TWO-WEIGHT ESTIMATES FOR SINGULAR AND STRONGLY SINGULAR INTEGRAL OPERATORS

V. KOKILASHVILI, N. LYALL, AND A. MESKHI

Abstract. In this article we consider conditional *two-weight* estimates for singular and strongly singular integral operators. The conditions governing two-weight estimates shall be simultaneously necessary and sufficient for a quite large class of singular integrals.

1. Introduction

In the sequel we shall assume that K is a distributional kernel that satisfies the estimate

$$|K(x,y)| \leq \frac{A}{|x-y|^{n+\alpha}}, \tag{1}$$

whenever $x \neq y$ for some $\alpha \geq 0$. Moreover, we assume that the operator

$$Tf(x) = \int_{\mathbf{R}^n} K(x,y)f(y)\,dy\ ,\quad x \notin \operatorname{supp} f, \tag{2}$$

which is initially defined for function $f \in \mathcal{S}(\mathbf{R}^n)$, extends to a bounded operator on $L^2(\mathbf{R}^n)$.

We shall assume that a weight ρ is an almost everywhere positive function on $\mathbf{R}^n$ and denote by $L^p_\rho(\mathbf{R}^n)$, for $1 \leq p < \infty$, the space of all measurable functions $f : \mathbf{R}^n \to \mathbf{R}$ for which

$$\|f\|_{L^p_\rho(\mathbf{R}^n)} := \left(\int_{\mathbf{R}^n} |f(x)|^p \rho(x)\,dx\right)^{1/p} < \infty.$$

We denote by $L^{p,\infty}_\rho(\mathbf{R}^n)$, for $1 \leq p < \infty$, the space of all measurable functions $f : \mathbf{R}^n \to \mathbf{R}$ for which

$$\|f\|_{L^{p,\infty}_\rho(\mathbf{R}^n)} := \sup_{\lambda>0}\ \lambda \left(\int_{\{x:|f(x)|>\lambda\}} \rho(x)\,dx\right)^{1/p} < \infty.$$

For convenience we shall often abbreviate $L^p_\rho(\mathbf{R}^n)$ and $L^{p,\infty}_\rho(\mathbf{R}^n)$ by L^p_ρ and $L^{p,\infty}_\rho$ respectively.

We shall say that an operator is of *two-weight* strong-type (p,p) or *two-weight* weak-type (p,p), for $1 \leq p < \infty$, if it is bounded from $L^p_{\rho_1}$ to $L^p_{\rho_2}$ or from $L^p_{\rho_1}$ to $L^{p,\infty}_{\rho_2}$ respectively.

In this article we will be concerned with conditional *two-weight* estimates for operators T defined by (2) with kernel satisfying (1). In our arguments we use known boundedness properties of appropriate singular integrals and two-weight criteria for the Hardy transforms. We also establish necessary conditions for such estimates to hold in the case where $\alpha = 0$.

For *one-weight* estimates it is a well known result of Stein [33] that operators given by (2) with kernels satisfying condition (1) for $\alpha = 0$ that are bounded on L^p for $1 < p < \infty$ will also be bounded on $L^p_\rho(\mathbf{R}^n)$, with $\rho(x) = |x|^\lambda$ and $-n < \lambda < n(p-1)$. For related topics when $p = 1$ see Hoffman [18]. The results of [33] were later extended by Soria and Weiss [32] to the case of general

Date: October 24, 2006.
2000 *Mathematics Subject Classification* 42B20 (primary).
The second author was supported by the European Commission through the IHP Network *HARP* 2002-2006.

A_p weights and to certain maximal singular integrals. *One-weight* estimates have been obtained in the case where $\alpha > 0$ by Chanillo [3].

