Bivariate Splines for Surface Design

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Existing Methods

The following methods for fitting a given set of data are available in the literature (cf. [1]).

- Minimal Energy Method;
- Discrete Least Squares Method;
- Penalized Least Squares Spline Method;
- L_1 Spline Method;
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Quick Comparison of New Methods

Table: Nonlinear Model Results

Function	Case 1	Case 2	Case 3	Case 4
$z = \frac{1}{9}(\tanh(9x - 9y) + 1)$	0.0206	0.0171	0.0593	
$z = \sin x + \sin y$	0.0021	2.55×10^{-4}	0.1561	
$z = 2x^4 + 5y^4$	0.1084	0.0188	1.5982	
$z = (x^2 + 3y^2)e^{-x^2-y^2}$	0.0109	0.0013	0.0942	

- Case 1: Minimal Energy Method
- Case 2: Minimal Triharmonic Energy Method
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Overview

- The Triharmonic Energy function is defined by the equation $H(f) = \sum_{t_i \in \triangle} \int_{t_i} \left[\left(\frac{\partial^3}{\partial x^3} f \right)^2 + 3 \left(\frac{\partial^3}{\partial x^2 \partial y} f \right)^2 + 3 \left(\frac{\partial^3}{\partial x \partial y^2} f \right)^2 + \left(\frac{\partial^3}{\partial y^3} f \right)^2 \right] dxdy.$
- Let $\Lambda(f) = \{s \in S_d^r(\partial), s(x_i, y_i) = f, i = 1, \ldots, N\}$. Find $S_f \in \Lambda(f)$ such that

$$H(S_f) = min\{H(s), s \in \Lambda(f)\} \quad (1)$$

Theorem

If $\Lambda(f)$ is not empty, then there exists a unique interpolatory spline $S_f \in \Lambda(f)$ satisfying (1).

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Outline

First Show Existence

- We first show the existence. Let $S_o \in \Lambda(f)$.
- Consider $D = \{s \in \Lambda(f), H(s) \leq H(S_o)\}$.
- Clearly D is not empty. We want to show that D is closed.
- Let $S_n \in D,\ n = 1, \dots, \infty$ and $S_n \to S^*$ in the maximum norm.
- Claim $S^* \in D$, then D is closed.
- $||S_n - S^*|| \to 0$ for each triangle t_i.
- $Sn - S^*|_{t_i}$ is a polynomial of degree d.

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Proof of Existence Continued

- $\left|\frac{\partial}{\partial x}S_n(x) - \frac{\partial}{\partial x}S^*(x)\right| \leq \frac{c}{|t_i|}||S_n - S^*||_\infty$ by Markov Inequality (c.f. [?])
- $\left|\frac{\partial^2}{\partial x^2}S_m(x) - \frac{\partial^2}{\partial x^2}S^*(x)\right| \leq \frac{c}{|t_i|}||\frac{\partial}{\partial x}S_n(x) - \frac{\partial}{\partial x}S^*(x)|| \leq \frac{c^2}{|t_i|^2}||S_m - S^*||_\infty$
- $\left|\frac{\partial^3}{\partial x^3}S_k(x) - \frac{\partial^3}{\partial x^3}S^*(x)\right| \leq \frac{c}{|t_i|}||\frac{\partial^2}{\partial x^2}S_k - \frac{\partial^2}{\partial x^2}S^*|| \leq \frac{c^2}{|t_i|^2}||\frac{\partial}{\partial x}S_k - \frac{\partial}{\partial x}S^*|| \leq \frac{c^3}{|t_i|^3}||S_k - S^*||_\infty$
- From this it follows that

$$\sum_{t_i \in \triangle} \int_{t_i} \left|\frac{\partial^3}{\partial x^3}S_k(x) - \frac{\partial^3}{\partial x^3}S^*(x)\right|^2 dx \leq \sum_{t_i \in \triangle} \int t_i \frac{c^3}{|t_i|^3}(||S_k - S^*||_\infty)^2 \to 0$$

