coincides with the topology obtained by pulling back the product topology on $K^{n}$ via any isomorphism $V \cong K^{n}$.
Exercise: Prove Theorem 14.
Remark: This is a standard result which appears early on in a "conventional" functional analysis course, i.e., for Banach spaces over $\mathbb{R}$ or $\mathbb{C}$. The proof in the Archimedean case carries over verbatim to the non-Archimedean case. By not giving the proof here, we are encouraging the reader to book up on his general mathematical knowledge.

Theorem 15. Let $(K,| |)$ be a complete normed field and $(V,\| \|)$ a normed $K$ linear space. Let $W$ be a finite-dimensional $K$-subspace of $V$. Then $W$ is closed.

Proof. Indeed, the restriction of $\|\|$ to $W$. Then $(W,\| \|)$ is a finite-dimensional normed linear space over $K$. By Theorem 14, $W$ is complete with respect to \|\|, and a complete subspace of a metric space is closed.

When one thinks of "Archimedean functional analysis", Theorems 14 and 15 are probably not the first two which come to mind, perhaps because they are not very interesting! On the other hand, for our purpose these results are remarkably useful: they are just what we need to prove the uniqueness in Theorem 3, a topic to which we now turn.

### 2.9. Ostrowski's Theorem Revisited.

We claim that the Big Ostrowski Theorem (Theorem 1.XX) is equivalent to the following version (which is in fact a more traditional statement):
Theorem 16. Let $(K,| |)$ be a complete Archimedean field with Artin constant 2 (equivalently, $|2|=2$ ). Then $(K,| |)$ is either equal to $\mathbb{R}$ with its standard absolute value or $\mathbb{C}$ with its standard absolute value.

Let us show the equivalence of these two versions of Big Ostrowski. Assume Theorem 1.XX, and let $\left(K,| |_{1}\right)$ be an Archimedean normed field with Artin constant 2. Then we have an embeddding $\iota: K \hookrightarrow \mathbb{C}$ such that for all $x \in K,|x|_{1}=|\iota(x)|$. In other words, $\iota$ is an (isometric!) embedding of normed fields. As we have seen, there is an induced map $\hat{\imath}:\left(\hat{K},| |_{1}\right) \rightarrow(\hat{\mathbb{C}},| |)=(\mathbb{C},| |)$, since the standard norm on $\mathbb{C}$ is of course complete. To simplify notation, we may as well identify $\hat{K}$ with its isometric image under $\hat{\iota}$. Thus $\hat{K}$ is an isometrically embedded complete subfield of $\mathbb{C}$. In particular, since $\hat{K}$ contains $\mathbb{Q}$, it contains the closure of $\mathbb{Q}$ in $\mathbb{C}$, namely $\mathbb{R}$. Thus $\mathbb{R} \subset \hat{K} \subset \mathbb{C}$. Since $[\mathbb{C}: \mathbb{R}]=2$, we have little choice: either $\hat{K}=\mathbb{R}$ or $\hat{K}=\mathbb{C}$.

Conversely, assume Theorem X.X, and let $(K,| |)$ be an Archimedean normed field with Artin constant 2. Then $(K,| |)$ is a normed subfield of its completion $\hat{K}$, which is either $(\hat{R},| |)$ or $(\hat{\mathbb{C}},| |)$. Of course, $(\mathbb{R},| |)$ is a normed subfield of $(\hat{C},| |)$, so either way $(K,| |)$ is a normed subfield of $(\mathbb{C},| |)$.