For convenience we recall that ρ is an A_p weight for $1 < p < \infty$, or more succinctly $\rho \in A_p$, if

$$\sup_{B \subset \mathbf{R}^n} \left(\frac{1}{|B|} \int_B \rho(x)\, dx \right)^{1/p} \left(\frac{1}{|B|} \int_B \rho^{-p'/p}(x)\, dx \right)^{1/p'} < \infty,$$

where $p' = \frac{p}{p-1}$ and the supremum is taken over all balls in $\mathbf{R}^n$. Passing to the limit in the definition above we obtain the following characterization of the class A_1, namely that $\rho \in A_1$ if

$$\sup_{B \subset \mathbf{R}^n} \left(\frac{1}{|B|} \int_B \rho(x)\, dx \right) \left\| \rho^{-1} \right\|_{L^\infty(B)} < \infty.$$

Recall also that if $\rho \in A_p$, then $\rho^{-p'/p} \in A_{p'}$, where again $p' = \frac{p}{p-1}$.

2. Main Results

2.1. **Positive results in the case where $\alpha \geq 0$.** Our first result establishes a sufficient condition for our operators T to be of *two-weight* strong-type (p, p) when $1 < p < \infty$.

Theorem 1. *Let $1 < p < \infty$ and T be an operator defined by (2) with kernel satisfying (1) with $\alpha \geq 0$ that is bounded on $L^p(\mathbf{R}^n)$. If v_0 and w_0 are positive monotonic functions on $(0, \infty)$ such that the weights $v(x) = v_0(|x|)$ and $w(x) = w_0(|x|)$ satisfy the condition*

$$B_{\alpha,d}(v, w) := \sup_{t>0} \left(\int\limits_{t \leq |x|} v(x)(|x|^{-\alpha} + 1)^p |x|^{-np} dx \right)^{1/p} \left(\int\limits_{|x| \leq t/d} w^{-p'/p}(x)\, dx \right)^{1/p'} < \infty$$

if v_0 and w_0 are increasing or

$$B'_{\alpha,d}(v, w) := \sup_{t>0} \left(\int\limits_{|x| \leq t/d} v(x)\, dx \right)^{1/p} \left(\int\limits_{t \leq |x|} w^{-p'/p}(x)(|x|^{-\alpha} + 1)^{p'} |x|^{-np'} dx \right)^{1/p'} < \infty$$

if v_0 and w_0 are decreasing, for some $d > 1$, then T is bounded from L^p_w to L^p_v. Moreover

$$\|Tf\|_{L^p_v} \leq C_1 B_{\alpha,d}(v, w) \ \left[\text{or } B'_{\alpha,d}(v, w)\right] \|f\|_{L^p_w},$$

where $C_1 = C_1(\|T\|_{L^p \to L^p}, A, p, n, \alpha, d)$.

Remark 1. If w satisfies the doubling condition:

$$\int_{|x| \leq 2t} w(x)\, dx \leq c' \int_{|x| \leq t} w(x)\, dx,$$

then so does $w^{-p'/p}$, and as a consequence $B_{\alpha,d}(v, w) \leq A_\alpha(v, w)$, where

$$A_\alpha(v, w) := \sup_{t>0} \ t^{-n}(t^{-\alpha} + 1) \left(\int_{|x| \leq t} v(x)\, dx \right)^{1/p} \left(\int_{|x| \leq t} w^{-p'/p}(x)\, dx \right)^{1/p'}.$$

We include the proof of this statement as an appendix.

Our second result establishes a sufficient condition for our operators T to be of *two-weight* weak-type (p, p) when $1 \leq p < \infty$.

Theorem 2. *Let $1 \le p < \infty$ and T be an operator defined by (2) with kernel satisfying (1) with $\alpha \ge 0$ that is bounded from $L^p(\mathbf{R}^n)$ to $L^{p,\infty}(\mathbf{R}^n)$. If v_0 and w_0 are positive increasing functions on $(0,\infty)$ such that the weights $v(x) = v_0(|x|)$ and $w(x) = w_0(|x|)$ satisfy the condition*

$$B^{(p)}_{\alpha,d}(v,w) := \sup_{0<t<\tau} \frac{\tau^{-\alpha}+1}{\tau^n} \left(\int\limits_{t\le|x|\le\tau} v(x)\,dx \right)^{1/p} \left(\int\limits_{|x|\le t/d} w^{-p'/p}(x)\,dx \right)^{1/p'} < \infty$$

