A Class of Compactly Supported Orthonormal B-Spline Wavelets

Okkyung Cho and Ming-Jun Lai

Abstract. We continue the study of constructing compactly supported orthonormal B-spline wavelets originated by T.N.T. Goodman. We simplify his constructive steps for compactly supported orthonormal scaling functions and provide an inductive method for constructing compactly supported orthonormal wavelets. Three examples of compactly supported orthonormal B-spline wavelets are included for demonstrating our constructive procedure.

§1. Introduction

After the seminal construction of compactly supported orthonormal univariate wavelets (cf. [3]), there have been many attempts to construct compactly supported orthonormal wavelets using B-spline functions due to the facts that B-splines have nice refinement properties and explicit representations. Three major research works along this direction are worthy mentioning. In [1] and [2], two kinds of semi-orthonormal B-spline wavelets were constructed. One of them is compactly supported although the orthonormality among the translates is lost. In [5] the researchers initiated a fractal functional approach to construct compactly supported orthonormal wavelets from B-spline functions. Examples of C^0 and C^1 compactly supported B-spline wavelets were constructed. In [4], the researchers used the interwining multiresolution analysis to show the existence of compactly supported orthonormal B-spline wavelets using multi-wavelet technique. In [6], the researchers used orthogonal polynomials to construct compactly supported smooth wavelets. An example of C^2 multiwavelets was shown. Furthermore, in [7], the researchers extended the interwinding multiresoluton analysis to the bivariate setting. Examples of compactly supported continuous piecewise linear spline wavelets are given. These approaches have an obvious difficulity that the number of wavelets is dependent on the

Splines and Wavelets: Athens 2005
Guanrong Chen and Ming-Jun Lai Editors pp. 123–151.

ISBN 0-0-9728482-6-6

size of the support of the scaling functions. Recently, another approach of multi-wavelets was given in [9] where Goodman showed how to construct compactly supported scaling functions using B-splines of any degree and indicated how to construct associated wavelets. One of the advantages of the approach in [9] is that the number of wavelets is always 3 for B-splines of any degree. Although the construction of orthonormal scaling functions is clearly described, a constructive method of wavelets was given without any supporting examples. One of reasons is that the construction is dependent on the factorization of positive definite matrices. The techniques in [11] were used to factorize nonnegative Laurent polynomial matrices. They require a lot of manual computation.

Followed the Goodman approach, we worked through his steps and found out that the computation of orthonormal scaling functions can be simplified by introducing a numerical approximation method of factorization of Laurent polynomial matrices and a new inductive method of constructing wavelets is given so that whole constructive procedure becomes much simpler. The purpose of this paper is to describe this new and simpler constructive procedure. One of our aims is to make these compactly supported orthonormal B-spline wavelets to become available to wavelet analysists as well as general wavelet practicianers.

The paper is organized as follows. We first describe a general procedure in the preliminary section below. The procedure is similar to the one given in [9]. Then we explain how to factorize Laurent polynomial matrices by using a symbol approximation method similar to the one in [12] The convergence analysis of the method in the setting of Laurent polynomial matrices is given in [8]. This consists of Section 2. In Section 3, an inductive method for constructing compactly supported B-spline wavelets is introduced. In §4, we summarize the computational steps and present three examples of compactly supported B-spline wavelets to illustrate the computation procedure.

§2. Preliminaries

Fix integers $r > 1$ and $d \geq 1$. Let $\phi_1, \cdots, \phi_r$ be compactly supported continuous real-valued functions in $\mathbb{R}^d$ and

$$\Phi = (\phi_1, \cdots, \phi_r)^T.$$

We suppose that Φ is refinable. That is, there exist matrices A_k's of size $r \times r$ such that

$$\Phi(x) = \sum_{k \in \mathbb{Z}^d} A_k \Phi(2x - k), \quad x \in \mathbb{R}^d.$$

Also, we say Φ is orthonormal if

$$\int_{\mathbb{R}^d} \phi_i(x)\phi_j(x-k)dx = \begin{cases} 1, & \text{if } i=j \text{ and } k=0, \\ 0, & \text{otherwise} \end{cases}$$

for all $i, j = 1, \cdots, r$. Φ generates a space $\mathcal{S}$ if $\mathcal{S}$ consists of all finitely linear combination of integer translates of entries of Φ.

Next we define a Grammian matrix $G = (G_{ij})_{i,j=1,\cdots,r}$ of size $r \times r$ associated with Φ by

$$G_{ij}(z) = \sum_{k\in\mathbb{Z}^d} z^k \int_{\mathbb{R}^d} \phi_i(x)\phi_j(x-k)dx$$

for all $i, j = 1, \cdots, r$ with $z \in \mathbb{C} \setminus \{0\}$. We note that Φ is orthonormal if and only if its Grammian matrix G is the identity.

We suppose that Φ generates a space $\mathcal{S}$. Then for any compactly supported functions $\psi_1, \cdots, \psi_s$ in $\mathcal{S}$, there exists a finitely many nonzero matrices C_k of size $s \times r$ such that

$$\Psi(x) = (\psi_1(x), \cdots, \psi_s(x))^T = \sum_{k\in\mathbb{Z}^d} C_k\Phi(x-k).$$

In terms of Fourier transform, we have

$$\widehat{\Psi}(\omega) = C(z)\widehat{\Phi}(\omega)$$

where $C(z)$ denotes the $s \times r$ matrix of Laurent polynomials, i.e.,

$$C(z) := \sum_{k\in\mathbb{Z}^d} C_k z^k.$$

A square matrix $C(z)$ is said to be invertible if $\det(C(z))$ is a monomial of z, e.g., αz^m for a scalar $\alpha \neq 0$ and an integer $m \in \mathbb{Z}$. It is clear that if $C(z)$ is invertible, Ψ generates the same $\mathcal{S}$. A proof of the following result can be found in literature (cf. e.g., [9]).

Lemma 1. *Fix $d = 1$. Suppose that Ψ is compactly supported and generates a space $\mathcal{S}$. Let $\mathcal{G}(z) = (G_{ij}(z))_{i,j=1,\cdots,r}$ of size $r \times r$ by*

$$G_{ij}(z) = \sum_{k\in\mathbb{Z}} z^k \int_{\mathbb{R}} \psi_i(x)\psi_j(x-k)dx$$

for all $i, j = 1, \cdots, r$ be the Grammian matrix associated with Ψ. If the determinant of the Grammian matrix $\mathcal{G}(z)$ is a nonzero constant, then there exists a Φ which is orthonormal and generates $\mathcal{S}$. The converse is also true.

The above lemma reveals a key for constructing orthonormal vector of scaling functions: find $\psi_1, \cdots, \psi_r$ which generate a space $\mathcal{S}$ such that its Grammian matrix has a constant determinant.

We now follow the steps in [9] to use B-splines for constructing an orthonormal vector of scaling functions with $r = 3$. Let N_m be the normalized B-spline of order m, in terms of Fourier transform,

$$\widehat{N}_m(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^m$$

Let $V_0 = \text{span}\{N_m(x-k), k \in \mathbb{Z}\}$ be the spline space. Since N_m is a refinable function, for V_1 being spanned by the integer translates of $N_m(2x-k), k \in \mathbb{Z}$, we have $V_0 \subset V_1$. Thus, letting $\psi_1(x) = N_m(2x)$ and $\psi_2(x) = N_m(2x-1)$, ψ_1 and ψ_2 generate V_1. On the other hand, by the dilation equation, there exist two finite sequences a_{2k} and a_{2k+1} such that

$$N_m(x) = \sum_{k\in\mathbb{Z}} a_{2k}\psi_1(x-k) + \sum_{k\in\mathbb{Z}} a_{2k+1}\psi_2(x-k).$$

Note that the Fourier transform of the above equation is

$$\widehat{N}_m(2\omega) = \frac{1}{2}A(z)\widehat{N}_m(\omega)$$

and

$$\begin{aligned}\widehat{N}_m(\omega) =& A_0(z)\widehat{\psi}_1(\omega) + A_1(z)\widehat{\psi}_2(\omega)\\ =& A_0(z)\frac{1}{2}\widehat{N}_m(\frac{\omega}{2}) + A_1(z)\frac{1}{2}z^{\frac{1}{2}}\widehat{N}_m(\frac{\omega}{2})\end{aligned}$$

where

$$A_0(z) = \sum_{k\in\mathbb{Z}} a_{2k}z^k \text{ and } A_1(z) = \sum_{k\in\mathbb{Z}} a_{2k+1}z^k.$$

It follows that

$$A(z) = A_0(z^2) + zA_1(z^2)$$

It is known that $A(z) = 2\left(\dfrac{1+z}{2}\right)^m$. Note that the proof of the following lemma is constructive.

Lemma 2. *There exist two Laurent polynomials $B_0(z)$ and $B_1(z)$ of degree $\leq m$ such that*

$$A_0(z)B_0(z) + A_1(z)B_1(z) = 1.$$

Proof: Recall

$$\begin{aligned}1 =& \left(\frac{1+z}{2}+\frac{1-z}{2}\right)^{2m-1}\\ =& \sum_{j=0}^{m-1}\binom{2n-1}{j}\left(\frac{1+z}{2}\right)^{2m-1-j}\left(\frac{1-z}{2}\right)^{j}\\ &+\sum_{j=0}^{m-1}\binom{2m-1}{j}\left(\frac{1-z}{2}\right)^{2m-1-j}\left(\frac{1+z}{2}\right)^{j}\\ =:& \ell(z)A(z)+\ell(-z)A(-z)\end{aligned}$$

by using binomial expansion, where $\ell(z)$ is a polynomial of degree $\leq m-1$. Thus, we have

$$\begin{aligned}1 =&(A_0(z^2)+zA_1(z^2))\ell(z)+(A_0(z^2)-zA_1(z^2))\ell(-z)\\ =&A_0(z^2)(\ell(z)+\ell(-z))+A_1(z^2)z(\ell(z)-\ell(-z)).\end{aligned}$$

That is, $B_0(z^2)=\ell(z)+\ell(-z)$ while $B_1(z^2)=z(\ell(z)-\ell(-z))$. $\square$

We now define a new spline function in terms of Fourier transform by

$$\widehat{M}_m(\omega)=-B_1(z)\widehat{\psi}_1(\omega)+B_0(z)\widehat{\psi}_2(\omega). \tag{1.1}$$

Recall that

$$\widehat{N}_m(\omega)=A_0(z)\widehat{\psi}_1(\omega)+A_1(z)\widehat{\psi}_2(\omega)$$

It follows that N_m and M_m generate V_1 since the determinant of the following matrix

$$\begin{bmatrix}\widehat{N}_m(\omega)\\ \widehat{M}_m(\omega)\end{bmatrix}=\begin{bmatrix}A_0(z) & A_1(z)\\ -B_1(z) & B_0(z)\end{bmatrix}\begin{bmatrix}\widehat{\psi}_1(\omega)\\ \widehat{\psi}_2(\omega)\end{bmatrix} \tag{1.2}$$

is constant 1. Furthermore, $N_m(2x), N_m(2x-1), M_m(2x), M_m(2x-1)$ generate V_2.