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Proof of Existence Continued

Then we have the equation

$$\begin{aligned} H(S^*) &= \sum_{t_i \in \triangle} \int_{t_i} \left| \frac{\partial^3}{\partial x^3} S^*(x) \right|^2 dxdy \\ &= \sum_{t_i \in \triangle} \int_{t_i} \left| \frac{\partial^3}{\partial x^3} S^*(x) - \frac{\partial^3}{\partial x^3} S_k(x) + \frac{\partial^3}{\partial x^3} S_k(x) \right|^2 dx \\ &= \sum_{t_i \in \triangle} \int_{t_i} \left(\left| \frac{\partial^3}{\partial x^3} S^*(x) - \frac{\partial^3}{\partial x^3} S_k(x) \right|^2 + \left| \frac{\partial^3}{\partial x^3} S_k(x) \right|^2 \right) dx \\ &\quad + \sum_{t_i \in \triangle} 2 \int_{t_i} \left(\frac{\partial^3}{\partial x^3} S^*(x) - \frac{\partial^3}{\partial x^3} S_k(x) \right) \frac{\partial^3}{\partial x^3} S_k(x) dx \end{aligned}$$

Proof of Existence Continued

By the Cauchy-Schwarz inequality

$$\begin{aligned}
&\leq \quad \epsilon + H(S_k) + 2 \sum_{t_i \in \triangle} \left(\int_{t_i} \left| \frac{\partial^3}{\partial x^3} S^*(x) - \frac{\partial^3}{\partial x^3} S_k(x) \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{t_i} \left| \frac{\partial^3}{\partial x^3} S_k(x) \right|^2 dx \right)^{\frac{1}{2}} \\
&\leq \quad \epsilon + H(S_k) + 2 \sqrt{\sum_{t_i \in \triangle} \left(\int_{t_i} \left| \frac{\partial^3}{\partial x^3} S^*(x) - \frac{\partial^3}{\partial x^3} S_k(x) \right|^2 dx \right)} \sqrt{H(S_k)} \\
&\leq \quad \epsilon + H(S_k) + 2\sqrt{\epsilon}\sqrt{H(S_k)} \\
&\leq \quad H(S_o) + \epsilon + 2\sqrt{\epsilon}\sqrt{H(S_o)}
\end{aligned}$$

Proof of Existence Concluded

- Consequently, $H(S^*) \leq H(S_o)$.
- It follows that $S^* \in D$ and hence D is closed.
- Therefore we can claim that $H(S_f)$ is continuous and has a limit over D.

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Make Titles Informative.

- Next we will show uniqueness.
- Suppose that we have two solutions $S_1, S_2 \in \Lambda_f$ such that $H(S_1) = H(S_2),\ S_1 \neq S_2$.
- Let $S_\alpha = \alpha S_1 + (1-\alpha)S_2,\ 0 \leq \alpha \leq 1$.
- Clearly, $S_\alpha \in \Lambda_f$, which means $S_\alpha \in S_d^r(\Delta)$.
- $S_\alpha(t_i) = \alpha S_1(t_i) + (1-\alpha)S_2(t_i) = \alpha f_i + (1-\alpha)f_i = f_i$
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Proof of Uniqueness Continued

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- $= \sum_{t_i \in \triangle} \int_{t_i} \left|\alpha \frac{\partial^3}{\partial x^3} S_1 + (1-\alpha)\frac{\partial^3}{\partial x^3} S_2\right|^2 dx$
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- $\alpha H(S_1) + (1-\alpha)H(S_2)$ from the previous slide
- $H(S_1)(\alpha + 1 - \alpha)$, since $H(S_1) = H(S_2)$
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  $\therefore \; F(\alpha) = H(S_1)$.
- Since $F(\alpha) = H(S_1)$, $F(\alpha)$ is a constant function.
- Therefore $F'(\alpha) = 0$.
  $F'(\alpha) =$
  $\sum_{t_i \in \triangle} 2 \int_{t_i} \left(\alpha \frac{\partial^3}{\partial x^3} S_1 + (1-\alpha) \frac{\partial^3}{\partial x^3} S_2 \right) \left(\frac{\partial^3}{\partial x^3} S_1 - \frac{\partial^3}{\partial x^3} S_2 \right) dxdy$

