A NEW PROOF OF BOURGAIN'S THEOREM ON SIMPLICES IN $\mathbb{R}^d$

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Abstract. We present a new direct proof, in fact two, of Bourgain's theorem on simplices in $\mathbb{R}^d$ in which he established that any subset of $\mathbb{R}^d$ of positive upper Banach density necessarily contains an isometric copy of all sufficiently large dilates of any fixed non-degenerate k-dimensional simplex provided $d \geq k+1$.

1. Introduction

1.1. **Background.** A result of Katznelson and Weiss [2] states that if $A \subseteq \mathbb{R}^2$ has positive upper Banach density, then its distance set

$$\mathrm{dist}(A) = \{|x-x'| \,:\, x, x' \in A\}$$

contains all large numbers. Recall that the *upper Banach density* of a measurable set $A \subseteq \mathbb{R}^d$ is defined by

$$\delta^*(A) = \lim_{N\to\infty} \sup_{t\in\mathbb{R}^d} \frac{|A\cap(t+Q_N)|}{|Q_N|}, \tag{1}$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^d$ and Q_N denotes the cube $[-N/2, N/2]^d$.

This result was later reproved using Fourier analytic techniques by Bourgain in [1] where he established the following more general result for arbitrary non-degenerate k-dimensional simplices.

Theorem 1.1 (Bourgain [1])**.** *Let $\Delta_k \subseteq \mathbb{R}^k$ be a fixed non-degenerate k-dimensional simplex.*

If $A \subseteq \mathbb{R}^d$ has positive upper Banach density and $d \geq k+1$, then there exists a threshold $\lambda_0 = \lambda_0(A, \Delta_k)$ such that A contains an isometric copy of $\lambda \cdot \Delta_k$ for all $\lambda \geq \lambda_0$.

Recall that a set $\Delta_k = \{0, v_1, \ldots, v_k\}$ of $k+1$ points in $\mathbb{R}^k$ is a non-degenerate k-dimensional simplex if the vectors $v_1, \ldots, v_k$ are linearly independent and that a configuration Δ_k' is an isometric copy of $\lambda\cdot\Delta_k$ in $\mathbb{R}^d$ if $\Delta_k' = x + \lambda \cdot U(\Delta_k)$ for some $x \in \mathbb{R}^d$ and $U \in SO(d)$ when $d \geq k+1$.

2. Uniformly Distributed Subsets of $\mathbb{R}^d$ and a New Proof of Theorem 1.1 when $k=1$

In this section we introduce a precise notion of uniform distribution for subsets of $\mathbb{R}^d$ and prove an (optimal) result, Proposition 2.1 below, on distances in uniformly distributed subsets of $[0,1]^d$. Proposition 2.1 immediately implies Theorem 1.1 when $k=1$ and hence provides a new direct proof of

Theorem 2.1 (Katznelson and Weiss [2])**.** *If $A \subseteq \mathbb{R}^d$ has positive upper Banach density and $d \geq 2$, then there exists a threshold $\lambda_0 = \lambda_0(A)$ such that for all $\lambda \geq \lambda_0$ there exist a pair of points*

$$\{x, x'\} \subseteq A \quad \textit{with} \quad |x-x'| = \lambda.$$

2.1. Uniform Distribution and Distances.

Definition 2.1 ((ε, L)-uniform distribution)**.** Let $0 < L \leq \varepsilon \ll 1$ and $Q_L = [-L/2, L/2]^d$.