Define $\psi_3=\sum_{k\in\mathbb{Z}}\alpha_k M_m(2x-k)$ for some finitely many nonzero coefficients α_k. We will show how to find such α_k that the Grammian matrix associated with $\{\psi_1,\psi_2,\psi_3\}$, $\mathcal{G}(z)=\left(\sum_{k\in\mathbb{Z}}z^k\int_{\mathbb{R}}\psi_i(x)\psi_j(x-k)dx\right)_{i,j=1,2,3}$ has a constant determinant. Put

$$r(z)=\sum_{k\in\mathbb{Z}}\alpha_k z^k.$$

The computation in [8] shows

$$4\det\mathcal{G}(z^2)=D(z)r(z)r(1/z)+D(-z)r(-z)r(-1/z), \tag{1.3}$$

where

$$D(z) = \frac{1}{2}(a(z^2) - zb(z^2))\,(a(z)^2 - zb(z)^2), \tag{1.4}$$

with

$$a(z) = \sum_{k\in\mathbb{Z}} z^k \int N_m(2x)N_m(2x-k)dx,$$
$$b(z) = \sum_{k\in\mathbb{Z}} z^k \int N_m(2x)N_m(2x-2k-1)dx.$$

Now we claim that there exists a polynomal $p(z) \geq 0$ such that

$$D(z)p(z) + D(-z)p(-z) = 1. \tag{1.5}$$

Once we have such a $p(z)$, it follows from the Riesz-Féjer lemma that there exists a polynomial $r(z)$ such that $r(z)r(1/z) = p(z)$. This $r(z)$ is the polynomial we look for such that the determinant (1.3) of Grammian matrix $\mathcal{G}(z)$ is a nonzero constant.

To prove the claim (1.5), we need the following lemma (see a constructive proof in [10]).

Lemma 3. *Let p be a polynomal of degree n with all its zeros in $[1,\infty)$ having a positive leading coefficient. Then there exists a unique polynomial q with real coefficients of degree $n-1$ such that*

$$p(x)q(x) + p(1-x)q(1-x) = 1.$$

for $x \in [0,1]$. Moreover, $(-1)^n q(x) > 0$ for $x \in (0,1)$.

To use the lemma above, we need to examine the zeros of $D(z) = \frac{1}{2}(a(z^2) - zb(z^2))(a(z)^2 - zb(z)^2)$. Let us simplify $a(z^2) - zb(z^2)$ a little bit more.

$$\begin{aligned}
a(z^2) - zb(z^2) =& \sum_{j\in\mathbb{Z}} z^{-2j} \int_{\mathbb{R}} N_m(2x)N_m(2x-2j)dx \\
&+ \sum_{j\in\mathbb{Z}} z^{-(2j+1)} \int_{\mathbb{R}} N_m(2x)N_m(2x-2j-1)dx \\
=& \frac{1}{2} \sum_{-j\in\mathbb{Z}} z^j \int_{\mathbb{R}} N_m(x)N_m(x-j)dx \\
=& \frac{1}{2} \sum_{j\in\mathbb{Z}} z^j \int_{\mathbb{R}} N_m(x)N_m(x+j)dx \\
=& \frac{1}{2} \sum_{j\in\mathbb{Z}} z^j \int_{\mathbb{R}} N_m(x)N_m(m-j-x)dx \\
=& \frac{1}{2} \sum_{j\in\mathbb{Z}} N_{2m}(m-j)z^j = \frac{1}{2} \sum_{j\in\mathbb{Z}} N_{2m}(j)z^{m-j}
\end{aligned}$$

where we have used the symmetric property of B-spline functions, i.e., $N_m(x) = N_m(m-x)$. It is well-known that $E_{2m}(z) := \sum_{j\in\mathbb{Z}} N_{2m}(j)z^j$ is an Euler-Frobinus polynomial which is never zero for $e^{-i\omega}$ for any ω. The zeros of $E_{2m}(z)$ are in $(-\infty, 0)$ since all coefficients of $E_{2m}(z)$ are positive. By the following Lemma 4, $E_{2m}(z)$ can be written in terms of $p(x)$ with $x = \sin^2(\omega/2)$ and $p(x)$ has only zeros in $[1, +\infty)$. Next we consider $a(z)^2 - zb(z)^2$. As above,

$$
\begin{aligned}
a(z) =&\sum_{j\in\mathbb{Z}} z^j \int_{\mathbb{R}} N_m(2x)N_m(2x-2j)dx = \sum_{j\in\mathbb{Z}} \frac{1}{2} N_{2m}(m+2j)z^j \\
=&\frac{1}{2}(N_{2m}(m) + \sum_{j=1}^{[m/2]} N_{2m}(m+2j)(z^j + 1/z^j))
\end{aligned}
$$

is a real polynomial in $\cos(\omega)$ which can be converted to a polynomial in terms of $x = \sin^2(\omega/2)$. So is $a(z)^2$. Similarly,

$$
\begin{aligned}
b(z) =&\sum_{j\in\mathbb{Z}} z^j \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} N_m(2x)N_m(2x-2j-1)dx \\
=&\frac{1}{2}\sum_{j\in\mathbb{Z}} N_{2m}(m+2j+1)z^j \\
=&\frac{z^{-1/2}}{2} \sum_{j=-[m/2]}^{[m/2]-1} N_{2m}(m+2j+1)z^{(2j+1)/2} \\
=&\frac{1}{2z^{1/2}}(N_{2m}(m+1)z^{1/2} + N_{2m}(m-1)z^{-1/2} \\
&+ N_{2m}(m+3)z^{3/2} + N_{2m}(m-3)z^{-3/2} + \cdots) \\
=&\frac{1}{2z^{1/2}} \sum_{j=0}^{[m/2]-1} N_{2m}(m+2j+1)(z^{(2j+1)/2} + z^{-(2j+1)/2}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
zb(z)^2 =&\frac{1}{4}\left(\sum_{j=0}^{[m/2]-1} N_{2m}(m+2j+1)(z^{(2j+1)/2} + z^{-(2j+1)/2})\right)^2 \\
=& \sum_{i,j=0}^{[m/2]-1} N_{2m}(m+2j+1)N_{2m}(m+2i+1)\times \\
&\frac{z^{i+j+1} + z^{-(i+j+1)} + z^{i-j} + z^{j-i}}{4}
\end{aligned}
$$

is again a real polynomial in $\cos(\omega)$ which can be converted to a polynomial in terms of $x = \sin^2(\omega/2)$ by Lemma 4 below. The zeros of $a(z)^2 - zb(z)^2$

are contained in the zeros of $a(z)$ which are located in $[1,+\infty)$. Therefore, Lemma 3 implies that a polynomial $p(z)$ exists such that

$$D(z)p(z) + D(-z)p(-z) = 1$$

and $p(z) > 0$. This completes the proof of our claim.

Lemma 4. *Let*

$$c(z) = \sum_{-m}^{m} c_j z^j$$

be a polynomial which has only zeros in $(-\infty, 0)$ *with real coefficients* c_j *and* $c_j = c_{-j}$. *Then there is a polynomal* p *of degree* m *such that*

$$p(x) = c(e^{-i\omega}), \text{ with } x = \sin^2(\omega/2)$$

and p *has only zeros in* $[1, \infty)$.

Proof: Clearly, $c(z)$ can be written as

$$c(z) = c_0 + \sum_{j=1}^{m} 2c_j \cos(j\omega) = \sum_{j=0}^{m} d_j \left(\frac{z + 1/z}{2}\right)^j$$

for some real coefficients $d_0, \cdots, d_m$. Then we define

$$p(x) = \sum_{j=0}^{m} d_j (1 - 2x)^j$$

Then we can see that $c(z) = p(1/2 - (z + 1/z)/4) = p(\sin 2(\omega/2))$. If $p(x) = 0$ with $x = 1/2 - (z + 1/z)/4$ for $z \in (-\infty, 0)$, then $z + 1/z \leq -2$ implies that $x \geq 1$. $\square$

A major step in the computation of orthonormal scaling function vector is to factorize Grammian matrix $\mathcal{G}(z)$ which will be discussed in the following section. This finishes the construction steps for compactly supported orthonormal scaling functions based on B-splines.

§3. A Computational Method for Matrix Factorization

Let ψ_1, ψ_2, ψ_3 be three compactly supported functions defined in the previous section. Since the determinant of the Grammian matrix $\mathcal{G}(z)$ associated with $\Psi = (\psi_1, \psi_2, \psi_3)^T$ is nonzero monomial, it can be factored into $\mathcal{G}(z) = B(z)B(z)^*$ with invertible polynomial matrix $B(z)$, where $B(z)^*$ stands for the transpose and conjugate of $B(z)$. In this section we discuss a computational method for the matrix factorization. Although the method in [11] is constructive, it requires a technique to factor a positive definite Hermitian matrix into matrices with rational Laurent polynomials, a method to identify the location of poles, an expansion of the rational entries into a special format, and construction of unitary matrices to cancel these poles. It is really not an easy task. To simplify the factorization, we describe a straightforward computational method to do such factorizations.

