Convergence of the tri-harmonic spline method

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The Tri Harmonic Spline

- Given the data set $\{(x_i, y_i, f_i), i = 1, \ldots, V\}$, with $f_i = f(x_i, y_i)$, we consider the spline space

$$S_d^r(\triangle) = \{s \in C^r(\Omega) : s|_t \in \mathbb{P}_d, \forall t \in \triangle\}$$

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- d = degree of the spline space
  r = smoothness (number of times differentiable)
  $\triangle$ = a triangulation of the data sites $(x_i, y_i), i = 1, \ldots, V$
  Ω = the union of all triangles in $\triangle$
  $\mathbb{P}_d$ = the space of all polynomials of degree $\leq d$.

The Tri Harmonic Spline

we are looking for the spline $Sf \in S_d^r(\triangle)$ such that $Sf(x_i, y_i) = f_i, i = 1, \ldots, V$, and

$$H(Sf) = \text{min}\{H(s), s \in S_d^r(\triangle)\},$$

where

$$H(s) = \sum_{T \in \triangle} \int_T \left((D_x^3 s)^2 + 3(D_x^2 D_y s)^2 + 3(D_x D_y^2 s)^2 + (D_y^3 s)^2 \right) dxdy$$

Purpose of Proving Convergence

We want to show that Sf will converge to the data function f as the number of data sites increases.

The Convergence Theorem

Let S_f be the spline interpolating f at the vertices of $\triangle$. Suppose that $f \in C^3(\Omega)$. Then there exists a constant C dependent on d and $\theta_\triangle$ as well as the Lipschitz constant associated with the boundary $\partial\Omega$ if Ω is not convex such that

$$\|f - Sf\|_{L_2(\Omega)} \leq C|\triangle|^3 |f|_{3,\infty,\Omega}.$$

Lemmas

Lemma 1:

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$$\|D_x^\alpha D_y^\beta (f - Qf)\|_{q,T} \leq K|T|^{m+1-\alpha-\beta} |f|_{m+1,q,\Omega_T},$$

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for all $0 \leq \alpha + \beta \leq m$.

Lemma 2: Suppose that g is continuously three times differentiable over a triangle T. Suppose that g is zero at six vertices in Star(T) which do not lie on a conic section. Then

$$\|g\|_{L_\infty(T)} \leq C_1|T|^3|g|_{3,\infty,T}$$

for a positive constant C_1 independent of g and T.

Lemmas

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Lemma 3: Let T be a triangle and let A_T be its area. Then for all $p \in \mathbb{P}_d$ and all $1 \leq q \leq \infty$,

$$|p|_T \leq K A_T^{-1/q} |p|_{q,T}$$

If we pick $q = 2, K = C_2$, and $p = Sf'''$, we get

$$|Sf|_{3,\infty,T} \leq \frac{C_2}{\sqrt{A_T}} |Sf|_{3,2,T}$$

where

$$|Sf|_{3,2,T} := \sqrt{\int_T \left((D_x^3 Sf)^2 + 3(D_x^2 D_y Sf)^2 + 3(D_x D_y^2 Sf)^2 + (D_y^3 Sf)^2\right) dxdy}.$$

Proof of Convergence

Since, by definition, $Sf - f = 0$ at the vertices of T, we can apply Lemma 2 and get

$$|Sf - f| \leq C_1 |T|^3 |S_f - f|_{3,\infty,T}.$$

Also note that

$$H(Sf) = \sum_{T \in \triangle} |Sf|^2_{3,2,T}.$$

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By Lemma 1, with $m = 1$, $p = \infty$, and $|\alpha| = 2$,

$$H(Qf) = |Qf|^2_{3,2,\Omega} \leq C_3 A_\Omega |f|^2_{3,\infty,\Omega}$$

Proof of Convergence

- Therefore,

$$\int_\Omega |Sf - f|^2 dxdy \leq 2C_1|\triangle|^6 \left(A_\Omega |f|^2_{3,\infty,\Omega} + C_3 A_\Omega |f|^2_{3,\infty,\Omega})\right)$$

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- Therefore,

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- This implies

$$\sqrt{\int_\Omega |Sf - f|^2 dxdy} \leq C_4 A_\Omega |\triangle|^3 |f|_{3,\infty,\Omega}.$$