A set $A \subseteq [0,1]^d$ is said to be (ε, L)-uniformly distributed if

$$\int_{[0,1]^d} \left| \frac{|A\cap(t+Q_L)|}{|Q_L|} - |A| \right|^2 dt \leq \varepsilon^2. \tag{2}$$

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Proposition 2.1 (Distances in uniformly distributed sets)**.** *Let $0<\lambda\leq\varepsilon\ll 1$ and $d\geq 2$*

If $A\subseteq[0,1]^d$ is $(\varepsilon,\varepsilon^4\lambda)$-uniformly distributed with $\alpha=|A|>0$, then there exist a pair of points

$$\{x,x'\}\subseteq A \quad \text{with} \quad |x-x'|=\lambda.$$

In fact,

$$\iint 1_A(x)1_A(x-\lambda x_1)\,d\sigma(x_1)\,dx=\alpha^2+O(\varepsilon^{2/3}).$$

where σ denotes the normalized measure on the sphere $\{x\in\mathbb{R}^d\,:\,|x|=1\}$ induced by Lebesgue measure.

Before proving Proposition 2.1 we will first show that it immediately implies Theorem 2.1. To the best of our knowledge this observation, which gives a direct proof of Theorem 2.1, is new.

2.2. **Proof that Proposition 2.1 implies Theorem 2.1.** Let $\varepsilon>0$ and $A\subseteq\mathbb{R}^d$ with $\delta^*(A)>0$.

The following two facts follow immediately from the definition of upper Banach density, see (1):

(i) There exist $M_0=M_0(A,\varepsilon)$ such that for all $M\geq M_0$ and all $t\in\mathbb{R}^d$

$$\frac{|A\cap(t+Q_M)|}{|Q_M|}\leq(1+\varepsilon^4/3)\,\delta^*(A).$$

(ii) There exist arbitrarily large $N\in\mathbb{R}$ such that

$$\frac{|A\cap(t_0+Q_N)|}{|Q_N|}\geq(1-\varepsilon^4/3)\,\delta^*(A)$$

for some $t_0\in\mathbb{R}^d$.

Combining (i) and (ii) above we see that for any $\lambda\geq\varepsilon^{-4}M_0$, there exist $N\geq\varepsilon^{-4}\lambda$ and $t_0\in\mathbb{R}^d$ such that

$$\frac{|A\cap(t+Q_{\varepsilon^4\lambda})|}{|Q_{\varepsilon^4\lambda}|}\leq(1+\varepsilon^4)\frac{|A\cap(t_0+Q_N)|}{|Q_N|}$$

for all $t\in\mathbb{R}^d$. Consequently, Theorem 2.1 reduces, via a rescaling of $A\cap(t_0+Q_N)$ to a subset of $[0,1]^d$, to establishing that if $0<\lambda\leq\varepsilon\ll 1$ and $A\subseteq[0,1]^d$ is measurable with $|A|>0$ and the property that

$$\frac{|A\cap(t+Q_{\varepsilon^4\lambda})|}{|Q_{\varepsilon^4\lambda}|}\leq(1+\varepsilon^4)\,|A|$$

for all $t\in\mathbb{R}^d$, then there exist a pair of points $x,x'\in A$ such that $|x-x'|=\lambda$. Now since $A\cap(t+Q_{\varepsilon^4\lambda})$ is only supported in $[-\varepsilon^4\lambda,1+\varepsilon^4\lambda]^d$ it follows that

$$|A|=\int_{\mathbb{R}^d}\frac{|A\cap(t+Q_{\varepsilon^4\lambda})|}{|Q_{\varepsilon^4\lambda}|}\,dt=\int_{[0,1]^d}\frac{|A\cap(t+Q_{\varepsilon^4\lambda})|}{|Q_{\varepsilon^4\lambda}|}\,dt+O(\varepsilon^4|A|),$$

from which one can easily deduce that

$$\left|\left\{t\in[0,1]^d\,:\,\frac{|A\cap(t+Q_{\varepsilon^4\lambda})|}{|Q_{\varepsilon^4\lambda}|}\leq(1-\varepsilon^2)\,|A|\right\}\right|=O(\varepsilon^2)$$

and hence that A is $(\varepsilon,\varepsilon^4\lambda)$-uniformly distributed. The result therefore follows, provided $d\geq 2$. □

