3200 PROBLEM SET 8: TYPED PROBLEMS

Recall that a relation R between sets X and Y is, formally, given by a subset R of the Cartesian product $X \times Y$. In many interesting cases we have $X = Y$, and then instead of saying "a relation between X and X" we generally abbreviate to "a relation **on** X".

1) Let R be a relation on X, and let Y be any subset of X. Then there we can define a relation $R|_Y$ on Y, called the **restriction** of R to Y, simply as $R \cap (Y \times Y)$. In other words, it consists of all ordered pairs $(y, y') \in Y \times Y$ such that yRy'.
a) Let R the relation on $\mathbb{R}$ given by $xRy \iff x^2 + y^2 = 1$. Describe the restriction of this relation to the subset $R^{>0}$ of positive real numbers; describe the restriction of this relation to the subset $\mathbb{Z}$ of integers.
b) Let R be a relation on X, and let D be its domain. Show that the domain of $R|_D$ is simply D. Describe this process explicitly for the relation R on $\mathbb{R}$ given in part a).
c) Show that any restriction of a reflexive relation is reflexive.
d) Show that any restriction of a symmetric relation is symmetric.
e) Show that any restriction of an anti-symmetric relation is anti-symmetric.
f) Show that any restriction of a transitive relation is transitive.
g) Conclude that the restriction of an equivalence relation (respectively, a partial ordering) on X to any subset Y is an equivalence relation (respectively, a partial ordering) on Y.
h) Equivalence relations correspond to partitions. So by part g), given a partition on a set X and a subset Y of X, there exists a natural "restricted partition" on Y. Describe this explicitly.

2) Let R be a relation on X which is symmetric and transitive.
a) Show by example that R need not be reflexive.
b) Show however that the restriction of X to its domain D is reflexive, hence is an equivalence relation on D. In particular, a symmetric transitive relation with domain X is an equivalence relation.

3) Let R be a relation on X which is both symmetric and anti-symmetric. Show that the restriction of R to the domain of X is the equality relation. Must R actually be equality on X?

4) A relation may or may not be reflexive, may or may not be symmetric, and may or may not be transitive, so that there are altogether $2 \cdot 2 \cdot 2 = 8$ possibilities. Give examples to show that all 8 of these possibilities actually occur.

5) A relation may or may not be anti-symmetric, so in conjunction with the three properties of 3) above, this gives 16 independent possibilities. Figure out which of these 16 possibilities actually occur, and give examples of those which do.