### 2.10. Proof of Theorem 3 Part I: Uniqueness.

Theorem 17. Let $(K,| |)$ be a complete normed field, let $L / K$ be a field extension of finite degree d, and let || be a norm on $L$ extending the given norm on $K$. Then we must have that for all $x \in L$,

$$
\begin{equation*}
|x|=\left|N_{L / K}(x)\right|^{\frac{1}{d}} \tag{1}
\end{equation*}
$$

Proof. Step 1: We may assume without loss of generality that $L / K$ is normal. This reduction is left as an exercise.
Step 2: Suppose first that $L / K$ is separable, so WLOG $L / K$ is Galois and $N_{L / K}(x)=$ $\prod_{\sigma \in \operatorname{Aut}(L / K)} \sigma(x)$. Then by the preceding theorem we have

$$
\left|N_{L / K}(x)\right|=\left|\prod_{\sigma \in \operatorname{Aut}(L / K)} \sigma(x)\right|=\prod_{\sigma \in \operatorname{Aut}(L / K)}|x|=|x|^{d}
$$

Step 3: In the general case, let $d_{s}$ be the number of distinct $K$-embeddings of $L$ into an algebraic closure $\bar{K}$ of $K$ (the "separable degree") and let $d_{i}=\frac{d}{d_{s}}$ (the "inseparable degree"). As a piece of basic field theory, we have that - under the
assumption that $L / K$ is normal $-N_{L / K}(x)=\left(\prod_{\sigma \in \operatorname{Aut}(L / K)} \sigma(x)\right)^{d_{i}}$. The proof now proceeds as in Step 2 above.

Exercise X.X: Work out the details of Step 1 of the proof of Theorem X.X.
Corollary 18. Let $(K, \mid)$ be a complete normed field, and let $L / K$ an algebraic extension.
a) There exists at most one norm on $L$ extending || on $K$.
b) Suppose that for every finite subextension $M$ of $L / K$, the mapping $x \in M \mapsto$ $\left|N_{M / K}(x)\right|^{\frac{1}{M: K]}}$ of (1) is indeed a norm on $M$. Then the map

$$
\begin{equation*}
x \in L \mapsto\left|N_{K[x] / K}(x)\right|^{\frac{1}{\mid K[x]: K]}} \tag{2}
\end{equation*}
$$

is a norm on $L$.
Exercise X.X: Prove Corollary X.X. (Hint: use Exercise 2.2.)
Exercise: Suppose that $(K,| |)$ is Archimedean and $L / K$ is an algebraic field extension of $K$. Use Theorem X.X to show that (2) is the unique norm on $L$ extending || on $K$ (i.e., reprove uniqueness and verify existence!).

### 2.11. Proof of Theorem 5.

We come now to the most technically complicated of the basic extension theorems, Theorem 5. The reader will surely have noticed that we have taken some time to build up suitable tools and basic facts. Now our hard work comes to fruition: given what we already know, the proof of Theorem 5 (modulo the existence part of Theorem 3 in the non-Archimedean case, which we will treat last) is rather straightforward and elegant.

Let us begin by recalling the setup and what we already know. Let $(K,| |)$ be a normed field (note that we are certainly interested in the Archimedean case, and even the case of the standard Archimedean norm on $K=\mathbb{Q}!$ ). Let $L / K$ be a degree $n$ extension. Let $\overline{\hat{K}}$ be the algebraic closure of the completion of $(K,| |)$. We know:

- There is a unique norm on $\overline{\hat{K}}$ extending the given norm on $K$.
- Every norm on $L$ extending $|\mid$ comes from an embedding $\iota: L \hookrightarrow \overline{\hat{K}}$.

Since $L / K$ is algebraic and $\overline{\hat{K}}$ is an algebraically closed field containing $K$, certainly there exists at least one $K$-algebra embedding $\iota: L \hookrightarrow \overline{\hat{K}}$, thus at least one extended norm on $L$. Since $[L: K]=n<\infty$, the number of such embeddings $\iota$ is at most $n$, in particular it is finite. Therefore, let $g$ be the number of norms on $L$ extending $K$. We have:

$$
1 \leq g \leq n=[L: K]
$$

and the problem is to compute $g$ exactly in terms of $L, K$ and ||.
For $1 \leq i \leq g$, let $\left|\left.\right|_{i}\right.$ be the norms on $L$ extending $| \mid$ on $K$. Then $\left(L,| |_{i}\right)$ is a normed field and we may take the completion, say $\hat{L}_{i}$.