2.3. **Proof of Proposition 2.1.**

Definition 2.2 (Counting Function for Distances)**.** For $0<\lambda\ll 1$ and functions

$$f_0,f_1:[0,1]^d\to\mathbb{R}$$

with $d\geq 2$ we define

$$T(f_0,f_1)(\lambda)=\iint f_0(x)f_1(x-\lambda x_1)\,d\sigma(x_1)\,dx.$$

Definition 2.3 ($U^1(L)$-norm)**.** For $0<L\ll 1$ and functions $f:[0,1]^d\to\mathbb{R}$ we define

$$\|f\|^2_{U^1(L)}=\int_{[0,1]^d}\Big|\frac{1}{L^d}\int_{t+Q_L}f(x)\,dx\Big|^2dt=\int_{[0,1]^d}\left(\frac{1}{L^{2d}}\iint_{x,x'\in t+Q_L}f(x)f(x')\,dx'\,dx\right)dt$$

where $Q_L=[-L/2,L/2]^d$.

It is an easy, but important, observation that

$$\|f\|^2_{U^1(L)} = \iint f(x)f(x-x_1)\psi_L(x_1)\,dx_1\,dx + O(L) \tag{3}$$

where $\psi_L = L^{-2d}\,1_{Q_L} * 1_{Q_L}$. Note also that if $A \subseteq [0,1]^d$ with $\alpha = |A| > 0$ and we define

$$f_A := 1_A - \alpha 1_{[0,1]^d}$$

then

$$\int\limits_{[0,1]^d} \Big|\frac{1}{L^d}\int\limits_{t+Q_L} f_A(x)\,dx\Big|^2 dt = \int\limits_{[0,1]^d} \left|\frac{|A\cap(t+Q_L)|}{|Q_L|} - |A|\right|^2 dt + O(L). \tag{4}$$

Evidently the $U^1(L)$-norm is measuring the mean-square uniform distribution of A on scale L. Specifically if A is (ε, L)-uniformly distributed, then $\|f_A\|_{U^1(L)} \leq 2\varepsilon$ provided $0 < L \ll \varepsilon$.

At the heart of this short proof of Proposition 2.1 is the following "generalized von-Neumann inequality".

Lemma 2.1 (Generalized von-Neumann for Distances). *For any $c > 0$, $0 < \varepsilon, \lambda \ll \min\{1, c^{-1}\}$ and functions*

$$f_0, f_1 : [0,1]^d \to [-1,1]$$

with $d \geq 2$ we have

$$|T(f_0,f_1)(c\lambda)| \leq \prod_{j=0,1} \|f_j\|_{U^1(\varepsilon^4\lambda)} + O(c^{-1/6}\varepsilon^{2/3}).$$

Indeed, if $A \subseteq [0,1]^d$ with $d \geq 2$ and $\alpha = |A| > 0$, then Lemma 2.1 (with $c = 1$) implies

$$\big|T(1_A, 1_A)(\lambda) - T(\alpha 1_{[0,1]^d}, \alpha 1_{[0,1]^d})(\lambda)\big| \leq 3\,\|f_A\|_{U^1(\varepsilon^4\lambda)} + O(\varepsilon^{2/3})$$

for any $0 < \varepsilon, \lambda \ll 1$. Since $T(\alpha 1_{[0,1]^d}, \alpha 1_{[0,1]^d})(\lambda) = \alpha^2 + O(\lambda)$ it follows that

$$T(1_A, 1_A)(\lambda) = \alpha^2 + O(\varepsilon^{2/3})$$

provided $0 < \lambda \leq \varepsilon \ll 1$.

To finish the proof of Proposition 2.1 we are therefore left with the task of proving Lemma 2.1.