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Proof of Uniqueness Continued

At $\alpha = 0^+$, we have

$$
\begin{aligned}
0 = F'(0) &= \sum_{t_i \in \triangle} 2 \int_{t_i} \frac{\partial^3}{\partial x^3} S_2 \left(\frac{\partial^3}{\partial x^3} S_1 - \frac{\partial^3}{\partial x^3} S_2 \right) dxdy \\
&= \sum_{t_i \in \triangle} 2 \int_{t_i} \left[\left(\frac{\partial^3}{\partial x^3} S_2 \frac{\partial^3}{\partial x^3} S_1 \right) - \left(\frac{\partial^3}{\partial x^3} S_2 \right)^2 \right] dxdy \\
&\Longrightarrow \sum_{t_i \in \triangle} 2 \int_{t_i} \frac{\partial^3}{\partial x^3} S_2 \frac{\partial^3}{\partial x^3} S_1 \, dxdy = \sum_{t_i \in \triangle} 2 \int_{t_i} \left(\frac{\partial^3}{\partial x^3} S_2 \right)^2 dxdy \\
&\Longrightarrow \sum_{t_i \in \triangle} \int_{t_i} \frac{\partial^3}{\partial x^3} S_1 \, dxdy = \sum_{t_i \in \triangle} \int_{t_i} \frac{\partial^3}{\partial x^3} S_2 \, dxdy
\end{aligned}
$$

Proof of Uniqueness Continued

At $\alpha = 1$, we have

$$\begin{aligned}
F'(1) &= \sum_{t_i \in \triangle} 2 \int_{t_i} \left[\left(\frac{\partial^3}{\partial x^3} S_1 \right) - \left(\frac{\partial^3}{\partial x^3} S_1 \frac{\partial^3}{\partial x^3} S_2 \right) \right] dxdy \\
&= \sum_{t_i \in \triangle} 2 \int_{t_i} \left(\frac{\partial^3}{\partial x^3} S_1^2 - \frac{\partial^3}{\partial x^3} S_1 \frac{\partial^3}{\partial x^3} S_2 \right) = 0 \\
\Longrightarrow & \sum_{t_i \in \triangle} 2 \int_{t_i} \left(\frac{\partial^3}{\partial x^3} S_1 \right)^2 dx = \sum_{t_i \in \triangle} 2 \int_{t_i} \frac{\partial^3}{\partial x^3} S_1 \frac{\partial^3}{\partial x^3} S_2 dx \\
\Longrightarrow & \sum_{t_i \in \triangle} 2 \int_{t_i} \frac{\partial^3}{\partial x^3} S_1 dx = \sum_{t_i \in \triangle} 2 \int_{t_i} \frac{\partial^3}{\partial x^3} S_2 dx
\end{aligned}$$

Proof of Uniqueness Continued

- From those two equations we get $\sum_{t_i \in \triangle} 2 \int_{t_i} \left[\left(\frac{\partial^3}{\partial x^3} S_1 \right)^2 - 2 \frac{\partial^3}{\partial x^3} S_1 \frac{\partial^3}{\partial x^3} S_2 + \left(\frac{\partial^3}{\partial x^3} S_2 \right)^2 \right] dx = 0$
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Proof of Uniqueness Concluded

- We can see that $\frac{\partial^3}{\partial x^3}(S_1 - S_2) = 0$, $\frac{\partial^3}{\partial xy^2}(S_1 - S_2) = 0$, and $\frac{\partial^3}{\partial y^3}(S_1 - S_2) = 0$ are similar cases.
- Since $\frac{\partial^3}{\partial x^3}(S_1 - S_2) = 0$, we know that $S_1 - S_2$ is a polynomial of degree 2.
- If $S_1 - S_2 = 0$ on at least 6 points, then it is not on a conic section and must equal 0. Then $S_1 = S_2$.
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Overview