Now I claim that there is a canonical ring homomorphism

$$
\Phi: L \otimes_{K} \hat{K} \rightarrow \prod_{i=1}^{g} \hat{L}_{i}
$$

Indeed, to define it, we will use the universal properties of the direct product and the tensor product to reduce to a situation where we can easily guess what the definition should be. First, just by writing out $\Phi$ in coordinates we have $\Phi=\left(\Phi_{i}\right)_{i=1}^{g}$ where $\Phi_{i}: L \otimes_{K} \hat{K} \rightarrow \hat{L}_{i}$. In other words, to define $\Phi$, it is necessary and sufficient to define each $\Phi_{i}$. Moreover, by the universal property of the tensor product, to define $\Phi_{i}$ what we need is precisely a $K$-bilinear map $\varphi_{i}: L \times \hat{K} \rightarrow \hat{L}_{i}$. What is the "obvious" map here? Well, observe that $\iota_{i}(L)$ and $\hat{K}$ are both subfields of $\hat{L}_{i}$, so given an element $x \in L$ and $y \in \hat{K}$, we may use $\iota_{i}$ to map $L$ into $\hat{L}_{i}$ and then multiply them in $\hat{L}_{i}$. Explicitly,

$$
\varphi_{i}(x, y):=\iota_{i}(x) \cdot y
$$

and thus $\Phi$ is defined (on "simple tensors" $x \otimes y$, and then uniquely extended by linearity) as $\Phi(x \otimes y)=\left(\iota_{i}(x) y\right)_{i=1}^{g}$.

Let us stop and note that $\Phi$ is a map between two objects each with a lot of structure. Both the source and target of $\Phi$ are finite dimensional $\hat{K}$-algebras, and $\Phi$ is a $\hat{K}$-linear map. Indeed, another perspective on the definition of $\Phi$ is to define the diagonal map $\Delta: L \hookrightarrow \prod_{i=1}^{g} \hat{L}_{i}, x \mapsto\left(\iota_{i}(x)\right)$, note that $\Delta$ is $K$-linear and that $\prod_{i=1}^{g} \hat{L}_{i}$ is also a $\hat{K}$-algebra, so that $\Phi$ is the unique map corresponding to $\Delta$ under the canonical "adjunction" isomorphism

$$
\operatorname{Hom}_{K}\left(L, \prod_{i} \hat{L}_{i}\right)=\operatorname{Hom}_{\hat{K}}\left(L \otimes_{K} \hat{K}, \prod_{i} \hat{L}_{i}\right)
$$

Moreover, like any two finite-dimensional $\hat{K}$-vector spaces, $L \otimes_{K} \hat{K}$ and $\prod_{i=1}^{g} \hat{L}_{i}$ each come with a canonical topology, such that any $\hat{K}$-linear map between them (for instance, $\Phi!$ ) is necessarily continuous.

The precise result we want to prove, a sharpening of Theorem X.X, is the following:
Theorem 19. Let $\Phi: L \otimes_{K} \hat{K} \rightarrow \prod_{i=1}^{g} \hat{L}_{i}$ be the homomorphism defined above.
a) $\Phi$ is surjective.
b) The kernel of $(\Phi)$ is the Jacobson radical of the Artinian ring $L \otimes_{K} \hat{K}$, i.e., the intersection of all the maximal ideals. More precisely, there are precisely $g$ maximal ideals in $L \otimes_{K} \hat{K}$; suitably labelled as $\mathfrak{m}_{1}, \ldots, \mathfrak{m}_{g}$, the map $\Phi$ can be identified as the Chinese Remainder Theorem homomorphism

$$
L \otimes_{K} \hat{K} \rightarrow \prod_{i=1}^{g}\left(L \otimes_{K} \hat{K}\right) / \mathfrak{m}_{i}
$$

d) It follows that if $L / K$ is separable, $\Phi$ is an isomorphism.