The basic ideas are as follows. Let $\mathcal{A}$ be a bi-infinite matrix with entries $\mathcal{A}_{ij} = c_{i-j}$. where $\{c_j\}$ is a finite sequence. Let x be a bi-infinite sequence. Then $y = \mathcal{A}x$ is another bi-infinite sequence. Formally, the discrete Fourier transform of y can be given by

$$Y(\omega) = \sum_j y_j e^{-ij\omega} = A(\omega)X(\omega)$$

with $X(\omega) = \sum_k x_k e^{-ik\omega}$ and $A(\omega) = \sum_k c_k e^{-ik\omega}$. This is an identification of the bi-infinite matrix $\mathcal{A}$ and Laurent polynomial $A(\omega)$. If $A(\omega)$ is symmetric, i.e., $A(-\omega) = A(\omega)$ and positive, we know that it can be factored into a polynomial B in $e^{-i\omega}$ such that $A(\omega) = B(\omega)B(-\omega)$ by Riesz-Féjer factorization. Then the matrix $\mathcal{A}$ can be factored into a product of two matrices $\mathcal{B}\mathcal{B}^T$, where $\mathcal{B}$ is a lower-trangular bi-infinite matrix. This is indeed the case as discussed in [12]. For a positive definite matrix $M(\omega)$ of size $r \times r$, we can write it as

$$M(\omega) = \sum_k m_k e^{-ik\omega}$$

with $r \times r$ matrices m_k. For simplicity, we assume that m_k are matrices with real entries. Then we can identify $M(\omega)$ by a bi-infinite block matrix $\mathcal{M} = (M_{ij})_{-\infty<i,j<\infty}$ with $M_{ij} = m_{i-j}$. When $M(\omega)$ can be factored into a product of two polynomial matrices $N(\omega)$ and $N(-\omega)$, $\mathcal{M}$ can be factored into a product of two bi-infinite matrices $\mathcal{N}$ and $\mathcal{N}^T$. The converse is also true.

Our computational method is to compute the Cholesky decomposition of a central section $\mathcal{M}_\ell := [M_{ij}]_{1\le i,j\le \ell}$ of the bi-infinite matrix $\mathcal{M}$ with $\ell > 1$ being an integer. That is, let

$$[M_{ij}]_{1\le i,j\le \ell} = \mathcal{N}_\ell \mathcal{N}_\ell^T$$

with lower triangular matrix $\mathcal{N}_\ell = [a_{ij}]_{1\le i,j\le \ell}$ of size $r\ell \times r\ell$. Let

$$n_0 = [a_{ij}]_{r(\ell-1)+1\le i,j\le r\ell}, n_1 = [a_{ij}]_{r(\ell-1)+1\le i\le r\ell, r(\ell-2)+1\le j\le r(\ell-1)}, \cdots$$

and define $N_\ell(\omega) = \sum_{k\ge 0} n_k e^{ik\omega}$, Then we can show that $N_\ell(\omega)$ converges to $N(\omega)$ as $\ell \to +\infty$. (See [8] for a proof.)

Example 1. *Let*

$$M(\omega) = \begin{bmatrix} 8+z+1/z & 1+z \\ 1+1/z & 1 \end{bmatrix}$$

be a Hermitian and positive definite matrix. Then $M(\omega) = m_0 + m_1 z + m_{-1} 1/z$ *with three matrices* m_{-1}, m_0, m_1 *of size* 2×2*. Let* $\mathcal{M}$ *be of size of* 20×20 *with* m_0 *on the diagonal blocks* m_1 *on the upper diagonal blocks and* m_{-1} *on the sub-diagonal blocks. The remaining entries are zero. Using computer software MATLAB, we find*

$$N_\ell(\omega) = \begin{bmatrix} 2.64575131106459 & 0 \\ 0.377964473009 & 0.9258200997725 \end{bmatrix} + \begin{bmatrix} 0.3779644730092 & 0.9258200997725 \\ 0 & 0 \end{bmatrix} z$$

It can be verified by using MAPLE to see that $M(\omega) = N_\ell(\omega) N_\ell(-\omega)^T + o(1)$ *with* $o(1) = 10^{-9}$.

Our main result in this section is the following :

Theorem 1. *Let* $\mathcal{M}$ *be a bi-infinite matrix associated with a positive definite Hermitian matrix* $M(\omega)$ *and let* $\mathcal{N}_\ell$ *be the Cholesky factorization of the central section* $\mathcal{M}_\ell$*. Then the block entry* $[a_{ij}]_{r(\ell-1)+1 \le i,j \le r\ell}$ *converges to the* m_0 *exponentially faster. Similar for the other block entries.*

We refer the reader to [8] for a proof and more numerical examples.

§4. Construction of the Associated Wavelets

For the Grammian matrix $\mathcal{G}$ associated with ψ_1, ψ_2, ψ_3, let

$$\mathcal{G}(z) = B(z)^* B(z)$$

be the Riesz-Fejér factorization as discussed in the previous section. Letting

$$\widehat{\Phi}(z) = B(z)^{-1} \widehat{\Psi}(z), \tag{3.1}$$

we know that the Grammian matrix of Φ is $B(z)^{-1}\mathcal{G}(z)B^{-1}(z)^*$ which is the identity matrix and hence $\Phi = (\phi_1, \phi_2, \phi_3)^T$ is an orthonormal refinable function vector. In this section we discuss how to compute the associated wavelets. We begin with

Lemma 5. Φ *is refinable. That is, letting*

$$\widetilde{\Phi}(x) = \sqrt{2}(\phi_1(2x), \phi_2(2x), \phi_3(2x), \phi_1(2x-1), \phi_2(2x-1), \phi_3(2x-1))^T,$$

there exists matrix coefficients p_i *of size* 3×6 *such that*

$$\Phi(x) = \sum_{k \in \mathbb{Z}} p_k \widetilde{\Phi}(x-k) \text{ or } \widehat{\Phi}(\omega) = P(z)\widehat{\widetilde{\Phi}}(\omega), \tag{3.2}$$

where $P(z)$ *is a matrix mask of size* 3×6.

Proof: Indeed, since Ψ is refinable,

$$\widehat{\Psi}(2\omega) = C(z)\widehat{\Psi}(\omega) \tag{3.3}$$

for a matrix mask $C(z)$ of size 3×3. If we denote by

$$\widetilde{\Psi}(x) = (\psi_1(2x), \psi_2(2x), \psi_3(2x), \psi_1(2x-1), \psi_2(2x-1), \psi_3(2x-1))^T,$$

the dilation relation of Ψ can be rewritten in terms of $\widetilde{\Psi}$. That is, by (3.2),

$$\begin{aligned} \Psi(x) &= \sum_{k \in \mathbb{Z}} c_k \Psi(2x-k) \\ &= \sum_{k \in 2\mathbb{Z}} c_k \Psi(2x-k) + \sum_{k \in 2\mathbb{Z}} c_{k+1} \Psi(2x-k-1) \sum_{k \in \mathbb{Z}} \tilde{c}_k \widetilde{\Psi}(x-k). \end{aligned}$$

In terms of Fourier transform, $\widehat{\Psi}(z) = \tilde{C}(z)\widehat{\widetilde{\Psi}}(z)$ with matrix $\tilde{C}(z)$ of size 3×6. By (3.1), $\Phi(x) = \sum_{k \in \mathbb{Z}} b_k \Psi(x-k)$ for matrix coefficients b_k of size 3×3. It follows that

$$\begin{aligned} \Phi(2x) &= \sum_{k \in \mathbb{Z}} b_k \Psi(2x-k) \\ \Phi(2x-1) &= \sum_{k \in \mathbb{Z}} b_k \Psi(2x-k-1) \end{aligned}$$

and

$$\widetilde{\Phi}(x) = \sum_{k \in \mathbb{Z}} \tilde{b}_k \widetilde{\Psi}(x-k).$$

In terms of Fourier transform, $\widehat{\widetilde{\Phi}}(z) = \tilde{B}(z)\widehat{\widetilde{\Psi}}(z)$. Note that $\tilde{B}(z)$ is invertible because that both $\widetilde{\Phi}$ and $\widetilde{\Psi}$ generates the same space $\mathcal{S}_1$. We have

$$\widehat{\widetilde{\Psi}}(\omega) = \tilde{B}(z)^{-1}\widetilde{\Phi}(\omega).$$

Therefore, we have

$$\begin{aligned} \widehat{\Phi}(\omega) =& B(z)^{-1}\widehat{\Psi}(\omega) \\ =& B(z)^{-1}\tilde{C}(z)\widehat{\widetilde{\Psi}}(\omega) \\ =& B(z)^{-1}\tilde{C}(z)\tilde{B}(z)^{-1}\widehat{\widetilde{\Phi}}(\omega) \end{aligned}$$

which completes the proof. □

Next using the dilation relation (3.2), the orthonormality of $\phi_i, i = 1, 2, 3$ implies

$$\begin{aligned} I_{3\times 3} &= \int_{\mathbb{R}} \Phi(x)\Phi(x)^T dx \\ &= \sum_{i,j\in\mathbb{Z}} p_i \int_{\mathbb{R}} \widetilde{\Phi}(x-i)\widetilde{\Phi}^T(x-j)dx p_j^T \\ &= \sum_{i,j\in\mathbb{Z}} p_i \delta_{2i,2j} I_{6\times 6} p_j^T = \sum_{i\in\mathbb{Z}} p_i p_i^T. \end{aligned} \tag{3.4}$$

Since Φ is of compact compact, we may assume that only $m+1$ terms $p_0, p_1, \cdots, p_m$ are nonzero matrix coefficients. Furthermore,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \Phi(x)\Phi(x-k)^T dx \\ &= \sum_{i,j=1}^{m} p_i \int_{\mathbb{R}} \widetilde{\Phi}(x-i)\widetilde{\Phi}^T(x-j-k)dx p_j^T \\ &= \sum_{i,j=1}^{m} p_i \delta_{2i,2j+2k} I_{6\times 6} p_j^T = \sum_{i=k}^{m} p_i p_{i-k}^T, \end{aligned} \tag{3.5}$$

for $k = 1, \cdots, m$. In particular, we have

$$p_m p_0^T = 0 \tag{3.6}$$

We now use induction on m to show how to construct three compactly supported orthonormal wavelets $h_1, h_2, h_3 \in \mathcal{S}_1$ such that letting

$$\mathcal{W} := \text{span}\{h_1(\cdot - i), h_2(\cdot - j), h_3(\cdot - k), i, j, k \in \mathbb{Z}\},$$

$\mathcal{W}$ is the orthogonal completement of $\mathcal{S}$ in $\mathcal{S}_1$. That is, $\mathcal{S}_1 = \mathcal{S} \oplus \mathcal{W}$. More precisely, let $H = (h_1, h_2, h_3)^T$. (3.2) and the Fourier transform of H give

$$\begin{aligned} \widehat{\Phi}(\omega) =& P(z)\widehat{\widetilde{\Phi}}(\omega) \\ \widehat{H}(\omega) =& Q(z)\widehat{\widetilde{\Phi}}(\omega) \end{aligned}$$

where $Q(z) = \sum_{i\in\mathbb{Z}} q_i z^i$ is a Laurent polynomial matrix of size 3×6. The orthogonal completementness and the orthonormality of h_1, h_2, h_3 imply that the matrix

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix}$$

is a unitary matrix. That is, $Q(z)$ is an unitary extension of $P(z)$.

