REVIEW PROBLEMS FOR SECOND 3200 MIDTERM

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1)a) State Euclid's Lemma (the one involving prime numbers and divisibility).
b) Use Euclid's Lemma to show that $3^{1/5}$ and $5^{1/3}$ are both irrational.

2) Let $x, y \in \mathbb{Z}$, and suppose that x is of the form $9k + 3$ for some integer k.
a) Show that $2x^2 + 54y$ is divisible by 9.
b) Show that $2x^2 + 54y$ is not divisible by 27.

3) Let $x \in \mathbb{Z}$. Prove or disprove each of the following statements:
a) If $4 \mid x^2$, then $4 \mid x$.
b) If $5 \mid x^2$, then $5 \mid x$.
c) If $6 \mid x^2$ then $6 \mid x$.

4) Let $r \neq 1$ be a real number. Show that for all $n \in \mathbb{Z}^+$, $1 + r + \ldots + r^n = \frac{r^{n+1}-1}{r-1}$.

5) Consider the following statement: for all $n \in \mathbb{Z}^+$, $3 \mid n^3 + 2n$.
a) Prove the statement using congruences.
b) Prove the statement using induction.

6) A student has been asked to prove: $\forall x \in \mathbb{Z}, P(x) \implies Q(x)$.[1] For each of the following openers, comment on the proof technique, or explain why it is not a valid proof technique.

Example: "Let $x \in S$, and suppose $P(x)$ is true."
Comment: This is the beginning of a direct proof.

a) "Let $x \in S$, and suppose $P(x)$ is false."
b) "Let $x \in S$, and suppose that $Q(x)$ is true."
c) "Let $x \in S$, and suppose $Q(x)$ is false."
d) "Let $x = 1$. Then" [the student shows that $P(1)$ is true and $Q(1)$ is true].
e) "Let $x = 2$. Then" [the student shows that $P(2)$ is false and $Q(2)$ is false].
f) "Let $x = 3$. Then" [the student shows that $P(3)$ is true and $Q(3)$ is false].
g) Let $x \in S$, and suppose that $P(x)$ is true and $Q(x)$ is false.

7) a) State the principle of mathematical induction as it applies to subsets of $\mathbb{Z}^+$ and also as a proof technique.
b) True or false: Suppose that for $P(x)$ is an open sentence with domain the real numbers. Then it is simply not possible to use mathematical induction to show that for all $x \in \mathbb{R}$, $P(x)$ holds.

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[1] Here $P(x)$ and $Q(x)$ are sentences involving an arbitrary integer x.

8) a) Show: $n! > 2^n$ for all $n \geq 4$.
b) Show: $n! > 3^n$ for all $n \geq 7$.
(You may use that $7! = 5040$ and $3^7 = 2187$.)

9) Show: for all integers $n \geq 0$, we have $\int_0^\infty x^n e^{-x} dx = n!$

10) Let A be a set. Prove or disprove: if for every set B, $A \setminus B = \varnothing$, then $A = \varnothing$.

11) Prove or disprove:
a) For all rational numbers a and b, $a + b$ and ab are both rational.
b) For all irrational (real) numbers a and b, $a + b$ is irrational.
c) For all irrational (real) numbers a and b, $a + b$ is rational.
d) For all irrational (real) numbers a and b, ab is irrational.
e) For all irrational (real) numbers a and b, ab is rational.