Proof. For brevity, we put $A=L \otimes_{K} \hat{K}$.
a) Let $W=\Phi(A)$. We wish to show that $W=\prod_{i=1}^{g} \hat{L}_{i}$. Since $\Phi$ is $\hat{K}$-linear, $W$ is a $\hat{K}$-subspace of the finite-dimensional $\hat{K}$-subspace $\prod_{i=1}^{g} \hat{L}_{i}$. By Theorem $15, W$ is closed. On the other hand, by Artin-Whaples, the image of of $L$ iunder
$\Delta: L \hookrightarrow \prod_{i=1}^{g} L_{i}$ is dense, and by definition of completion (and an easy verification involving the topology on a finite product of metric spaces) $\prod_{i=1}^{g} L_{i}$ is dense in $\prod_{i=1}^{g} \hat{L}_{i}$. Certainly "is dense in" is a transitive relation among subspaces of a topological space, so $\Phi(L)=\Phi(L \otimes 1)$ is dense in $\prod_{i=1}^{g} \hat{L}_{i}$. Thus $\Phi$ is surjective.
b) Since $A$ is a finite-dimensional $\hat{K}$-algebra, it is an Artinian ring and therefore has finitely many maximal ( $=$ prime, here) ideals, say $\mathfrak{m}_{1}, \ldots, \mathfrak{m}_{N}$. So the Jacobson ( $=$ nil, here) radical $J=\bigcap_{i=1}^{N} \mathfrak{m}_{i}$. We have a finite set of pairwise comaximal ideals in a commutative ring, so the Chinese Remainder Theorem gives an isomorphism

$$
\Psi: A / J \xrightarrow{\sim} \prod_{j=1}^{N} A / \mathfrak{m}_{j} .
$$

For each $i, A / \mathfrak{m}_{i}$ is a finitely generated $\hat{K}$-module and also (since we modded out by a maximal ideal) a field, hence is a finite degree field extension of $\hat{K}$, say $L\left(\mathfrak{m}_{j}\right)$. Now our map $\Phi$ factors through $\Psi$ and we get

$$
A \rightarrow A / J \xrightarrow{\sim} \prod_{j=1}^{N} L\left(\mathfrak{m}_{j}\right) \xrightarrow{q} \prod_{i=1}^{g} \hat{L}_{i} .
$$

Thus by part a) we have one finite product of finite field extensions of $\hat{K}$ surjecting onto another finite product of finite field extensions of $\hat{K}$. A little thought shows that the surjectivity means that we must have $N \geq g$ and can relabel the $j$ 's such that for all $1 \leq j \leq g, \mathfrak{m}_{j}=\operatorname{Ker}\left(\Phi_{j}\right)$ and thus

$$
q: \prod_{j=1}^{g} \hat{L}_{j} \oplus \prod_{j>g} L\left(\mathfrak{m}_{j}\right) \rightarrow \prod_{j=1}^{g} \hat{L}_{j}
$$

is projection onto the first factor. In other words, what we wish to show is that we have put enough factors on the right hand side: $g=N$.

So let's try. What we have put on the right hand side is, precisely, one factor for each inequivalent norm on $L$ extending || of $K$. Each $L\left(\mathfrak{m}_{j}\right)$ is a finite degree extension of the complete field $\hat{K}$ so has a unique norm, say $\|_{j}$, which restricts to a norm on $L$. So if $N>g$ there exists $j_{1} \leq g$ and $j_{2}>g$ such that $\left|\left.\right|_{j_{1}}=| |_{j_{2}}\right.$ as norms on $L$. Now consider the projection of $\Psi$ onto just these two factors, i.e.,

$$
\Psi_{j_{1}, j_{2}}: L \otimes_{K} \hat{K} \rightarrow\left(\widehat{L,| |_{j_{1}}}\right) \times\left(\widehat{L,| |_{j_{2}}}\right)
$$