It is trivial when $m = 0$. Indeed, in this case, $P(z) = p_0$ is a scalar matrix. We simply choose $Q(z)$ to be a scalar matrix which is an orthonormal extension of p_0. Assume that for $m \geq 1$, when $P_m(z) = \sum_{k=0}^{m} p_k z^k$ is an orthonormal matrix of 3×6, we can find $Q_m(z)$ such that

$$\begin{bmatrix} P_m(z) \\ Q_m(z) \end{bmatrix}$$

is unitary. We now consider the case of $m+1$: $P_{m+1}(z) = \sum_{k=0}^{m+1} p_k z^k$ satisfying orthonormal properties in (3.4) and (3.5). In particular, (3.6) implies that there exists a unitary matrix U_0 of size 6×6 such that $p_0 U_0 = [0_{3\times 3}\ \tilde{p}_0^b]$ and $p_{m+1} U_0 = [\tilde{p}_{m+1}^a\ 0_{3\times 3}]$, where $\tilde{p}_0^b$ is of size 3×3 and the same for $\tilde{p}_{m+1}^a$. Writing $p_k U_0 = [\tilde{p}_k^a, \tilde{p}_k^b]$ with $\tilde{p}_k^a$ and $\tilde{p}_k^b$ being of size 3×3. Then

$$P_{m+1}(z) U_0 = [\sum_{k=1}^{m+1} \tilde{p}_k^a z^k, \sum_{k=0}^{m} \tilde{p}_k^b z^k]$$

Let

$$U_1 := \begin{bmatrix} \frac{1}{z} I_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix}.$$

Then it follows that

$$\begin{aligned} P_{m+1}(z) U_0 U_1 =& [\sum_{k=0}^{m} \tilde{p}_{k+1}^a z^k, \sum_{k=0}^{m} \tilde{p}_k^b z^k] \\ =& \sum_{k=0}^{m} [\tilde{p}_{k+1}^a, \tilde{p}_k^b] z^k. \end{aligned}$$

That is, $\tilde{P}_m(z) := P_{m+1}(z) U_0 U_1$ has only $m+1$ terms and is unitary. By induction, we can find an unitary extension $\tilde{Q}_m(z)$ such that

$$\begin{bmatrix} P_{m+1}(z) U_0 U_1 \\ \tilde{Q}_m(z) \end{bmatrix}$$

is unitary. Clearly,

$$\begin{bmatrix} P_{m+1}(z) U_0 U_1 \\ \tilde{Q}_m(z) \end{bmatrix} U_1^* U_0^* = \begin{bmatrix} P_{m+1}(z) \\ \tilde{Q}_m(z) U_1^* U_0^* \end{bmatrix}$$

is also unitary. It follows that $Q_{m+1}(z) := \tilde{Q}_m(z) U_1^* U_0^*$ is an unitary extension of $P_{m+1}(z)$. This completes the induction procedure. Therefore, we conclude the following :

Theorem 2. . *For the given refinable orthonormal functions ϕ_1, ϕ_2, ϕ_3, we can construct three associated wavelets h_1, h_2, h_3 such that $h_i(x-k)$'s are orthogonal to $\phi_j(x-m)$ for all $j = 1,2,3$ and $m \in \mathbb{Z}$, $h_i(x-k)$'s are orthonormal among each other for all $i = 1,2,3$ and $k \in \mathbb{Z}$, and the linear span of $h_i(x-k), i = 1,2,3$ and $k \in \mathbb{Z}$ forms a space $\mathcal{W}$ which is an orthogonal completement of $\mathcal{S}$ in $\mathcal{S}_1$.*

§5. Examples

In this section we want to provide a few examples based on the construction method in the previous sections. Recall that we have considered the B-spline of order m, $N_m(x)$. Thus $\psi_1(x) = N_m(2x), \psi_2(x) = N_m(2x-1)$. Now we organize our computation in the following four major steps:

Step 1. Computation of $M_m(x)$.

First we find $B_0(z), B_1(z)$ satisfying the equation as in Lemma 2. And then we define $M_m(x)$ in terms of Fourier transform according to (1.1).

Step 2. Computation of $\psi_3(x)$.

We have to begin with the computation of the determinant of $\mathcal{G}(z)$, mainly we compute $D(z)$ satisfying (1.4). Then by Lemma 3 we find a polynomial $p(z) \geq 0$ such that

$$D(z)p(z) + D(-z)p(-z) = 1.$$

A straightforward computation in [12] gives $r(z) = \sum \alpha_k z^k$ such that $p(z) = r(z)r(1/z)$ and so we get $\psi_3(x) = \sum \alpha_k M_m(2x-k)$.

Step 3. Computation of ϕ_1, ϕ_2, ϕ_3.

We need to find the entries for the Grammian matrix $\mathcal{G}(z)$ using ψ_1, ψ_2, ψ_3. Then by using the method in Section 2 we factorize into $\mathcal{G}(z) = B(z)B(z)^*$ with the help from the computer software MAPLE. Therefore we can define the orthonormal scaling vector $\Phi = (\phi_1, \phi_2, \phi_3)$ in terms of Fourier transform :

$$\widehat{\Phi}(\omega) = B(z)^{-1}\widehat{\Psi}(\omega).$$

Step 4. Computation of the associated wavelets h_1, h_2, h_3.

In this step, we follow Lemma 3.1 so that we have the dilation relation for ϕ_1, ϕ_2, ϕ_3 :

$$\widehat{\Phi}(\omega) = P(z)\widehat{\tilde{\Phi}}(\omega)$$

where $P(z) = B(z)^{-1}\tilde{C}(z)\tilde{B}(z)^{-1}$. Then by induction on m, i.e., steps in the proof of Theorem 3.2 we find the unitary extension $Q(z)$ of $P(z)$. Hence we define h_1, h_2, h_3 in Fourier transform:

$$\widehat{H}(\omega) = Q(z)\widehat{\tilde{\Phi}}(\omega),$$

where $H = (h_1, h_2, h_3)^T$.

Following the above steps, we have the orthonormal scaling functions ϕ_1, ϕ_2, ϕ_3 and the corresponding wavelet functions h_1, h_2, h_3 for $m = 2, 3, 4$ listed as follows :

5.1. m=2 : Linear B-spline case

We have the following scaling functions :

$$\begin{aligned}
\phi_1(x) =& \sqrt{3} N_2(2x), \\
\phi_2(x) =& \frac{\sqrt{165}}{11} N_2(2x) - \frac{4(2+\sqrt{5})}{\sqrt{33}} N_2(4x) + \frac{4(2-\sqrt{5})}{\sqrt{33}} N_2(4x-2), \\
\phi_3(x) =& \sum_{j=0}^{2} \alpha_j N_2(2x-j) + \sum_{k=0}^{3} \beta_k N_2(4x-2k)
\end{aligned}$$

with $\alpha_j's$ and $\beta_k's$ defined as follows:

$$\begin{aligned}
\alpha_0 &= -\frac{\sqrt{231}\left(3+2\sqrt{5}\right)}{154}, & \beta_0 &= \frac{\sqrt{231}\left(3+2\sqrt{5}\right)}{231}, \\
\alpha_1 &= \frac{\sqrt{231}}{7}, & \beta_1 &= -\frac{\sqrt{231}\left(4-\sqrt{5}\right)\left(2-\sqrt{5}\right)}{231}, \\
\alpha_2 &= -\frac{\sqrt{231}\left(3-2\sqrt{5}\right)}{154}, & \beta_2 &= -\frac{\sqrt{231}\left(13+6\sqrt{5}\right)}{231}, \\
& & \beta_3 &= \frac{\sqrt{231}\left(4+\sqrt{5}\right)\left(2-\sqrt{5}\right)}{231}.
\end{aligned}$$