Proof of Lemma 2.1. An application of Parseval followed by Cauchy-Schwarz implies that

$$\begin{aligned} T(f_0,f_1)(c\lambda)^2 &= \Big(\iint f_0(x)f_1(x-c\lambda x_1)\,d\sigma(x_1)\,dx\Big)^2 \\ &\leq \Big(\int_{\mathbb{R}^d} |\widehat{f}_0(\xi)||\widehat{f}_1(\xi)||\widehat{\sigma}(c\lambda\xi)|\,d\xi\Big)^2 \\ &\leq \prod_{j=0,1}\int_{\mathbb{R}^d} |\widehat{f}_j(\xi)|^2|\widehat{\sigma}(c\lambda\xi)|\,d\xi \end{aligned}$$

where

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}\,d\mu(x)$$

denotes the Fourier transform of any complex-valued Borel measure $d\mu$ and $\widehat{g}(\xi)$ is the Fourier transform of the measure $d\mu = g\,dx$. Combining the basic fact (see for example [3]) that

$$|\widehat{\sigma}(\xi)| \leq \min\{1, C|\xi|^{-(d-1)/2}\}$$

with the simple observation that $|1 - \widehat{\psi}(\xi)| \leq \min\{1, C|\xi|\}$ gives

$$|\widehat{\sigma}(c\lambda\xi)| = |\widehat{\sigma}(c\lambda\xi)|\widehat{\psi}(\varepsilon^4\lambda\xi) + |\widehat{\sigma}(c\lambda\xi)|(1-\widehat{\psi}(\varepsilon^4\lambda\xi)) \leq \widehat{\psi}(\varepsilon^4\lambda\xi) + O(\min\{\varepsilon^4\lambda|\xi|, (c\lambda|\xi|)^{-1/2}\}).$$

The result now follows, since $\|f_j\|_2^2 \leq 1$,

$$\min\{\varepsilon^4\lambda|\xi|, (c\lambda|\xi|)^{-1/2}\} \leq c^{-1/3}\varepsilon^{4/3}$$

and a further application of Parseval (and appeal to (3)) reveals that

$$\int |\widehat{f}_j(\xi)|^2\widehat{\psi}(\varepsilon^4\lambda\xi)\,d\xi = \iint f_j(x)f_j(x-x_1)\psi_{\varepsilon^4\lambda}(x_1)\,dx_1\,dx = \|f_j\|^2_{U^1(\varepsilon^4\lambda)} + O(\varepsilon^4\lambda). \qquad \square$$

3. A New Proof of Theorem 1.1

In light of the reduction argument presented in Section 2.2 it is clear that in order to prove Theorem 1.1 it would suffice to establish the following result for uniformly distributed subsets of $[0,1]^d$.

Proposition 3.1 (Simplices in uniformly distributed sets)**.** *Let $\Delta_k = \{0, v_1, \dots, v_k\}$ be a fixed non-degenerate k-dimensional simplex with $c_{\Delta_k} = \min_{1\leq j\leq k} \operatorname{dist}(v_j, \operatorname{span}\{\{v_1,\dots,v_k\}\setminus v_j\})$.*

Let $0 < \lambda \leq \varepsilon \ll \min\{1, c_{\Delta_k}^{-1}\}$ and $A \subseteq [0,1]^d$ with $d \geq k+1$ and $\alpha = |A| > 0$. If A is $(\varepsilon, \varepsilon^4\lambda)$-uniformly distributed, then A contains an isometric copy of $\lambda \cdot \Delta_k$ and in fact

$$\iint 1_A(x) 1_A(x - \lambda \cdot U(v_1)) \cdots 1_A(x - \lambda \cdot U(v_k))\, d\mu(U)\, dx = \alpha^{k+1} + O_k(c_{\Delta_k}^{-1/6} \varepsilon^{2/3}) \tag{5}$$

where μ denotes the Haar measure on $SO(d)$.

Note that Proposition 2.1 is the special case of Proposition 3.1 with $k = 1$ and $v_1 = 1$.