if $1 < p < \infty$, and

$$B^{(1)}_{\alpha,d}(v,w) := \sup_{0<t<\tau} \frac{\tau^{-\alpha}+1}{\tau^n} \left(\int\limits_{t\le|x|\le\tau} v(x)\,dx \right) \left\| w^{-1} \right\|_{L^\infty(\{|\cdot|<t/d\})} < \infty$$

if $p = 1$, for some $d > 1$, then T is bounded from L^p_w to $L^{p,\infty}_v$. Moreover

$$\|Tf\|_{L^{p,\infty}_v} \le C_2 B^{(p)}_{\alpha,d}(v,w) \|f\|_{L^p_w},$$

where $C_2 = C_2(\|T\|_{L^p \to L^{p,\infty}}, A, p, n, \alpha, d)$.

2.1.1. **Examples**. Let $1 < p < \infty$ and recall that $|x|^\gamma \in A_p(\mathbf{R}^n)$ if and only if $-n < \gamma < n(p-1)$.

For simplicity we shall restrict our examples to the case where $n = 1$. It is known (see [8]) that if

$$v(x) = \begin{cases} |x|^{p-1} & \text{if } 0 < |x| \le 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

$$w(x) = \begin{cases} |x|^{p-1}(1-\log|x|)^p & \text{if } 0 < |x| \le 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

with $0 < \gamma < p-1$, then the Hilbert transform is bounded from L^p_w to L^p_v. Furthermore, if

$$v(x) = \begin{cases} |x|^{p-1}(1-\log|x|)^p & \text{if } 0 < |x| \le 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

$$w(x) = \begin{cases} |x|^{p-1}(1-\log|x|) & \text{if } 0 < |x| \le 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

with $0 < \gamma < p-1$, then the Hilbert transform is bounded from L^p_w to $L^{p,\infty}_v$, but is not bounded from L^p_w to L^p_v. See [10] page 557.

The following two examples are an immediate consequence of Theorem 1.

Example 1. Suppose that T is an operator defined by (2) with kernel satisfying (1) with $\alpha \ge 0$ that is bounded on $L^p(\mathbf{R})$. If we set

$$v(x) = \begin{cases} |x|^{\gamma+\alpha p} & \text{if } 0 < |x| \le 1 \\ |x|^\gamma & \text{if } |x| > 1 \end{cases}$$

$$w(x) = |x|^\gamma \quad \text{if } |x| > 0$$

with $0 < \gamma < p-1$, then T is bounded from L^p_w to L^p_v.

Example 2. Suppose that T is an operator defined by (2) with kernel satisfying (1) with $\alpha \geq 0$ that is bounded on $L^p(\mathbf{R})$. If we set

$$v(x) = \begin{cases} |x|^{p-1+\alpha p} & \text{if } 0 < |x| \leq 1 \\ |x|^{\gamma} & \text{if } |x| > 1 \end{cases}$$

$$w(x) = \begin{cases} |x|^{p-1}(1-\log|x|)^p & \text{if } 0 < |x| \leq 1 \\ |x|^{\gamma} & \text{if } |x| > 1 \end{cases}$$

with $0 < \gamma < p-1$, then T is bounded from L^p_w to L^p_v.

The following is an immediate consequence of Theorem 2.

Example 3. Suppose that T is an operator defined by (2) with kernel satisfying (1) with $\alpha \geq 0$ that is bounded from $L^p(\mathbf{R})$ to $L^{p,\infty}(\mathbf{R})$. If we set

$$v(x) = \begin{cases} |x|^{p-1+\alpha p}(1-\log|x|)^p & \text{if } 0 < |x| \leq 1 \\ |x|^{\gamma} & \text{if } |x| > 1 \end{cases}$$

$$w(x) = \begin{cases} |x|^{p-1}(1-\log|x|) & \text{if } 0 < |x| \leq 1 \\ |x|^{\gamma} & \text{if } |x| > 1 \end{cases}$$

with $0 < \gamma < p-1$, then T is bounded from L^p_w to $L^{p,\infty}_v$.