- The Surface Area function is defined by the equation
  $A(f) = \sum_{t_i \in \triangle} \int_{t_i} \sqrt{1 + \left(\frac{\partial}{\partial x} f\right)^2 + \left(\frac{\partial}{\partial y} f\right)^2} dxdy$
- Let $\Lambda(f) = \{s \in S_d^r(\Delta), s(x_i, y_i) = f_, i = 1, \ldots, N\}$. Find $S_f \in \Lambda(f)$ such that

$$A(S_f) = min\{A(s), s \in \Lambda(f)\} \quad (2)$$

Theorem

If $\Lambda(f)$ is not empty, then there exists a unique interpolatory spline $S_f \in \Lambda(f)$ satisfying (2).

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First Show Existence

- First we will prove the existence. Let $S_o \in \Lambda(f)$.
- Consider $D = \{s \in \Lambda(f), A(s) \leq A(S_o)\}$.
- Clearly D is not empty.
- Let $S_n \in D,\ n = 1, \ldots, \infty$ and $S_n \to S^*$ in the maximum norm.
- We claim that D is closed. We need to show that $S^* \in D$.
- $||S_n - S^*|| \to 0$ for each triangle t_i.
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Proof of Existence Continued

$\left|\left|\frac{\partial}{\partial x}(S_n - S^*)\right|\right| \leq \frac{c}{|t_i|}||S_n - S^*|| \rightarrow$
0 By Markov Inequality (c.f. [1])
Next we show that $A(S_n) \rightarrow A(S^*)$, since

$$\begin{aligned}
S_n &\rightarrow S^* \text{ , from assumption} \\
\Longrightarrow &\left(\frac{\partial}{\partial x}S_n\right) \rightarrow \left(\frac{\partial}{\partial x}S^*\right) \text{ as shown above by } (\textbf{??}) \\
\Longrightarrow &\left(\frac{\partial}{\partial x}S_n\right)^2 \rightarrow \left(\frac{\partial}{\partial x}S^*\right)^2 \\
\Longrightarrow &\ 1 + \left(\frac{\partial}{\partial x}S_n\right)^2 + \left(\frac{\partial}{\partial y}S_n\right)^2 \rightarrow 1 + \left(\frac{\partial}{\partial x}S^*\right)^2 + \left(\frac{\partial}{\partial y}S^*\right)^2 \\
\Longrightarrow &\sqrt{1 + \left(\frac{\partial}{\partial x}S_n\right)^2 + \left(\frac{\partial}{\partial y}S_n\right)^2} \rightarrow \sqrt{1 + \left(\frac{\partial}{\partial x}S^*\right)^2 + \left(\frac{\partial}{\partial y}S^*\right)^2}
\end{aligned}$$

Proof of Existence Concluded

- Thus, since $A(S_n) \leq A(S_o)$, $A(S^*) \leq A(S_o)$.
- Therefore $S^* \in D$ and D is closed.
- Therefore we can claim that $A(S_f)$ is continuous and has a limit over D.

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Outline

Proof of Uniqueness

- Next we will show uniqueness.
- Suppose that we have two solutions $S_1, S_2 \in \Lambda_f$ such that $A(S_1) = A(S_2),\ S_1 \neq S_2$.
- Let $S_\alpha = \alpha S_1 + (1-\alpha)S_2,\ 0 \leq \alpha \leq 1$.
- Clearly, $S_\alpha \in \Lambda_f$, which means $S_\alpha \in S_d^r(\Delta)$.
- $S_\alpha(t_i) = \alpha S_1(t_i) + (1-\alpha)S_2(t_i) = \alpha f_i + (1-\alpha) f_i = f_i$
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Proof of Uniqueness Continued