3.1. Proof of Proposition 3.1.

Definition 3.1 (Counting Function for Simplices)**.** For any $0 < \lambda \ll 1$ and functions

$$f_0, f_1, \dots, f_k : [0,1]^d \to \mathbb{R}$$

with $d \geq k+1$ we define

$$T_{\Delta_k}(f_0, f_1, \dots, f_k)(\lambda) = \iint f_0(x) f_1(x - \lambda \cdot U(v_1)) \cdots f_k(x - \lambda \cdot U(v_k))\, d\mu(U)\, dx. \tag{6}$$

Proposition 3.1 is an immediate consequence of the following "generalized von-Neumann inequality".

Lemma 3.1 (Generalized von-Neumann for Simplices)**.** *For any $0 < \varepsilon, \lambda \ll \min\{1, c_{\Delta_k}^{-1}\}$ and functions*

$$f_0, f_1, \dots, f_k : [0,1]^d \to [-1,1]$$

$$|T_{\Delta_k}(f_0, f_1, \dots, f_k)(\lambda)| \leq \min_{j=0,1,\dots,k} \|f_j\|_{U^1(\varepsilon^4\lambda)} + O(c_{\Delta_k}^{-1/6} \varepsilon^{2/3}).$$

Indeed, if $A \subseteq [0,1]^d$ with $d \geq k+1$ and $\alpha = |A| > 0$, then Lemma 3.1 implies

$$\left|T_{\Delta_k}(1_A, \dots, 1_A)(\lambda) - T_{\Delta_k}(\alpha 1_{[0,1]^d}, \dots, \alpha 1_{[0,1]^d})(\lambda)\right| \leq (2^{k+1} - 1)\|f_A\|_{U^1(\varepsilon^4\lambda)} + O_k(c_{\Delta_k}^{-1/6} \varepsilon^{2/3})$$

for any $0 < \varepsilon, \lambda \ll \min\{1, c_{\Delta_k}^{-1}\}$. Since $T_{\Delta_k}(\alpha 1_{[0,1]^d}, \dots, \alpha 1_{[0,1]^d})(\lambda) = \alpha^{k+1} + O(\lambda)$ it follows that

$$T_{\Delta_k}(1_A, \dots, 1_A)(\lambda) = \alpha^{k+1} + O_k(c_{\Delta_k}^{-1/6} \varepsilon^{2/3})$$

provided $0 < \lambda \leq \varepsilon \ll \min\{1, c_{\Delta_k}^{-1}\}$.

To finish the proof of Proposition 3.1 we are therefore left with the task of proving Lemma 3.1.

Proof of Lemma 3.1. By symmetry it suffices to show that

$$|T_{\Delta_k}(f_0, f_1, \dots, f_k)(\lambda)| \leq \|f_k\|_{U^1(\varepsilon^4\lambda)} + O(c_{\Delta_k}^{-1/6} \varepsilon^{2/3}). \tag{7}$$

As in [1] we start by writing

$$T_{\Delta_k}(f_0, f_1, \dots, f_k)(\lambda) = \iint \cdots \int f_0(x) f_1(x - \lambda x_1) \cdots f_k(x - \lambda x_k)\, d\sigma^{(d-k)}_{x_1,\dots,x_{k-1}}(x_k) \cdots d\sigma^{(d-2)}_{x_1}(x_2)\, d\sigma(x_1)\, dx$$

where σ now denotes the normalized measure on the sphere $S^{d-1}(0, |v_1|)$ and $\sigma^{(d-j)}_{x_1,\dots,x_{j-1}}$ denotes, for each $2 \leq j \leq k$, the normalized measure on the spheres

$$S^{d-j}_{x_1,\dots,x_{j-1}} = S^{d-1}(0, |v_j|) \cap S^{d-1}(x_1, |v_j - v_1|) \cap \cdots \cap S^{d-1}(x_{j-1}, |v_j - v_{j-1}|) \tag{8}$$