The wavelet functions associated with the above scaling functions are:

$$h_1(x) = \sum_{j=0}^{3} \alpha_j \sqrt{2}\phi_1\left(2x-j\right) + \sum_{k=0}^{3} \beta_k \sqrt{2}\phi_2\left(2x-k\right) + \sum_{l=0}^{3} \gamma_l \sqrt{2}\phi_3\left(2x-l\right)$$

with $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$$\begin{aligned}
\alpha_0 &= -0.000458857008, & \beta_0 &= -0.000732123098, & \gamma_0 &= -0.004392495, \\
\alpha_1 &= 0.008233854626, & \beta_1 &= 0.004743860499, & \gamma_1 &= 0.01293269968, \\
\alpha_2 &= 0.03396060301, & \beta_2 &= 0.09061226061, & \gamma_2 &= 0.5436434207, \\
\alpha_3 &= -0.6093982729, & \beta_3 &= 0.03009628816, & \gamma_3 &= 0.5679245459.
\end{aligned}$$

$$h_2(x) = \sum_{j=0}^{3} \alpha_j \sqrt{2}\phi_1\left(2x-j\right) + \sum_{k=0}^{3} \beta_k \sqrt{2}\phi_2\left(2x-k\right) + \sum_{l=0}^{2} \gamma_l \sqrt{2}\phi_3\left(2x-l\right)$$

with $\alpha_j' s$, $\beta_k' s$ and $\gamma_l' s$ defined as follows:

$$\begin{array}{lll}
\alpha_0 = -0.01131064902, & \beta_0 = -0.01804655319, & \gamma_0 = -0.1082733148, \\
\alpha_1 = 0.2029613542, & \beta_1 = 0.1169343393, & \gamma_1 = 0.3187860801, \\
\alpha_2 = -0.0001241837662, & \beta_2 = 0.8977104423, & \gamma_2 = -0.1590117827, \\
\alpha_3 = 0.002228387187, & \beta_3 = -0.01256974349. &
\end{array}$$

$$h_3(x) = \sum_{j=0}^{3} \alpha_j \sqrt{2}\phi_1\left(2x-j\right) + \sum_{k=0}^{3} \beta_k \sqrt{2}\phi_2\left(2x-k\right) + \sum_{l=0}^{2} \gamma_l \sqrt{2}\phi_3\left(2x-l\right)$$

with $\alpha_j' s$, $\beta_k' s$ and $\gamma_l' s$ defined as follows:

$$\begin{array}{lll}
\alpha_0 = 0.0009518791574, & \beta_0 = 0.001518757925, & \gamma_0 = 0.009112042237, \\
\alpha_1 = -0.01708077780, & \beta_1 = -0.009840934872, & \gamma_1 = -0.02682833007, \\
\alpha_2 = 0.007635145592, & \beta_2 = -0.06338384300, & \gamma_2 = -0.3802819728, \\
\alpha_3 = -0.1370071236, & \beta_3 = -0.8697092234, & \gamma_3 = 0.2737702148.
\end{array}$$

The graphs for three linear B-spline scaling functions and wavelets can be seen from Fig. 4.1. And the masks associated with scaling functions ϕ_1, ϕ_2, ϕ_3 and wavelet functions h_1, h_2, h_3 are

$$P(z) = \sum_{j=0}^{1} p_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{1} q_j z^j$$

where the matrix coefficients p_0, p_1, q_0, and q_1 are listed as follows :

$$p_0 = \begin{bmatrix} 0.530330085 & 0.0940178926 & 0.564076074 & 0.530330085 & -0.33238353 & 0.0 \\ -0.84662849 & 0.0633868503 & 0.380300011 & 0.290441988 & -0.22409293 & 0.0 \\ -0.02508841 & -0.040029477 & -0.24016354 & 0.450193276 & 0.259374764 & 0.707106781 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.000275454 & -0.408778340 & 0.0473162416 & 0.0049428373 & -0.027881239 & 0.0 \end{bmatrix}$$

$$q_0 = \begin{bmatrix} -0.00045885 & -0.00073212 & -0.00439249 & 0.008233854 & 0.004743860 & 0.012932699 \\ -0.01131064 & -0.01804655 & -0.10827331 & 0.202961354 & 0.116934339 & 0.318786080 \\ 0.000951879 & 0.001518757 & 0.009112042 & -0.01708077 & -0.00984093 & -0.02682833 \end{bmatrix}$$

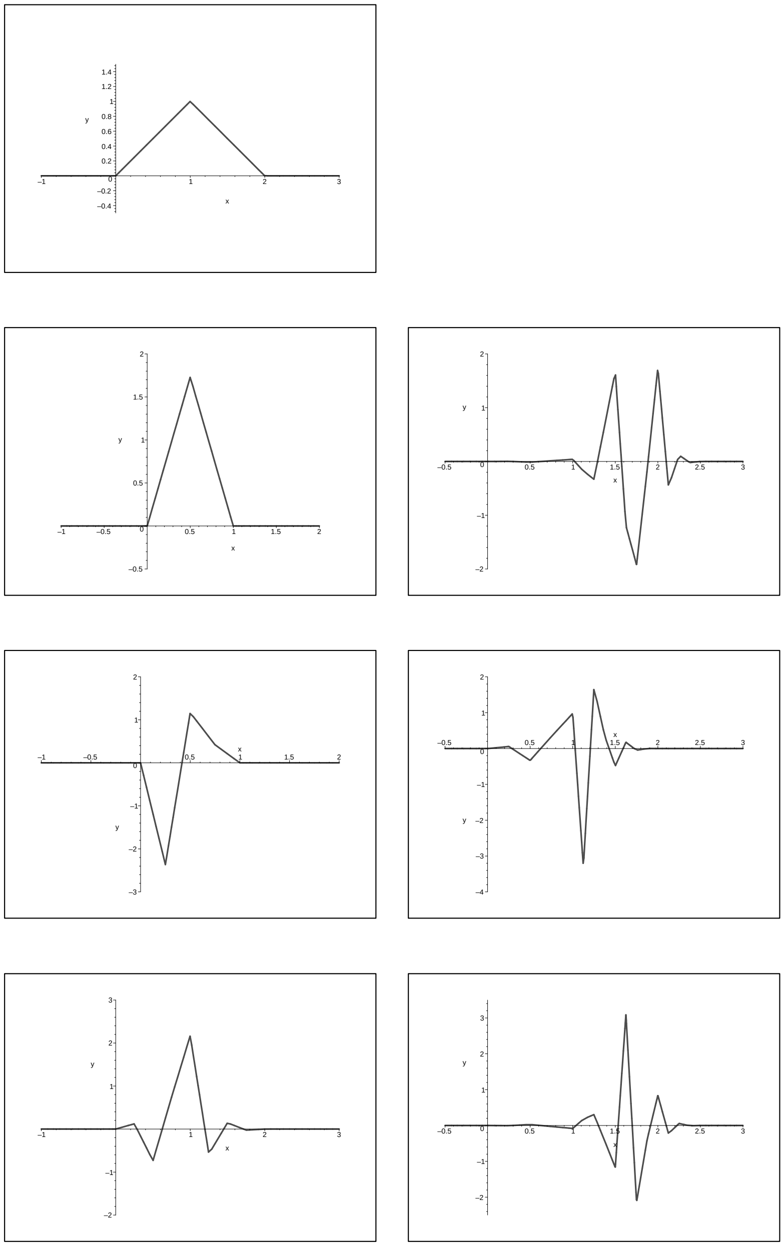

Figure 4.1. The scaling functions ϕ_1, ϕ_2, ϕ_3 on the left colume and the asociated wavelet functions ψ_1, ψ_2, ψ_3 on the right colume with the linear B-spline function $N_2(x)$ on the top.

$q_1 =$

$$\begin{bmatrix} 0.033960603 & 0.090612260 & 0.543643420 & -0.60939827 & 0.030096288 & 0.567924545 \\ -0.00012418 & 0.897710442 & -0.15901178 & 0.002228387 & -0.01256974 & 0.0 \\ 0.007635145 & -0.06338384 & -0.38028197 & -0.13700712 & -0.86970922 & 0.273770214 \end{bmatrix}.$$

5.2. m=3 : Quadratic B-spline case

Now by using the quadratic B-spline function N_3 we have the following scaling functions :

$$\phi_1(x) = \sum_{j=0}^{6} \alpha_j N_3(2x-j) + \sum_{k=4}^{15} \beta_k N_3(4x-k)$$

with $\alpha_j' s$ and $\beta_k' s$ defined as follows:

$\alpha_0 = 1.912780893,$ $\beta_4 = -0.2493006024,$ $\beta_{11} = 0.004401390839,$
$\alpha_1 = 0,$ $\beta_5 = -0.08310020075,$ $\beta_{12} = 0.000439817656,$
$\alpha_2 = 0.1423309930,$ $\beta_6 = -0.04069437823,$ $\beta_{13} = 0.000146605885,$
$\alpha_3 = -0.07936180142,$ $\beta_7 = -0.01356479274,$ $\beta_{14} = 0.000000771830,$
$\alpha_4 = -0.01603251929,$ $\beta_8 = 0.07955030605,$ $\beta_{15} = 0.000000257276,$
$\alpha_5 = -.00058359349,$ $\beta_9 = .02651676867,$
$\alpha_6 = -.00000102910,$ $\beta_1 0 = .01320417253,$

$$\phi_2(x) = \sum_{j=0}^{6} \alpha_j N_3(2x-j) + \sum_{k=4}^{15} \beta_k N_3(4x-k)$$

with $\alpha_j' s$ and $\beta_k' s$ defined as follows:

$\alpha_0 = -1.357081867,$ $\beta_4 = -4.443679090,$ $\beta_{11} = 0.007485508380,$
$\alpha_1 = 4.031609463,$ $\beta_5 = -1.481226363,$ $\beta_{12} = 0.000749281266,$
$\alpha_2 = 1.039047160,$ $\beta_6 = -0.7253602915,$ $\beta_{13} = 0.000249760422,$
$\alpha_3 = -0.1062123080,$ $\beta_7 = -0.2417867637,$ $\beta_{14} = 0.000001314905,$
$\alpha_4 = -0.0272621673,$ $\beta_8 = 0.1136754825,$ $\beta_{15} = 0.000000438301,$
$\alpha_5 = -0.0009942203,$ $\beta_9 = 0.03789182747,$
$\alpha_6 = -0.0000017532,$ $\beta_1 0 = 0.02245652515,$

$$\phi_3(x) = \sum_{j=0}^{6} \alpha_j N_3(2x-j) + \sum_{k=0}^{15} \beta_k N_3(4x-k)$$

with $\alpha_j's$ and $\beta_k's$ defined as follows:

$$\begin{array}{lll}
\alpha_0 = 2.444184380, & \beta_0 = -3.273033852, & \beta_8 = 0.005554086339, \\
\alpha_1 = -1.614417379, & \beta_1 = -1.091011283, & \beta_9 = 0.001851362113, \\
\alpha_2 = -0.3295461381, & \beta_2 = -0.5342709814, & \beta_{10} = -0.0005359258, \\
\alpha_3 = -0.0084637394, & \beta_3 = -0.1780903271, & \beta_{11} = -0.0001786419, \\
\alpha_4 = 0.00064886591, & \beta_4 = 1.625509507, & \beta_{12} = -0.0000183700, \\
\alpha_5 = 0.00002437516, & \beta_5 = 0.5418365020, & \beta_{13} = -0.0000061233, \\
\alpha_6 = 0.00000004298, & \beta_6 = 0.2682118760, & \beta_{14} = -0.0000000322, \\
 & \beta_7 = 0.0894039586, & \beta_{15} = -0.0000000107,
\end{array}$$