- $$\begin{aligned} F(\alpha) &= A(\alpha S_1 + (1-\alpha)S_2) \\ &\leq \alpha A(S_1) + (1-\alpha)A(S_2) \text{ by convexity} \\ &= (\alpha + 1 - \alpha)A(S_1) \text{ since } A(S_1) = A(S_2) \\ &= A(S_1) \text{ which implies that } F(\alpha) \leq A(S_1),\ \therefore F(\alpha) = A(S_1) \end{aligned}$$

- Since $F(\alpha) = A(S_1)$, $F(\alpha)$ is a constant function. Therefore $F'(\alpha) = 0$.

$$F'(\alpha) = \sum_{t_i \in \triangle} \int_{t_i} \frac{2\left[2\frac{\partial}{\partial x}(\alpha S_1 + (1-\alpha)S_2)\frac{\partial}{\partial x}(S_1 - S_2) + 2\frac{\partial}{\partial y}(\alpha S_1 + (1-\alpha)S_2)\right]\frac{\partial}{\partial y}(S_1 - S_2)}{\sqrt{1 + \left(\frac{\partial}{\partial x}(\alpha S_1 + (1-\alpha)S_2)\right)^2 + \left(\frac{\partial}{\partial y}\alpha S_1 + (1-\alpha)S_2\right)^2}} dxdy$$

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Proof of Uniqueness Continued

At $\alpha = 0^+$, we have

$$0 = F'(0) = \sum_{t_i \in \triangle} \int_{t_i} \frac{2\left[2\left(\frac{\partial}{\partial x} S_2\right) \frac{\partial}{\partial x}(S_1 - S_2) + 2\left(\frac{\partial}{\partial y} S_2\right] \frac{\partial}{\partial y}(S_1 - S_2)\right)}{\sqrt{1 + \left(\frac{\partial}{\partial x} S_2\right)^2 + \left(\frac{\partial}{\partial y} S_2\right)^2}} dxdy$$

$$= \sum_{t_i \in \triangle} \int_{t_i} \frac{4\left[\frac{\partial}{\partial x} S_2 \frac{\partial}{\partial x} S_1 - \left(\frac{\partial}{\partial x} S_2\right)^2\right] + 4\left[\frac{\partial}{\partial y} S_2 \frac{\partial}{\partial y} S_1 - \left(\frac{\partial}{\partial y} S_2\right)^2\right]}{\sqrt{1 + \left(\frac{\partial}{\partial x} S_2\right)^2 + \left(\frac{\partial}{\partial y} S_2\right)^2}} dxdy$$

$$\implies \sum_{t_i \in \triangle} \int_{t_i} 4 \frac{\partial}{\partial x} S_2 \frac{\partial}{\partial x} S_1 \, dx = \sum_{t_i \in \triangle} \int_{t_i} 4 \frac{\partial}{\partial x} S_2^2 \, dx$$

$$\implies \sum_{t_i \in \triangle} \int_{t_i} \frac{\partial}{\partial x} S_1 \, dx = \sum_{t_i \in \triangle} \int_{t_i} \frac{\partial}{\partial x} S_2 \, dx$$

Proof of Uniqueness Continued

At $\alpha = 1$, we have

$$0 = F'(1) = \sum_{t_i \in \triangle} \int_{t_i} \frac{2\left[2\left(\frac{\partial}{\partial x}S_1\right)\frac{\partial}{\partial x}(S_1 - S_2) + 2\left(\frac{\partial}{\partial y}S_1\right)\frac{\partial}{\partial y}(S_1 - S_2)\right]}{\sqrt{1 + \left(\frac{\partial}{\partial x}S_1\right)^2 + \left(\frac{\partial}{\partial y}S_1\right)^2}} dxdy$$

$$= \sum_{t_i \in \triangle} \int_{t_i} \frac{4\left[\left(\frac{\partial}{\partial x}S_1\right)^2 - \frac{\partial}{\partial x}S_1 \frac{\partial}{\partial x}S_2\right] + 4\left[\left(\frac{\partial}{\partial y}S_1\right)^2 - \frac{\partial}{\partial y}S_1 \frac{\partial}{\partial y}S_2\right]}{\sqrt{1 + \left(\frac{\partial}{\partial x}S_1\right)^2 + \left(\frac{\partial}{\partial y}S_1\right)^2}} dxdy$$