where $S^{d-1}(x, r) = \{x' \in \mathbb{R}^d : |x - x'| = r\}$. Since

$$|T_{\Delta_k}(f_0, f_1, \dots, f_k)(\lambda)| \leq \iint \cdots \int \left| \int f_k(x - \lambda x_k)\, d\sigma^{(d-k)}_{x_1,\dots,x_{k-1}}(x_k) \right| d\sigma^{(d-k+1)}_{x_1,\dots,x_{k-2}}(x_{k-1}) \cdots d\sigma^{(d-2)}_{x_1}(x_2)\, d\sigma(x_1)\, dx$$

it follows from an application of Cauchy-Schwarz that

$$|T_{\Delta_k}(f_0,f_1,\dots,f_k)(\lambda)|^2 \le \int\cdots\iiint\Big|\int f_k(x-\lambda x_k)\,d\sigma^{(d-k)}_{x_1,\dots,x_{k-1}}(x_k)\Big|^2\,dx \\ d\sigma^{(d-k+1)}_{x_1,\dots,x_{k-2}}(x_{k-1})\cdots d\sigma^{(d-2)}_{x_1}(x_2)\,d\sigma(x_1). \tag{9}$$

An application of Plancherel therefore shows that

$$|T_{\Delta_k}(f_0,f_1,\dots,f_k)(\lambda)|^2 \le \int |\widehat{f_k}(\xi)|^2 I(\lambda\,\xi)\,d\xi$$

where

$$I(\xi)=\int\cdots\int\big|\widehat{\sigma^{(d-k)}_{x_1,\dots,x_{j-1}}}(\xi)\big|^2\,d\sigma^{(d-k+1)}_{x_1,\dots,x_{k-2}}(x_{k-1})\cdots d\sigma^{(d-2)}_{x_1}(x_2)\,d\sigma(x_1). \tag{10}$$

Estimate (7) will follow if we can show that

$$I(\lambda\xi)=I(\lambda\xi)\widehat{\psi}(\varepsilon^4\lambda\xi)+I(\lambda\xi)(1-\widehat{\psi}(\varepsilon^4\lambda\xi))\le\widehat{\psi}(\varepsilon^4\lambda\xi)+O(c_{\Delta_k}^{-1/3}\varepsilon^{4/3}) \tag{11}$$

since $\|f_k\|_2\le 1$ and an application of Parseval and appeal to (3) reveals that

$$\int|\widehat{f_k}(\xi)|^2\widehat{\psi}(\varepsilon^4\lambda\xi)\,d\xi=\iint f_k(x)f_k(x-x_1)\psi_{\varepsilon^4\lambda}(x_1)\,dx\,dx_1=\|f_k\|^2_{U^1(\varepsilon^4\lambda)}+O(\varepsilon^4\lambda). \tag{12}$$

To establish (11) we argue as in [1], in particular we use the fact that in addition to being trivially bounded by 1 the Fourier transform of $\sigma^{(d-k)}_{x_1,\dots,x_{k-1}}$ also decays for large ξ in certain directions, specifically

$$\big|\widehat{\sigma^{(d-k)}_{x_1,\dots,x_{k-1}}}(\xi)\big|\le C\left(r(S^{d-k}_{x_1,\dots,x_{k-1}})\cdot\operatorname{dist}(\xi,\operatorname{span}\{x_1,\dots,x_{k-1}\})\right)^{-(d-k)/2} \tag{13}$$

where $r(S^{d-k}_{x_1,\dots,x_{k-1}})=\operatorname{dist}(v_k,\operatorname{span}\{v_1,\dots,v_{k-1}\})$ denotes the radius of the sphere $S^{d-k}_{x_1,\dots,x_{k-1}}$.

This estimate is a consequence of the well-known asymptotic behavior of the Fourier transform of the measure on the unit sphere $S^{d-k}\subseteq\mathbb{R}^{d-k+1}$ induced by Lebesgue measure, see for example [3].