We claim that $\Psi_{j_{1}, j_{2}}$ is not in fact surjective, thus we have a contradiction. But this map ${ }^{8}$ is not so mysterious: it is determined by the images of $L$ and of $\hat{K}$. In particular, consider $\Psi_{j_{1}, j_{2}}$ restricted to $L$ : this is just the diagonal map; since the norms are equivalent, the topologies are the same, and thus the image of $L$ is closed in $\left(L,| |_{j_{1}}\right) \times\left(L,| |_{j_{2}}\right)$. Moreover, tensoring this diagonal map with $\hat{K}$ has the effect of completing these normed spaces (to see this, all we have to check is that after tensoring with $\hat{K}$ we have complete spaces and that the image is dense in the tensorization with $\hat{K}$ ). We have the same topology on both factors, so the closure of the diagonal is the diagonal of the closure, and thus the image of $A$ under $\Psi_{j_{1}, j_{2}}$ has $\hat{K}$-dimension $\operatorname{dim} L\left(\mathfrak{m}_{j_{1}}\right)=\operatorname{dim} L\left(\mathfrak{m}_{j_{2}}\right)$ and hence not equal to $\operatorname{dim} L\left(\mathfrak{m}_{j_{1}}\right) \times L\left(\mathfrak{m}_{j_{2}}\right)=2 \operatorname{dim} L\left(\mathfrak{m}_{j_{1}}\right)$, contradiction.
c) If $L / K$ is a separable field extension, then $A=L \otimes_{K} \hat{K}$ is a separable $K$-algebra,

[^7]i.e., a product of finite separable field extensions. To see this, write $L=K[t] /(P(t))$ with $P(t)$ an irreducible separable polynomial (this is possible by the primitive element theorem). Being a separable polynomial is unaffected by extending the field: if $M$ is any extension of $K$, however large, then $P \in M[t]$ factors into distinct irreducible polynomials. (To see this, use e.g. the derivative criterion for separability: $P$ is separable iff $\operatorname{gcd}\left(P, P^{\prime}\right)=1$.) Applying this remark with $M=\hat{K}$, we get $P=P_{1} \cdots P_{g} \in \hat{K}[t]$ with the $P_{i}$ 's distinct irreducible polynomials; thus the set of ideals $\left(P_{i}\right)$ are pairwise comaximal, and the CRT isomorphism is
$$
A=\hat{K}[t] /(P)=\hat{K}[t] /\left(P_{1} \cdots P_{g}\right) \cong \prod_{i=1}^{g} \hat{K}[t] /\left(P_{i}\right) \cong \prod_{i=1}^{g} \hat{L}_{i}
$$

### 2.12. Proof of Theorem 3 Part II: Existence.

We return to the situation of a complete normed field ( $K,| |$ ) and a degree $n<\infty$ field extension $L / K$. We have seen that if there exists an extension of $|\mid$ to a norm on $L$, it must be the map

$$
x \in L \mapsto\left|N_{L / K}(x)\right|^{\frac{1}{[L: K]}} .
$$

In the Archimedean case, the Ostrowski theorem reduces us to checking that this is indeed the correct recipe for the standard norm on $\mathbb{C}$ as a quadratic extension of $\mathbb{R}$. Thus we are left to deal with the non-Archimedean case. As mentioned above, we really need to check the ultrametric triangle inequality.

From our study of Artin valuations in $\S 1$, we know that we do not change whether a mapping is a non-Archimedean norm by raising it to any power, so we might we well look at the mapping $x \mapsto\left|N_{L / K}(x)\right|$ instead. Moreover, we also know that the non-Archimedean triangle inequality is equivalent to: for all $x \in L$, $|x| \leq 1 \Longrightarrow|x+1| \leq 1$.

This is what we will check. In fact, as came out in the lecture, it is convenient to make a further reduction: since $N_{L / K}(x)=\left(N_{K[x] / K}(x)\right)^{[L: K[x]]}$, we may as well assume that $L=K[x]$.