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Proof of Uniqueness Concluded

- From those two equations we get
$\sum_{t_i \in \triangle} 2 \int_{t_i} \left[\left(\frac{\partial}{\partial x} S_1\right)^2 - 2 \frac{\partial}{\partial x} S_1 \frac{\partial}{\partial x} S_2 + \left(\frac{\partial}{\partial x} S_2\right)^2 \right] dx = 0$
- which is the same as
$$\sum_{t_i \in \triangle} 2 \int_{t_i} \left[\left(\frac{\partial}{\partial x} S_1\right)^2 - \left(\frac{\partial}{\partial x} S_2\right)^2 \right] dx = 0.$$
- The previous equation implies that $\frac{\partial}{\partial x} S_1 = \frac{\partial}{\partial x} S_2$.
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- Since $\frac{\partial^3}{\partial x^3}(S_1 - S_2) = 0$, we know that $S_1 - S_2$ is a polynomial of degree 2.
- If $S_1 - S_2 = 0$ on at least 6 points, then it is not on a conic section and must equal 0. Then $S_1 = S_2$.
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- The previous equation implies that $\frac{\partial}{\partial x} S_1 = \frac{\partial}{\partial x} S_2$.
- We can see that $\frac{\partial^3}{\partial x^3}(S_1 - S_2) = 0$, $\frac{\partial^3}{\partial xy^2}(S_1 - S_2) = 0$, and $\frac{\partial^3}{\partial y^3}(S_1 - S_2) = 0$ are similar cases.
- Since $\frac{\partial^3}{\partial x^3}(S_1 - S_2) = 0$, we know that $S_1 - S_2$ is a polynomial of degree 2.
- If $S_1 - S_2 = 0$ on at least 6 points, then it is not on a conic section and must equal 0. Then $S_1 = S_2$.
- Therefore the solution is unique.

The Difference

- The implementation the the Minimal Surface Area Method uses a different triangulation method from the other data fitting methods.
- This triangulation method is unique, because the interior points are discarded leaving only the exterior points.
- The following process is then used to determine the proper triangulation.

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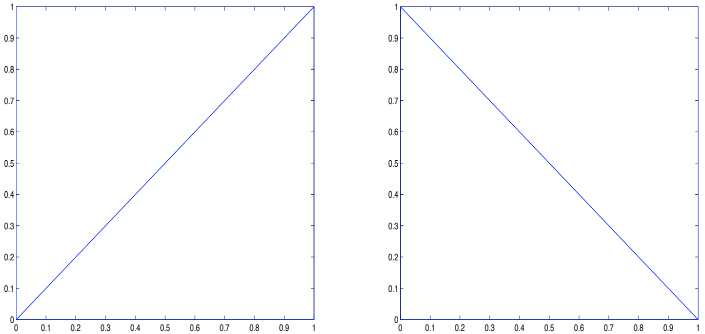
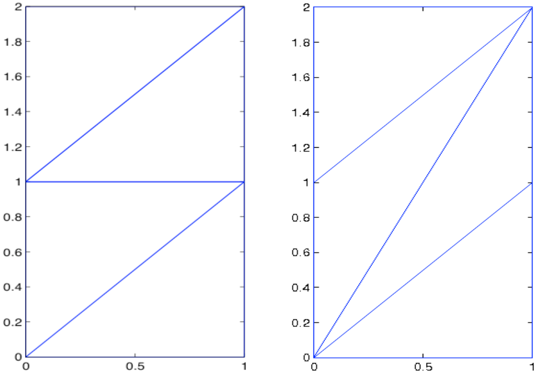
- First, the area for each of the 2 types of basic triangulations, as shown below, is calculated and compared. The triangulation with the smaller surface area is then saved in the set of triangles.