The wavelet functions associated with the above scaling functions are:

$$h_1(x) = \sum_{j=0}^{7} \alpha_j \sqrt{2}\phi_1(2x-j) + \sum_{k=0}^{7} \beta_k \sqrt{2}\phi_2(2x-k) + \sum_{l=0}^{7} \gamma_l \sqrt{2}\phi_3(2x-l)$$

with $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$$\begin{array}{lll}
\alpha_0 = -0.0000007843, & \beta_0 = -0.000001231, & \gamma_0 = 0, \\
\alpha_1 = 0.00002326677, & \beta_1 = 0.1959997269, & \gamma_1 = 0.00000173200, \\
\alpha_2 = 0.00030212971, & \beta_2 = 0.8933455321, & \gamma_2 = -0.0000275840, \\
\alpha_3 = 0.01245784786, & \beta_3 = -0.111372634, & \gamma_3 = -0.0012483258, \\
\alpha_4 = 0.05420980759, & \beta_4 = -0.138089893, & \gamma_4 = 0.01548791361, \\
\alpha_5 = 0.2681619223, & \beta_5 = 0.5672687664, & \gamma_5 = 0.1903509775, \\
\alpha_6 = -0.0000010976, & \beta_6 = 0.5276281207, & \gamma_6 = 0.4716892592, \\
\alpha_7 = -0.0857016717, & \beta_7 = 0.1210154680, & \gamma_7 = 0.1570383431,
\end{array}$$

$$h_2(x) = \sum_{j=0}^{7} \alpha_j \sqrt{2}\phi_1(2x-j) + \sum_{k=0}^{7} \beta_k \sqrt{2}\phi_2(2x-k) + \sum_{l=0}^{7} \gamma_l \sqrt{2}\phi_3(2x-l)$$

with $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$$\begin{array}{lll}
\alpha_0 = -0.00000018711, & \beta_0 = -0.0000002938, & \gamma_0 = 0, \\
\alpha_1 = 0.000005550499, & \beta_1 = 0.00000467575, & \gamma_1 = 0.00000041318, \\
\alpha_2 = 0.000025847814, & \beta_2 = 0.00014052160, & \gamma_2 = -0.0000065804, \\
\alpha_3 = -0.00160065404, & \beta_3 = -0.0015017279, & \gamma_3 = -0.0001957200, \\
\alpha_4 = 0.05420980759, & \beta_4 = -0.0186033464, & \gamma_4 = 0.00206905767, \\
\alpha_5 = 0.006341780457, & \beta_5 = 0.08857568915, & \gamma_5 = 0.02571026036, \\
\alpha_6 = -0.1012566025, & \beta_6 = -0.1679559187, & \gamma_6 = 0.1700264115, \\
\alpha_7 = -0.2658689347, & \beta_7 = 0.3754215397, & \gamma_7 = -0.8421656245,
\end{array}$$

$$h_3(x) = \sum_{j=0}^{7} \alpha_j \sqrt{2}\phi_1\left(2x-j\right) + \sum_{k=0}^{7} \beta_k \sqrt{2}\phi_2\left(2x-k\right) + \sum_{l=0}^{7} \gamma_l \sqrt{2}\phi_3\left(2x-l\right)$$

with $\alpha_j' s$, $\beta_k' s$ and $\gamma_l' s$ defined as follows:

$$\begin{aligned}
\alpha_0 &= 0.0000005292, & \beta_0 &= 0.00000083118, & \gamma_0 &= 0,\\
\alpha_1 &= -0.0000157007, & \beta_1 &= -0.0000132263, & \gamma_1 &= -0.0000001168,\\
\alpha_2 &= -0.0000967130, & \beta_2 &= -0.0004345512, & \gamma_2 &= 0.00001861413,\\
\alpha_3 &= 0.00522776122, & \beta_3 &= 0.00483761409, & \gamma_3 &= 0.00060574260,\\
\alpha_4 &= -0.0228524832, & \beta_4 &= 0.05751006748, & \gamma_4 &= -0.0066826235,\\
\alpha_5 &= -0.1105939898, & \beta_5 &= -0.2357048403, & \gamma_5 &= -0.0793615472,\\
\alpha_6 &= -0.0359813109, & \beta_6 &= 0.7187673745, & \gamma_6 &= -0.5033810216,\\
\alpha_7 &= -0.2209134870, & \beta_7 &= 0.3119419782, & \gamma_7 &= -0.06755834879,
\end{aligned}$$

The graphs for three quadratic B-spline scaling and wavelet functions can be seen from Fig. 4.2. And the masks associated with scaling functions ϕ_1, ϕ_2, ϕ_3 and wavelet functions h_1, h_2, h_3 are

$$P(z) = \sum_{j=0}^{3} p_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{3} q_j z^j$$

where the matrix coefficients $p_i' s$ and $q_i' s$ are listed as follows :

$$p_0 = \\ \begin{bmatrix} 0.355291343 & 0.251612976 & 0.0 & 0.802732835 & -0.05013566 & -0.36866764 \\ -0.25207248 & -0.17851464 & 0.0 & 0.179331447 & 0.565900388 & 0.261562717 \\ -0.89172203 & 0.130162169 & 0.0 & 0.327900813 & -0.19411942 & -0.10879571 \end{bmatrix}$$

$$p_1 = \\ \begin{bmatrix} 0.002088965 & -0.04561987 & -0.15879517 & 0.014677753 & -0.00790070 & 0.000162493 \\ 0.002454101 & -0.19347681 & -0.66438654 & 0.040735567 & -0.03435241 & -0.04308742 \\ 0.000361690 & 0.048939594 & 0.167388903 & 0.040735567 & 0.008818645 & 0.017613411 \end{bmatrix}$$

$$p_2 = \\ \begin{bmatrix} 0.000022246 & 0.002330481 & 0.007984577 & -0.00031666 & 0.000420592 & 0.000873075 \\ 0.000038030 & 0.003972917 & 0.014502263 & -0.00053953 & 0.000716605 & 0.001489205 \\ -0.00000098 & -0.00009842 & -0.00069899 & 0.000013248 & -0.00001759 & -0.00003720 \end{bmatrix}$$

$$p_3 = \begin{bmatrix} 0.0 & -0.000000053 & -0.000018109 & 0.0 & 0.0 & -0.000000036 \\ 0.0 & -0.000000091 & -0.000030851 & 0.0 & 0.0 & -0.000000062 \\ 0.0 & 0.0 & 0.0000007563 & 0.0 & 0.0 & 0.0 \end{bmatrix}.$$

$$q_0 =$$
$$\begin{bmatrix} -0.0000007843 & -0.000001231 & 0.0 & 0.0000232667 & 0.0000195999 & 0.0000017320 \\ -0.0000001871 & -0.000000293 & 0.0 & 0.0000055504 & 0.0000046757 & 0.0000004131 \\ 0.00000052929 & 0.0000008311 & 0.0 & -0.000015700 & -0.000013226 & -0.0000011687 \end{bmatrix}$$

$$q_1 =$$
$$\begin{bmatrix} 0.000302129 & 0.000893345 & -0.00002758 & -0.01245784 & -0.01113726 & -0.00124832 \\ 0.000025847 & 0.000140521 & -0.00000658 & -0.00160065 & -0.00150172 & -0.00019572 \\ -0.00009671 & -0.00043455 & 0.000018614 & 0.005227761 & 0.004837614 & 0.000605742 \end{bmatrix}$$

$$q_2 =$$
$$\begin{bmatrix} 0.054209807 & -0.13808989 & 0.015487913 & 0.268161922 & 0.567268766 & 0.190350977 \\ 0.006341780 & -0.01860334 & 0.002069057 & 0.053443777 & 0.088575689 & 0.025710260 \\ -0.02285248 & 0.057510067 & -0.00668262 & -0.11059398 & -0.23570484 & -0.07936154 \end{bmatrix}$$

$$q_3 =$$
$$\begin{bmatrix} -0.00000109 & 0.527628120 & 0.471689259 & -0.08570167 & 0.121015468 & 0.157038343 \\ -0.10125660 & -0.16795591 & 0.170026411 & -0.26586893 & 0.375421539 & -0.84216562 \\ -0.03598131 & 0.718767374 & -0.50338102 & -0.22091348 & 0.311941978 & -0.06755834 \end{bmatrix}.$$

5.3. m=4 : Cubic B-spline case

Finally, by using the cubic B-spline function, N_4 we have the following scaling functions :

$$\phi_1(x) = \sum_{j=0}^{8} \alpha_j N_3(2x-j) + \sum_{k=4}^{20} \beta_k N_3(4x-k)$$

