MATH 3200 THIRD MIDTERM EXAM

Directions: Do all five problems. No calculators are permitted.

1) Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove or disprove:
a) If f is injective and g is injective, then $g \circ f$ is injective.
b) If f is surjective and g is surjective, then $g \circ f$ is surjective.
c) If f is surjective and g is injective, then $g \circ f$ is either injective or surjective.

2) For each of the following functions $f : \mathbb{R} \to \mathbb{R}$, determine whether f is injective, surjective and/or bijective. Briefly justify your answers.
a) $f(x) = x^5 - x$.
b) $f(x) = e^x + x + 5$.
c) $f(x) = (x-1)(x-2)(x-3)(x-4)$.

3) a) Consider the relation $R_1 \subset \mathbb{Z} \times \mathbb{Z}$ defined by $xR_1y \iff x \mid y$ (i.e., the divisibility relation). Is R_1 reflexive? symmetric? transitive? Is it an equivalence relation? If so, describe explicitly the partition of $\mathbb{Z}$ into equivalence classes.

b) Consider the relation $R_2 \subset \mathbb{Z} \times \mathbb{Z}$ defined by $xR_2y \iff (x \mid y \text{ and } y \mid x)$. Is R_2 reflexive? symmetric? transitive? Is it an equivalence relation? If so, describe explicitly the partition of $\mathbb{Z}$ into equivalence classes.

4) Recall that the Fibonacci sequence is defined as follows $F_1 = F_2 = 1$ and for all $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$. Prove that for all positive integers n, $F_n \leq 2^n$. (Hint: strong induction on n.)

5) Let $f : X \to Y$ be a function. What does it mean to say that f has an inverse function? Give a necessary and sufficient condition for f to have an inverse function.