The Difference

- Second, if there are two adjacent triangulations of the same type, the area of those triangles and of the third type are compared, and the smaller one saved in the set of triangles.



Overview

- The Roughness function is defined by the equation $R(f) = \sum_{t_i \in \triangle} \int_{t_i} \left[\left(\frac{\delta}{\delta x} f \right)^2 + \left(\frac{\delta}{\delta y} f \right)^2 \right] dxdy$ (c.f [2]).
- Let $\Lambda(f) = \{s \in S_d^r(\Lambda), s(x_i, y_i) = f, i = 1, \ldots, N\}$.
- Find $S_f \in \Lambda(f)$ such that $R(S_f) = min\{R(s), s \in \Lambda(f)\}$
- This method is similar to the other methods discussed; however, when it was implemented, the numerical results were significantly worse compared to the results from the other methods.

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Summary

- The three new methods researched were Minimal Triharmonic Method, Minimal Surface Area Method, and Minimal Roughness Method.
- Each of these methods and the Minimal Energy Method were used for fitting data points from the a toy car.
- The contour maps from each piece of the vehicle display the smoothness of the function fitted to the data.

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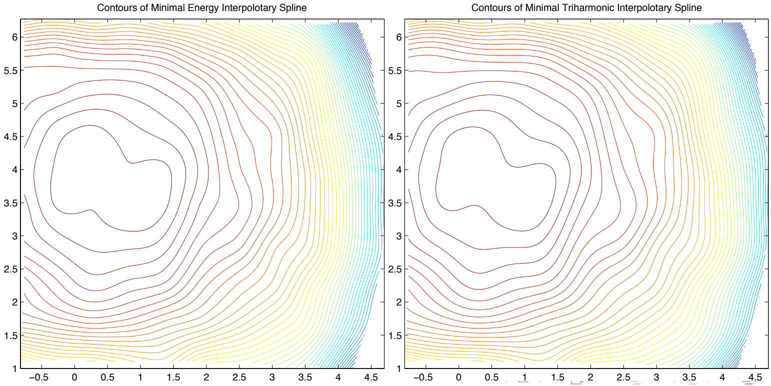
Summary

- Shown below is the Minimal Energy Method and Minimal Triharmonic Method, Minimal Surface Area Method and Minimal Roughness Method are show on the next slide.



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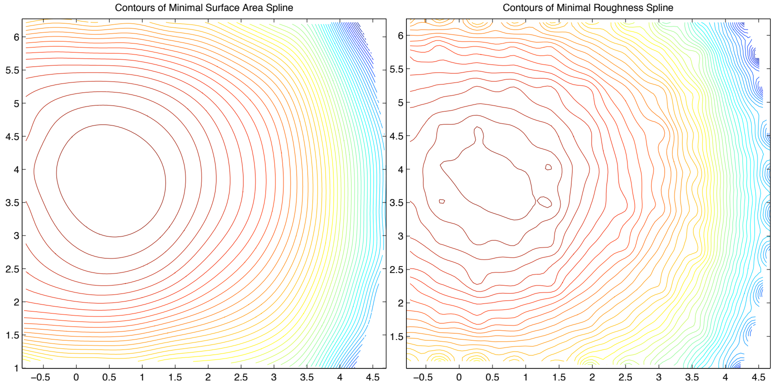

- It is easy to see that Minimal Surface Area is the best method.

References

[1] Lai, M. J., Multivariate splines for data fitting and approximation, Approximation Theory XII, San Antonio, 2007, Edited by M. Neamtu and L. L. Schumaker, Nashboro Press, Brentwood, TN., 210–228.
[2] S. Rippa, Minimal roughness property of the Delaunay triangulation, Comput. Aided Geom. Design 7 (1990) 489-497.