Lemma 20. (Hensel-Kürschák) Let $(K,| |)$ be a complete, non-Archimedean normed field. Suppose that $P(t)=t^{n}+a_{n-1} t^{n-1}+\ldots+a_{1} t+a_{0} \in K[t]$ is irreducible and such that $\left|a_{0}\right| \leq 1$. Then $\left|a_{i}\right| \leq 1$ for all $0<i<n$.

Let us defer the proof of Lemma 20 and see why it is useful for us. For $\alpha \in L$, let $P(t)$ be the minimal polynomial of $\alpha$ over $K$, so $P(t)$ is a monic irreducible polynomial of degree $m=[K[\alpha]: K]$ and has constant coefficient $a_{0}=(-1)^{m} N_{K[\alpha]: K}(x)$. By assumption,

$$
1 \geq\left|N_{L / K}(\alpha)\right|=\left|N_{K[\alpha] / K}(\alpha)^{n / m}\right|=\left|a_{0}\right|^{n / m}
$$

so $\left|a_{0}\right| \leq 1$. Now the minimal polynomial for $\alpha+1$ is $P(t-1)$ (note that $K[\alpha]=$ $K[\alpha+1])$. Plugging in $t=0$, we get

$$
(-1)^{m} N_{K[\alpha] / K}(\alpha+1)=P(-1)=(-1)^{m}+a_{m-1}(-1)^{m-1}+\ldots+(-1) a_{1}+a_{0}
$$

By Lemma 20 we have $\left|a_{i}\right| \leq 1$ for all $i$, and by the non-Archimedean triangle inequality in $K$ we conclude that

$$
\left|N_{L / K}(\alpha+1)=\left|N_{K[\alpha] / K}(\alpha+1)\right|^{n / m} \leq 1^{n / m} \leq 1 .\right.
$$

Now what about Lemma 20? It follows from the following result:
Theorem 21. (Hensel's Lemma, Version I) Let $(K,| |)$ be a complete, non-Archimedean normed field with valuation ring $R$, and let $f(t) \in R[t]$. Suppose there exists $\beta \in R$ such that $|f(\beta)|<1$ and $\left|f^{\prime}(\beta)\right|=1$. Then there exists $\alpha \in R$ such that $f(\alpha)=0$ and $|\alpha-\beta|<1$.
Exercise X.X: Deduce Lemma 20 from Theorem 21.
The proof of 21 brings us to the next chapter of our course, a study of Hensel's Lemma in its various forms. This is an all-important result in the study of nonArchimedean fields, and we will give many applications, from the abstract - the completion of the algebraic closure is algebraically closed - to the concrete - determination of the multiplicative structure of $p$-adic fields.


[^0]:    Thanks to John Doyle and David Krumm for pointing out typos.

[^1]:    ${ }^{1}$ Somewhat embarrassingly, the question of whether there exist infinitely many number fields of class number one remains open!

[^2]:    ${ }^{2}$ The letter $T$ will be used to denote an exercise that is - despite appearances, perhaps - trivial to prove, but useful to apply later. I do not guarantee that an exericse not so marked will be nontrivial.

[^3]:    ${ }^{3}$ This is just one of many possible choices of a product metric. The non-canonicity in the choice of the product is a clue that our setup is not optimal. But the remedy for this, namely uniform spaces, is not worth our time to develop.

[^4]:    ${ }^{4}$ Note that I am sidestepping the issue of whether a non-Hausdorff space should be called "compact" or just "quasi-compact" as is standard e.g. in algebraic geometry. The point is that all our spaces will be metrizable, hence Hausdorff, so no worries.
    ${ }^{5}$ I made up the term.

[^5]:    ${ }^{6}$ Completeness is formally defined in the next section.

[^6]:    ${ }^{7}$ This is a standard abuse of terminology: really we should refer to the map $\iota: X \hookrightarrow \hat{X}$ as the completion, but one rarely does so.

[^7]:    ${ }^{8}$ Despite its complication notation!