with $\alpha_j's$ and $\beta_k's$ defined as follows

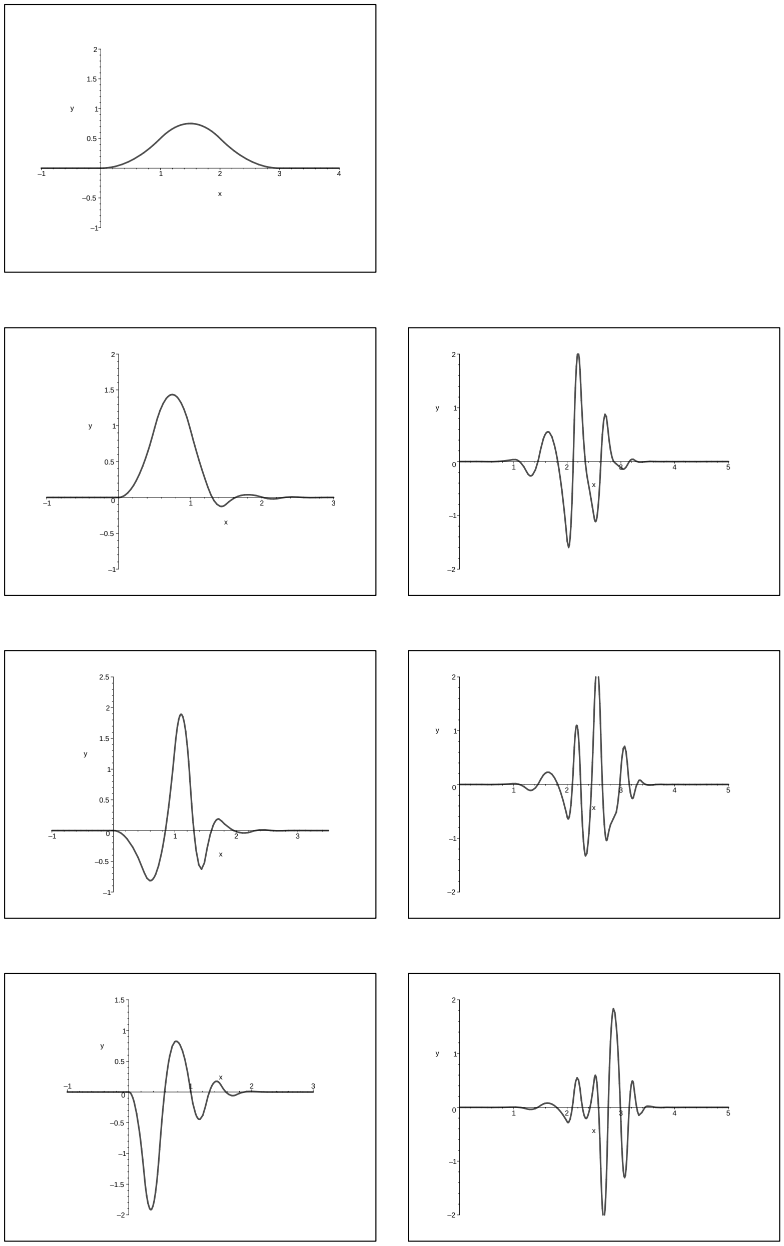

Figure 4.2. The scaling functions ϕ_1, ϕ_2, ϕ_3 on the left colume and the asociated wavelet functions ψ_1, ψ_2, ψ_3 on the right colume with the quadratic B-spline function $N_3(x)$ on the top.

$$\begin{array}{lll}
\alpha_0 = 2.047361858, & \beta_4 = -0.0057617048, & \beta_{13} = 0.007721081177, \\
\alpha_1 = 0, & \beta_5 = -0.0046093638, & \beta_{14} = 0.002152500393, \\
\alpha_2 = -0.162453281, & \beta_6 = -0.0830844980, & \beta_{15} = 0.000177784077, \\
\alpha_3 = 0.4546458529, & \beta_7 = -0.0655457256, & \beta_{16} = 0.000001457964, \\
\alpha_4 = 0.0209310834, & \beta_8 = -0.3256052473, & \beta_{17} = -0.00003439044, \\
\alpha_5 = -0.014913387, & \beta_9 = -0.2473750528, & \beta_{18} = -0.00000966735, \\
\alpha_6 = -0.000418176, & \beta_{10} = -0.0667432322, & \beta_{19} = -0.00000085579, \\
\alpha_7 = 0.0000671170, & \beta_{11} = -0.0039195752, & \beta_{20} = -0.00000022009, \\
\alpha_8 = 0.0000017017, & \beta_{12} = 0.00867145767, &
\end{array}$$

$$\phi_2(x) = \sum_{j=0}^{9} \alpha_j N_3(2x-j) + \sum_{k=4}^{20} \beta_k N_3(4x-k)$$

with $\alpha_j' s$ and $\beta_k' s$ defined as follows:

$$\begin{array}{lll}
\alpha_0 = -1.422340146, & \beta_4 = -0.0601939703, & \beta_{13} = 0.04535404437, \\
\alpha_1 = 3.133241229, & \beta_5 = -0.0481551763, & \beta_{14} = 0.01269929961, \\
\alpha_2 = -2.15509642, & \beta_6 = -0.8680045153, & \beta_{15} = 0.00108863081, \\
\alpha_3 = 4.847979095, & \beta_7 = -0.6847725771, & \beta_{16} = 0.00021138691, \\
\alpha_4 = 0.181803343, & \beta_8 = -3.402501470, & \beta_{17} = -0.0000486166, \\
\alpha_5 = -0.08830780, & \beta_9 = -2.585046661, & \beta_{18} = -0.0000138111, \\
\alpha_6 = -0.00226493, & \beta_{10} = -0.709137660, & \beta_{19} = -0.0000013255, \\
\alpha_7 = 0.000094656, & \beta_{11} = -0.050300797, & \beta_{20} = -0.0000003409, \\
\alpha_8 = 0.000002635, & \beta_{12} = 0.0441173561, & \\
\alpha_9 = 0.000000015, & &
\end{array}$$

$$\phi_3(x) = \sum_{j=0}^{8} \alpha_j N_3(2x-j) + \sum_{k=0}^{20} \beta_k N_3(4x-k)$$

with $\alpha_j's$ and $\beta_k's$ defined as follows:

$$\begin{array}{lll}
\alpha_0 = 0.3708734661, & \beta_0 = 0.1798049879, & \beta_{11} = -0.0528635547, \\
\alpha_1 = -10.86060071, & \beta_1 = 0.1438439903, & \beta_{12} = 0.03330326553, \\
\alpha_2 = -2.688916817, & \beta_2 = 2.592810217, & \beta_{13} = 0.03721532337, \\
\alpha_3 = 4.905934934, & \beta_3 = 2.045479376, & \beta_{14} = 0.01043629685, \\
\alpha_4 = 0.1735293597, & \beta_4 = 10.10617943, & \beta_{15} = 0.00090597280, \\
\alpha_5 = -0.072653067, & \beta_5 = 7.675847666, & \beta_{16} = 0.000229090023, \\
\alpha_6 = -0.001807250, & \beta_6 = 1.290416398, & \beta_{17} = 0.000002077457, \\
\alpha_7 = -0.000004300, & \beta_7 = -0.502836415, & \beta_{18} = 0.000000425529, \\
\alpha_8 = 0.0000001492, & \beta_8 = -3.377001462, & \beta_{19} = -0.00000007506, \\
 & \beta_9 = -2.601033888, & \beta_{20} = -0.00000001930, \\
 & \beta_{10} = -0.7163379153, &
\end{array}$$

The wavelet functions associated with the above scaling functions are:

$$h_1(x) = \sum_{j=1}^{9} \alpha_j \sqrt{2}\phi_1\,(2x-j) + \sum_{k=1}^{9} \beta_k \sqrt{2}\phi_2\,(2x-k) + \sum_{l=1}^{9} \gamma_l \sqrt{2}\phi_3\,(2x-l)$$

with $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$$\begin{array}{lll}
\alpha_1 = 0.0000001074, & \beta_1 = 0.000000095428, & \gamma_1 = 0, \\
\alpha_2 = 0.0000133441, & \beta_2 = 0.000025595903, & \gamma_2 = 0.000000035136, \\
\alpha_3 = -0.000471146, & \beta_3 = -0.00033211847, & \gamma_3 = 0.000009007608, \\
\alpha_4 = -0.029076217, & \beta_4 = -0.06947233055, & \gamma_4 = -0.00012341183, \\
\alpha_5 = -0.047096435, & \beta_5 = -0.4270657024, & \gamma_5 = -0.02422841891, \\
\alpha_6 = 0.2285027920, & \beta_6 = -0.6413196916, & \gamma_6 = -0.1522026344, \\
\alpha_7 = 0.1843782612, & \beta_7 = 0.1446451161, & \gamma_7 = -0.2559693458, \\
\alpha_8 = -0.054592451, & \beta_8 = 0.06058669615, & \gamma_8 = -0.4401057657, \\
\alpha_9 = 0.0042003730, & \beta_9 = -0.00452511830, & \gamma_9 = -0.04058760837,
\end{array}$$

$$h_2(x) = \sum_{j=1}^{9} \alpha_j \sqrt{2}\phi_1\,(2x-j) + \sum_{k=1}^{9} \beta_k \sqrt{2}\phi_2\,(2x-k) + \sum_{l=1}^{9} \gamma_l \sqrt{2}\phi_3\,(2x-l)$$

with $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$$\begin{aligned}
&\alpha_1 = 0.00000001897, && \beta_1 = 0.000000016780, && \gamma_1 = 0,\\
&\alpha_2 = 0.00000229941, && \beta_2 = 0.000004436742, && \gamma_2 = 0,\\
&\alpha_3 = -0.0000791584, && \beta_3 = -0.00005505550, && \gamma_3 = 0.000001560951,\\
&\alpha_4 = -0.0048883100, && \beta_4 = -0.01168005041, && \gamma_4 = -0.00002047031,\\
&\alpha_5 = -0.0081169175, && \beta_5 = -0.07200624470, && \gamma_5 = -0.00407331489,\\
&\alpha_6 = 0.02672411759, && \beta_6 = -0.1359890495, && \gamma_6 = -0.02566431523,\\
&\alpha_7 = 0.00904893415, && \beta_7 = -0.1514098512, && \gamma_7 = -0.05286310260,\\
&\alpha_8 = -0.0368356683, && \beta_8 = -0.2991751406, && \gamma_8 = 0.1580416645,\\
&\alpha_9 = 0.00057025294, && \beta_9 = -0.0006143411, && \gamma_9 = 0.9127237277,
\end{aligned}$$

$$h_3(x) = \sum_{j=1}^{9} \alpha_j \sqrt{2}\phi_1\,(2x-j) + \sum_{k=1}^{9} \beta_k \sqrt{2}\phi_2\,(2x-k) + \sum_{l=1}^{9} \gamma_l \sqrt{2}\phi_3\,(2x-l)$$

with $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$$\begin{aligned}
&\alpha_1 = 0.00000005285, && \beta_1 = 0.000000046663, && \gamma_1 = 0,\\
&\alpha_2 = 0.00000635096, && \beta_2 = 0.000012279585, && \gamma_2 = 0.000000017185,\\
&\alpha_3 = -0.0002166677, && \beta_3 = -0.00014995383, && \gamma_3 = 0.000004319856,\\
&\alpha_4 = -0.0133884492, && \beta_4 = -0.03198553984, && \gamma_4 = -0.00005576703,\\
&\alpha_5 = -0.0220461985, && \beta_5 = -0.1970118499, && \gamma_5 = -0.01115464488,\\
&\alpha_6 = 0.08401342292, && \beta_6 = -0.3463530596, && \gamma_6 = -0.07021651492,\\
&\alpha_7 = 0.04413417380, && \beta_7 = -0.2535171105, && \gamma_7 = -0.1356760027,\\
&\alpha_8 = -0.0149583912, && \beta_8 = 0.09722392396, && \gamma_8 = 0.8236228685,\\
&\alpha_9 = 0.00168765151, && \beta_9 = -0.00181812965, && \gamma_9 = -0.2340099559.
\end{aligned}$$