2.1.2. **Local Properties in the case where** $\alpha \geq 0$. Our third and final result in the generality of $\alpha \geq 0$ concerns the local properties of our operator T.

We make the assumption here that our operators T are local; that the boundedness of T on L^p is equivalent to the following estimate holding uniformly in x_0,

$$\int_{|x-x_0|\leq 1} |Tf(x)|^p dx \leq C_0 \int_{|x-x_0|\leq 10} |f(x)|^p dx. \tag{3}$$

Theorem 3. *Let $1 < p < \infty$ and T be an operator defined by (2) with kernel satisfying (1) with $\alpha \geq 0$ that satisfies (3). If v_0 and w_0 are positive monotonic functions on $(0, 10)$ such that the weights $v(x) = v_0(|x|)$ and $w(x) = w_0(|x|)$ satisfy the condition*

$$B^{\mathrm{loc}}_{\alpha,d}(v,w) := \sup_{0<t<1} \left(\int\limits_{t\leq|x|\leq 1} v(x)|x|^{-(n+\alpha)p} dx \right)^{1/p} \left(\int\limits_{|x|\leq t/d} w^{-p'/p}(x)\, dx \right)^{1/p'} < \infty$$

if v_0 and w_0 are increasing or

$$B'^{\mathrm{loc}}_{\alpha,d}(v,w) := \sup_{0<t<1} \left(\int\limits_{|x|\leq t/d} v(x)\, dx \right)^{1/p} \left(\int\limits_{t\leq|x|\leq 1} w^{-p'/p}(x)|x|^{-(n+\alpha)p'} dx \right)^{1/p'} < \infty$$

if v_0 and w_0 are decreasing, for some $d > 1$, then

$$\int\limits_{|x-x_0|\leq 1} |Tf(x)|^p\, v(x-x_0)\, dx \leq C_3 B^{\mathrm{loc}}_{\alpha,d}(v,w) \; \left[\text{or } B'^{\mathrm{loc}}_{\alpha,d}(v,w)\right] \int\limits_{|x-x_0|\leq 10} |f(x)|^p\, w(x-x_0)\, dx,$$

where $C_3 = C_3(C_0, A, p, n, \alpha, d)$ is independent of x_0 and f.

2.2. **Positive results in the case where $\alpha = 0$.** The restriction to $\alpha = 0$ in (1) enables us to formulate more general statements.

Again our first result establishes a sufficient condition for our operators T to be of *two-weight* strong-type (p, p) when $1 < p < \infty$. We introduce the following notation,

$$\begin{aligned} B_d(v,w) &:= B_{0,d}(v,w) = \sup_{t>0}\left(\int_{t\le|x|} v(x)|x|^{-np}dx\right)^{1/p}\left(\int_{|x|\le t/d} w^{-p'/p}(x)\,dx\right)^{1/p'} \\ B'_d(v,w) &:= B'_{0,d}(v,w) = \sup_{t>0}\left(\int_{|x|\le t/d} v(x)\,dx\right)^{1/p}\left(\int_{t\le|x|} w^{-p'/p}(x)|x|^{-np'}dx\right)^{1/p'}. \end{aligned}$$

Theorem 4. *Let $1 < p < \infty$, $\rho \in A_p$, and T be an operator defined by (2) with kernel satisfying (1) with $\alpha = 0$ that is bounded on $L^p_\rho(\mathbf{R}^n)$. If v_0 and w_0 are positive monotonic functions on $(0, \infty)$ such that the weights $v(x) = v_0(|x|)\rho(x)$ and $w(x) = w_0(|x|)\rho(x)$ satisfy the condition*

$$B_d(v,w) < \infty$$

if v_0 and w_0 are increasing or

$$B'_d(v,w) < \infty$$

if v_0 and w_0 are decreasing, for some $d > 1$, then T is bounded from L^p_w to L^p_v. Moreover

$$\|Tf\|_{L^p_v} \le C_4 B_d(v,w)\ \left[\text{or } B'_d(v,w)\right]\|f\|_{L^p_w},$$

where $C_4 = C_4(\|T\|_{L^p_\rho\to L^p_\rho}, A, p, n, d)$.