Together with the trivial uniform bound $I(\xi)\le 1$, and an appropriate conical decomposition (depending on ξ) of the configuration space over which the integral $I(\xi)$ is defined, this gives

$$I(\xi)\le\min\{1,C(c_{\Delta_k}|\xi|)^{-(d-k)/2}\}. \tag{14}$$

Combining (14) with the basic bound $|1-\widehat{\psi}(\xi)|\le\min\{1,C|\xi|\}$ we obtain the uniform bound

$$|1-\widehat{\psi}(\varepsilon^4\lambda\,\xi)|I(\lambda\,\xi)\ll\min\{(\lambda c_{\Delta_k}|\xi|)^{-1/2},\varepsilon^4\lambda|\xi|\}\le c_{\Delta_k}^{-1/3}\varepsilon^{4/3}$$

from which (11) follows. □

3.2. **A Second Approach to our New Proof of Theorem 1.1.** In this final subsection we present an alternative approach to proving Proposition 3.1, and hence Theorem 1.1, with the slightly worse error bound $O_k(c_{\Delta_k}^{-1/12}\varepsilon^{1/3})$. Specifically, we establish the following (slightly weaker) generalized von-Neumann inequality for simplices using only Lemma 2.1, namely the generalized von-Neumann inequality for distances.

Lemma 3.2 (Generalized von-Neumann for Simplices II)**.** *For any $0<\lambda\le\varepsilon\ll\min\{1,c_{\Delta_k}^{-1}\}$ and functions*

$$f_0,f_1,\dots,f_k:[0,1]^d\to[-1,1]$$

$$|T_{\Delta_k}(f_0,f_1,\dots,f_k)(\lambda)|\le\sqrt{2\pi}\min_{j=0,1,\dots,k}\|f_j\|^{1/2}_{U^1(\varepsilon^4\lambda)}+O(c_{\Delta_k}^{-1/12}\varepsilon^{1/3}).$$

In the proof below we will make use of the following straightforward observations:

(i) If we let $\Delta_{k-1}=\{0,v_1,\dots,v_{k-1}\}$, then

$$T_{\Delta_k}(f_0,f_1,\dots,f_{k-1},1_{[0,1]^d})(\lambda)=T_{\Delta_{k-1}}(f_0,f_1,\dots,f_{k-1})(\lambda)+O(\lambda). \tag{15}$$

(ii) If we let $\Delta'_k=\{0,v'_1,\dots,v'_k\}$ with $v'_j=v_{k-j}-v_k$ for $0\le j\le k-1$ and $v'_k=-v_k$, then

$$T_{\Delta_k}(f_0,f_1,\dots,f_k)(\lambda)=T_{\Delta'_k}(f_k,f_{k-1},\dots,f_0)(\lambda). \tag{16}$$

Proof of Lemma 3.2. By symmetry it suffices to show that

$$|T_{\Delta_k}(f_0,f_1,\dots,f_k)(\lambda)|^2 \le 2\pi\,\|f_k\|_{U^1(\varepsilon^4\lambda)} + O(c_{\Delta_k}^{-1/6}\varepsilon^{2/3}). \tag{17}$$

We initially follow the proof of Lemma 3.1, but after (9) we now proceed differently. Instead of applying Plancherel to the right hand side of

$$|T_{\Delta_k}(f_0,f_1,\dots,f_k)(\lambda)|^2 \le \iint\cdots\int\Big|\int f_k(x-\lambda x_k)\,d\sigma^{(d-k)}_{x_1,\dots,x_{k-1}}(x_k)\Big|^2 d\sigma^{(d-k+1)}_{x_1,\dots,x_{k-2}}(x_{k-1})\cdots d\sigma(x_1)\,dx.$$

we now "square out" the right hand side to obtain

$$\iint\cdots\iiint f_k(x-\lambda x_k)f_k(x-\lambda x_{k+1})\,d\sigma^{(d-k)}_{x_1,\dots,x_{k-1}}(x_{k+1})\,d\sigma^{(d-k)}_{x_1,\dots,x_{k-1}}(x_k)\,d\sigma^{(d-k+1)}_{x_1,\dots,x_{k-2}}(x_{k-1})\cdots d\sigma(x_1)\,dx. \tag{18}$$