The graphs of the C^2 cubic spline scaling functions and wavelets can be seen from Figure 4.3. And the masks associated with scaling functions ϕ_1, ϕ_2, ϕ_3 and wavelet functions h_1, h_2, h_3 are

$$P(z) = \sum_{j=0}^{4} p_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{4} q_j z^j$$

where the matrix coefficients $p_j's$ and $q_j's$ are as follows :

$$p_0 = \begin{bmatrix} 0.248884506 & 0.231023299 & 0.0 & 0.870325778 & 0.133179107 & 0.085315812 \\ -0.17290466 & -0.16049615 & 0.0 & -0.22374359 & 0.261031401 & -0.05927047 \\ 0.129737105 & 0.074311747 & 0.0 & 0.081293784 & -0.75519742 & 0.029034934 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} 0.267627261 & -0.13235734 & 0.099604822 & -0.049091666 & 0.023326949 & -0.00424509 \\ 0.840078804 & 0.028260428 & 0.061368284 & -0.302428509 & 0.086735491 & 0.056025532 \\ 0.197299566 & 0.481935211 & -0.24100157 & -0.230490111 & 0.033888200 & 0.090700657 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 0.008911364 & -0.00787672 & 0.004840723 & -0.000456290 & 0.000701718 & -0.00078800 \\ 0.072061152 & -0.05970174 & 0.036915667 & -0.005030185 & 0.005786401 & -0.00571634 \\ 0.064631029 & -0.05275082 & 0.032638853 & -0.004778442 & 0.005212076 & -0.00499658 \end{bmatrix}$$

$$p_3 = \begin{bmatrix} -0.00006343 & 0.000051715 & -0.00003117 & 0.0000047094 & -0.00000511 & 0.0000049016 \\ -0.00011145 & 0.000090817 & -0.00004961 & 0.0000082836 & -0.00000899 & 0.0000086499 \\ -0.00002000 & 0.000016263 & -0.00000418 & 0.0000014958 & -0.00000161 & 0.0000015876 \end{bmatrix}$$

$$p_4 = \begin{bmatrix} 0.0000000138 & -0.0000000112 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0000000214 & -0.0000000174 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$q_0 = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0000001074 & 0.0000000954 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0000000189 & 0.0000000167 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0000000528 & 0.0000000466 & 0.0 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0.000013344 & 0.000025595 & 0.000000035 & -0.00047114 & -0.000332118 & 0.000009007 \\ 0.000002299 & 0.000004436 & 0.00 & -0.00007915 & -0.000055055 & 0.000001560 \\ 0.000006350 & 0.000012279 & 0.000000017 & -0.00021666 & -0.000149953 & 0.000004319 \end{bmatrix}$$

$$q_2 = - \begin{bmatrix} 0.029076217 & 0.069472330 & 0.00012341 & 0.047096435 & 0.427065702 & 0.024228418 \\ 0.004888310 & 0.011680050 & 0.00002047 & 0.008116917 & 0.072006244 & 0.004073314 \\ 0.013388449 & 0.031985539 & 0.00005576 & 0.022046198 & 0.197011849 & 0.011154644 \end{bmatrix}$$

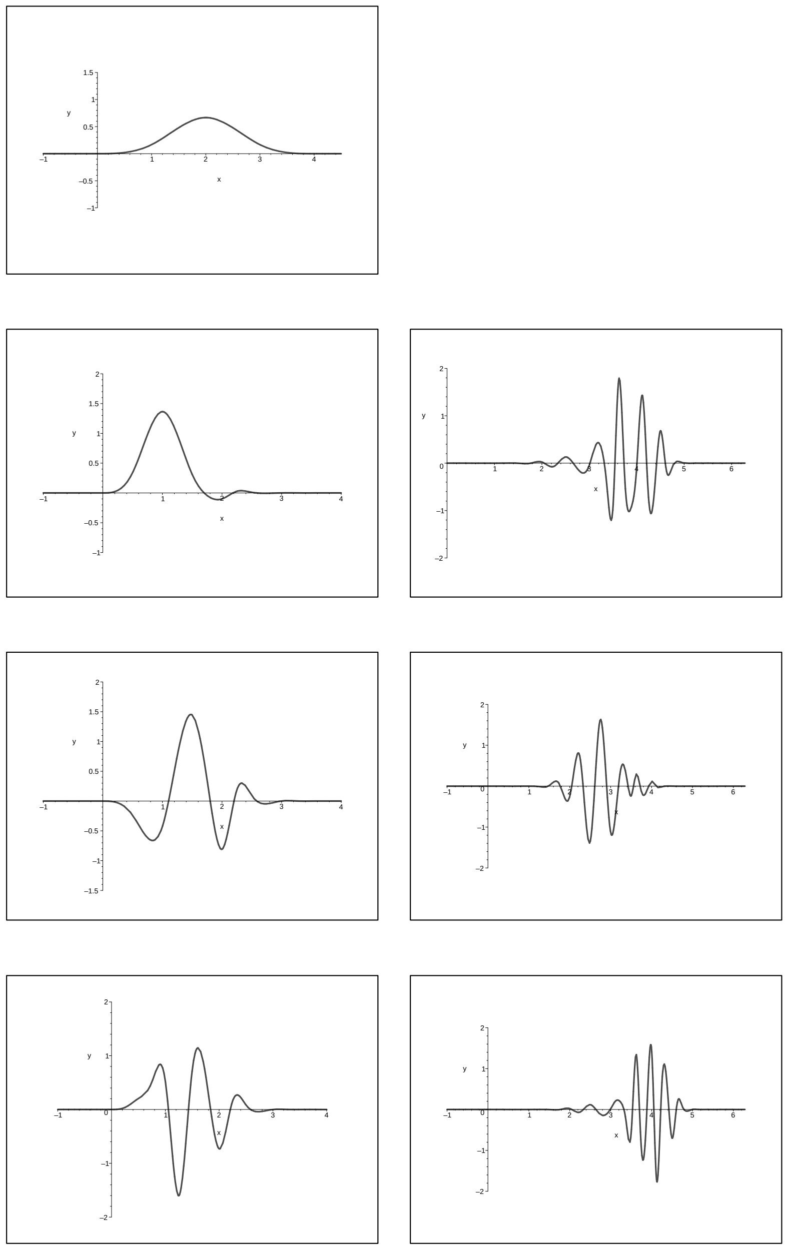

Figure 4.3. The scaling functions ϕ_1, ϕ_2, ϕ_3 on the left colume and the asociated wavelet functions ψ_1, ψ_2, ψ_3 on the right colume with the cubic B-spline function $N_4(x)$ on the top.

$q_3 =$

$$\begin{bmatrix} 0.22850279 & -0.64131969 & -0.15220263 & 0.184378261 & 0.144645116 & -0.25596934 \\ 0.02672411 & -0.13598904 & -0.02566431 & 0.009048934 & -0.15140985 & -0.05286310 \\ 0.08401342 & -0.34635305 & -0.07021651 & 0.044134173 & -0.25351711 & -0.13567600 \end{bmatrix}$$

$q_4 =$

$$\begin{bmatrix} 0.05459245 & 0.060586696 & -0.44010576 & 0.004200373 & -0.00452511 & -0.04058760 \\ -0.0368356 & -0.29917514 & 0.158041664 & 0.000570252 & -0.00061434 & 0.912723727 \\ -0.0149583 & 0.097223923 & 0.823622868 & 0.001687651 & -0.00181812 & -0.23400995 \end{bmatrix}.$$

Acknowledgement: Results in this paper are based on the research supported by the National Science Foundation under the grant No. 0327577.

References

1. C. K. Chui and J. Wang, A cardinal spline approach to wavelets, Proc. Amer. Math. Soc. 113(1991), pp. 785–793.
2. C. K. Chui and J. Wang, On compactly supported spline wavelets and a duality principle, Trans. Amer. Math. Soc. 330(1992), pp.903-916.
3. I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math., 41(1988), pp. 909–996.
4. G. C. Donovan, J. S. Geronimo, D. P. Hardin, Interwining multiresolution analyses and the construction of piecewise polynomial wavelets, SIAM J. Math. Anal., 27(1996), pp. 1791–1815.
5. G. C. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, Construction of orthogonal wavelets using fractal interpolation functions, SIAM J. Math. Anal. 27(1996), pp. 1158–1192.
6. G. C. Donovan, J. S. Geronimo, and D. P. Hardin, Orthogonal polynomials and the construction of piecewise polynomial smooth wavelets, SIAM J. Math. Anal., 30(1999), pp. 1029–1056.
7. G. C. Donovan, J. S. Geronimo, and D. P. Hardin, Compactly supported piecewise affine scaling functions on triangulation, Constr. Approx., 16(2000), 201–219.
8. J. S. Geronimo and M. J. Lai, Factorization of multivariate positive Laurent polynomials, submitted, 2005.
9. T. Goodman, A class of orthonormal refinable functions and wavelets, Constr. Approx. 19(2003), pp.525-540.

10. T. Goodman and C. A. Micchelli, Orthonormal cardinal functions, in Wavelets: Theory, Algorithms and Applications, C. K. Chui et. al. (eds.), Academic Press, San Diego, 1994, 53–88.
11. D. Hardin, A. Hogan, and Q. Sun, The matrix-valued Riesz lemma and local orthonormal bases in shift-invariant spaces, Adv. Comput. Math. 20(2004), pp.367-384.
12. M. J. Lai, On computation of Battle-Lemarie's wavelets, Math. Comp. 63(1994), pp. 689–699.

Okkyung Cho
Department of Mathematics
University of Georgia
Athens, GA 30602
`ocho@math.uga.edu`

and

Ming-Jun Lai
Department of Mathematics
University of Georgia
Athens, GA 30602
`mjlai@math.uga.edu`