Theorem 4 has been already been proven in the case of Calderón-Zygmund singular integrals; see [8]. The following corollary generalizes results presented in [9], see also [10], p517.

Corollary 5. *Let $1 < p < \infty$ and T be an operator defined by (2) with kernel satisfying (1) with $\alpha = 0$ that is bounded on $L^p_\varrho(\mathbf{R}^n)$ for $\rho \in A_p$. Let $\rho_1 \in A_1$, if v_0 and w_0 are positive monotonic functions on $(0, \infty)$ such that the weights $v(x) = v_0(|x|)$ and $w(x) = w_0(|x|)$ satisfy the condition*

$$B_d(v,w) < \infty$$

if v_0 and w_0 are increasing [or $B'_d(v,w) < \infty$ if v_0 and w_0 are decreasing], for some $d > 1$, then it follows that T is bounded from $L^p_{w\rho_1}$ to $L^p_{v\rho_1}$ [or from $L^p_{w\rho_1^{1-p}}$ to $L^p_{v\rho_1^{1-p}}$]. Moreover

$$\|Tf\|_{L^p_{v\rho_1}} \le C_5 B_d(v,w)\|f\|_{L^p_{w\rho_1}}$$

$$\left[\text{or }\ \|Tf\|_{L^p_{v\rho_1^{1-p}}} \le C_5 B'_d(v,w)\right]\|f\|_{L^p_{w\rho_1^{1-p}}}\Bigg],$$

where $C_5 = C_5(\|T\|_{L^p_\rho\to L^p_\rho}, A, p, n, d)$.

Proof. We shall assume that v_0 and w_0 are increasing. Using the fact that $\rho_1 \in A_1 \subset A_p$ it follows that

$$
\begin{aligned}
B_d(v\rho_1, w\rho_1) &= \sum_{k=0}^{\infty} \Bigg(\int\limits_{2^k t \le |x| < 2^{k+1} t} v(x)\rho_1(x)|x|^{-np} dx \Bigg)^{1/p} \Bigg(\int\limits_{|x| \le t/d} w^{-p'/p}(x)\rho_1^{-p'/p}(x)\, dx \Bigg)^{1/p'} \\
&\le [A_1(\rho_1)]^{1/p} \sum_{k=0}^{\infty} v^{1/p}(2^{k+1})(2^k t)^{-n+n/p} \Bigg(\int\limits_{|x| \le t/d} w^{-p'/p}(x)\, dx \Bigg)^{1/p'} \\
&\le [A_1(\rho_1)]^{1/p} \sum_{k=1}^{\infty} \Bigg(\int\limits_{2^k t \le |x| < 2^{k+1} t} v(x)|x|^{-np} dx \Bigg)^{1/p} \Bigg(\int\limits_{|x| \le t/d} w^{-p'/p}(x)\, dx \Bigg)^{1/p'} \\
&\le C B_d(v, w).
\end{aligned}
$$

The argument for v_0 and w_0 decreasing is similar, in this case one instead uses the fact that $\rho_1^{-p/p'} \in A_{p'}$ if $\rho_1 \in A_1$. □

Our second main result when $\alpha = 0$ establishes a sufficient condition for our operators T to be of *two-weight* weak-type (p, p) when $1 \le p < \infty$. We introduce the following notation,

$$
\begin{aligned}
B_d^{(p)}(v, w) \;&:=\; B_{0,d}^{(p)}(v, w) = \sup_{0 < t < \tau} \frac{1}{\tau^n} \Bigg(\int\limits_{t \le |x| \le \tau} v(x)\, dx \Bigg)^{1/p} \Bigg(\int\limits_{|x| \le t/d} w^{-p'/p}(x)\, dx \Bigg)^{1/p'} \\
B_d^{(1)}(v, w) \;&:=\; B_{0,d}^{(1)}(v, w) = \sup_{0 < t < \tau} \frac{1}{\tau^n} \Bigg(\int\limits_{t \le |x| \le \tau} v(x)\, dx \Bigg) \left\| w^{-1} \right\|_{L^\infty(\{|\cdot| < t/d\})}.
\end{aligned}
$$