If $d=k+1$, then for fixed $x_1,\dots,x_k$ we can use arc-length to parameterize of the circle $S^{d-k}_{x_1,\dots,x_{k-1}}$, with $\theta=0$ and $\theta=2\pi$ corresponding to the point x_k, to write

$$\int f_k(x-\lambda x_{k+1})\,d\sigma^{(d-k)}_{x_1,\dots,x_{k-1}}(x_{k+1}) = \int_0^{2\pi} f_k(x-\lambda x_{k+1}(x_1,\dots,x_k,\theta))\,d\theta. \tag{19}$$

For any fixed $\theta\in[0,2\pi]$ we then define $\Delta_{k+1}(\theta)=\{0,v_1,\dots,v_k,v_{k+1}(\theta)\}$ with $v_{k+1}=v_{k+1}(\theta)$ satisfying $|v_{k+1}|=|v_k|$, $|v_{k+1}-v_j|=|v_k-v_j|$ for all $1\le j\le k-1$ and use θ to determine the angle between v_{k+1} and v_k measured from the center of the circle $S^{d-k}_{x_1,\dots,x_{k-1}}$, consequently

$$|v_{k+1}-v_k| = 2\sin(\theta/2)\cdot\operatorname{dist}(v_k,\operatorname{span}\{v_1,\dots,v_{k-1}\}).$$

It follows that

$$|T_{\Delta_k}(f_0,f_1,\dots,f_k)(\lambda)|^2 \le \int_0^{2\pi} T_{\Delta_{k+1}(\theta)}(1_{[0,1]^d},\dots,1_{[0,1]^d},f_k,f_k)(\lambda)\,d\theta + O(\lambda)$$

and in light of (15) and (16) that

$$\begin{aligned}|T_{\Delta_k}(f_0,f_1,\dots,f_k)(\lambda)|^2 &\le \int_0^{2\pi} T_{\Delta'_{k+1}(\theta)}(f_k,f_k,1_{[0,1]^d},\dots,1_{[0,1]^d})(\lambda)\,d\theta + O(\lambda)\\ &= \int_0^{2\pi} T_{\Delta'_1(\theta)}(f_k,f_k)(\lambda)\,d\theta + O(\lambda)\end{aligned}$$

where

$$T_{\Delta'_1(\theta)}(f_k,f_k)(\lambda) = T(f_k,f_k)(c(\theta)\lambda) := \iint f_k(x)f_k(x-c(\theta)\lambda x_1)\,d\sigma(x_1)\,dx$$

with $c(\theta)=2\sin(\theta/2)\cdot\operatorname{dist}(v_k,\operatorname{span}\{v_1,\dots,v_{k-1}\})$. Lemma 2.1 now implies that

$$|T_{\Delta'_1(\theta)}(f_k,f_k)(\lambda)| \le \|f_k\|_{U^1(\varepsilon^4\lambda)} + O((\sin(\theta/2))^{-1/6}c_{\Delta_k}^{-1/6}\varepsilon^{2/3})$$

since $c(\theta)\ge 2\sin(\theta/2)\,c_{\Delta_k}$. This completes the proof, when $d=k+1$, as $\int_0^{2\pi}(\sin(\theta/2))^{-1/6}\,d\theta<\infty$, and in fact establishes the result in general, since if $d\ge k+2$, one can define a new non-degenerate simplex

$$\Delta_{d-1}=\{0,v_1,\dots,v_{k-1},v'_k,\dots,v'_{d-2},v'_{d-1}\}$$

with $v'_{d-1}=v_k$ and use the fact that

$$T_{\Delta_k}(f_0,f_1,\dots,f_k)(\lambda) = T_{\Delta_{d-1}}(f_0,\dots,f_{k-1},1_{[0,1]^d},\dots,1_{[0,1]^d},f_k)(\lambda)+O(\lambda). \qquad \square$$

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