Theorem 6. *Let $1 \le p < \infty$, $\rho \in A_p$, and T be an operator defined by (2) with kernel satisfying (1) with $\alpha = 0$ that is bounded from $L^p_\rho(\mathbf{R}^n)$ to $L^{p,\infty}_\rho(\mathbf{R}^n)$. If v_0 and w_0 are positive increasing functions on $(0, \infty)$ such that the weights*

$$v(x) = v_0(|x|)\rho(x) \quad \text{and} \quad w(x) = w_0(|x|)\rho(x)$$

satisfy the condition

$$B_d^{(p)}(v, w) < \infty$$

if $1 < p < \infty$, and

$$B_d^{(1)}(v, w) < \infty$$

if $p = 1$, for some $d > 1$, then T is bounded from L^p_w to $L^{p,\infty}_v$. Moreover

$$\|Tf\|_{L^{p,\infty}_v} \le C_6 B_d^{(p)}(v, w) \|f\|_{L^p_w},$$

where $C_6 = C_6(\|T\|_{L^p_\rho \to L^{p,\infty}_\rho}, A, p, n, d)$.

Remark 2. If $\rho \in A_p$ with $p > 1$, then *one-weight* weak-type (p, p) estimates for the Riesz transforms are equivalent to *one-weight* strong-type (p, p) estimates. It has however been shown that for $p > 1$ the class of weight pairs guaranteeing *two-weight* weak-type (p, p) estimates for the Hilbert transform is larger than the class that ensures *two-weight* strong-type (p, p) estimates; see [9] and [10], Chapter 8.

2.3. **Necessary conditions in the case where $\alpha = 0$.** Our first result establishes a necessary condition for our operators T to be of *two-weight* strong-type (p, p) when $1 < p < \infty$.

Theorem 7. *Let $1 < p < \infty$ and T be an operator defined by (2) with kernel $K(x, y) = \widetilde{K}(x - y)$ satisfying the estimates*

$$|\nabla \widetilde{K}(x)| \leq \frac{A_0}{|x|^{n+1}}, \tag{4a}$$

$$|\widetilde{K}(x)| \geq \frac{A_1}{|x|^n}, \tag{4b}$$

whenever $x \neq 0$, in addition to (1) with $\alpha = 0$. If T is bounded from L^p_w to L^p_v then this implies the condition

$$B_{\frac{4A_0 n}{A_1}}(v, w) < \infty.$$

Our second result of this section establishes a necessary condition for our operators T to be of *two-weight* weak-type (p, p) when $1 \leq p < \infty$.

Theorem 8. *Let $1 \leq p < \infty$ and T be an operator defined by (2) with kernel $K(x, y) = \widetilde{K}(x - y)$ satisfying estimates (4) and (1) with $\alpha = 0$. If T is bounded from L^p_w to $L^{p,\infty}_v$ then it follows that*

$$B^{(p)}_{\frac{4nA_0}{A_1}}(v, w) < \infty,$$

if $1 < p < \infty$, and

$$B^{(1)}_{\frac{4nA_0}{A_1}}(v, w) < \infty,$$

if $p = 1$.

Corollary 9. *Let $K(x, y) = \widetilde{K}(x - y)$ satisfy conditions (1) and (4) and P be a real polynomial on $\mathbf{R}^n \times \mathbf{R}^n$. If we, in the sense of (2), define*

$$T_P f(x) = \int K(x, y) e^{iP(x,y)} f(y)\, dy$$

then in order for T_P to be bounded from L^p_w to L^p_v with bounds independent of the coefficients of P it is necessary that

$$B_{\frac{4A_0 n}{A_1}}(v, w) < \infty.$$

Proof. For $\varepsilon > 0$ we denote $P_\varepsilon(x, y) := \varepsilon P(x, y)$. Let f be a non-negative belonging to $L^p_w$with support in $\chi_{B(0,t)}$, $t > 0$. It is then easy to see, using the Lebesgue dominated convergence theorem, that if $|x| > \frac{4nA_0}{A_1} t$, then

$$\lim_{\varepsilon \to 0} \int_{|y|<t} e^{iP_\varepsilon(x,y)} K(x, y) f(y) dy = \int_{|y|<t} K(x, y) f(y) dy. \tag{5}$$

□

3. Background

3.1. **Model Operators.** We now list some model operators that are of the form that we are considering, namely of the form (2) with kernels satisfying (1). Recall that we are also making the *a priori* assumption that our operators are bounded on $L^2(\mathbf{R}^n)$.

3.1.1. *Calderón-Zygmund Singular Integrals.* These operators, which we shall denote by S, are defined as in (2) with integral kernel that in addition to satisfying (1) for $\alpha = 0$ also satisfy the differential inequality

$$|\partial_x^\nu \partial_y^\mu K(x,y)| \leq A|x-y|^{-n-|\nu|-|\mu|}.$$

Key examples are the following;

(1) The key example when $n = 1$ is the *Hilbert transform* H, this is a convolution operator (defined as a principle value) with distributional kernel
$$K(x) = x^{-1}.$$
(2) The analogues of the Hilbert transform in higher dimensions are the *Reisz transforms* $R_1, \dots, R_n$, where each operator R_j is given by convolution with
$$K_j(x) = x_j|x|^{-n-1}.$$
(3) Another important example are the convolution kernels of the form
$$K(x) = |x|^{-n-it}, \text{ with } t \neq 0.$$

The following result is of course well known; see for example [34].

Theorem A. *If* $1 < p < \infty$ *then* S *extends to a bounded operator on* $L^p(\mathbf{R}^n)$ *and if* $p = 1$ *then* S *extends to an operator which is of weak-type* $(1, 1)$*.*

3.1.2. *Oscillatory Singular Integrals.* Let K be a Calderón-Zygmund kernel as described above and P be a real polynomial on $\mathbf{R}^n \times \mathbf{R}^n$. If we, in the sense of (2), define

$$T_P f(x) = \int K(x,y)e^{iP(x,y)} f(y)\, dy$$

then the following is known, see [30] and [4].

Theorem B. *If* $1 < p < \infty$ *then* T_P *extends to a bounded operator on* $L^p(\mathbf{R}^n)$ *and if* $p = 1$ *then* T_P *extends to an operator which is of weak-type* $(1, 1)$*. In both instances the bounds on* T_P *can be taken independent of the coefficients of* P*.*

For extensions of Theorem B to more general phase functions see [29], see also [34].

3.1.3. *Strongly Singular Integrals.* These are operators, which we shall denote by T_α, whose integral kernels take the form

$$K_\alpha(x,y) = a(x,y)e^{i\varphi(x,y)},$$

where the amplitude[1] and phase satisfy the differential inequalities

$$|\partial_x^\mu \partial_y^\nu a(x,y)| \leq C_{\mu,\nu}|x-y|^{-d-\alpha-|\mu|-|\nu|}$$
$$|\partial_x^\mu \partial_y^\nu \varphi(x,y)| \leq C_{\mu,\nu}|x-y|^{-\beta-|\mu|-|\nu|},$$

that φ is real-valued and furthermore that

$$|\nabla_x\varphi(x,y)|, |\nabla_y\varphi(x,y)| \geq C|x-y|^{-\beta-1} \tag{6}$$

with $\beta > 0$ and $0 \leq \alpha \leq n\beta/2$. In additions to these two assumptions one also makes the non-degeneracy assumption that

$$\left|\det\left(\frac{\partial^2\varphi_\lambda(x,y)}{\partial x_i \partial y_j}\right)\right| \geq C > 0$$

[1] In the case where $\alpha = 0$ we must make the further assumption that our amplitude a is compactly supported in a neighborhood of the diagonal $x = y$, this is of course also the only region of any interest when $\alpha > 0$.

uniformly in λ. Note that this non-degeneracy assumption ensures that T_α extends to a bounded operator on $L^2(\mathbf{R}^n)$.

The key example of such a kernels are $K_\alpha(x,y) = \widetilde{K}_\alpha(x-y)$ where $\widetilde{K}_\alpha$ is a distribution on $\mathbf{R}^d$ that away from the origin agrees with the function

$$\widetilde{K}_\alpha(x) = |x|^{-d-\alpha} e^{i|x|^{-\beta}} \chi(|x|),$$

with χ smooth and compactly supported in a small neighborhood of the origin. Operators of this type where first studied in one dimension by Hirschman [17] and then in higher dimensions by Wainger [36]. For the following result see [25].

Theorem C. *If $1 < p < \infty$, then T_α extends to a bounded operator on $L^p(\mathbf{R}^n)$ if and only if*

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{2} - \frac{\alpha}{n\beta}.$$

The prototype non-convoltion operator of this strongly singular type are the pseudo-differential operators with symbols in the class $S^m_{\rho,\delta}$ that where introduced by Hörmander. In the special case where $K_\alpha(x,y) = \widetilde{K}_\alpha(x-y)$ and $\alpha = 0$ it was shown by C. Fefferman [11] that T_α extends to an operator which is of weak-type $(1,1)$. In fact this result, and the one above, can be extended to the general class introduced by Hörmander, see [12].

We now recall some well known *one-weight* and *two-weight* estimates for these singular integrals.

3.2. **One-weight estimates for singular integrals.** For completeness we choose to state here some results which pre-date those in [32].

3.2.1. *Calderón-Zygmund singular integrals.*

Theorem D. *If $1 < p < \infty$ and $\rho \in A_p$ then S is bounded on L^p_ρ, while if $\rho \in A_1$ then S is bounded from L^1_ρ to $L^{1,\infty}_\rho$ (S is of* one-weight *weak-type $(1,1)$).*

In the case of the Hilbert transform H having $\rho \in A_p$ for $1 < p < \infty$ and $\rho \in A_1$ is also necessary for H to be of one-weight *strong-type (p,p) and* one-weight *weak-type $(1,1)$ respectively.*

Theorem D was proved for the Hilbert transform in [21] and for general Calderón-Zygmund intgrals in [7], see also [13].

Moreover, in [13] (page 417) it is shown that if the Reisz transforms R_j are of *one-weight* weak-type (p,p) for $1 \leq p < \infty$ then one must necessarily have $\rho \in A_p$.

3.2.2. *Oscillatory singular integrals.* In [31] it was shown that if $K(x,y) = \widetilde{K}(x-y)$ satisfies the conditions[2]

$$|\widetilde{K}(x)| \leq A|x|^{-n} \quad \text{and} \quad |\nabla \widetilde{K}(x)| \leq A_0 |x|^{-n-1}$$

and

$$\int_{\epsilon<|x|<N} \widetilde{K}(x)\, dx = 0,$$

for all $0 < \epsilon < N < \infty$, and $\rho \in A_1$, then T_P is *one-weight* weak-type $(1,1)$. See also [5].

Let $1 < p < \infty$, $P(x,y) = Q(x-y)$ and $\widetilde{K}(x,y) = (x-y)^{-1}$, then T_P is bounded on $L^p_\rho(\mathbf{R})$ if and only if $\rho \in A_p$, see [19]. See also [20].

[2] These are precisely the necessary and sufficient conditions in order for the Calderón-Zygmund singular integrals with this convolution kernel $\widetilde{K}$ to extend top a bounded operator on $L^2(\mathbf{R}^n)$.

3.2.3. *Strongly singular integrals.* The following results have been established in the in the 'model' convolution case.

Theorem E. *Let $K_\alpha(x,y) = \widetilde{K}_\alpha(x-y)$.*

(i) *If $\alpha = 0$, $1 < p < \infty$ and $\rho \in A_p$ then T_α is bounded on L^p_ρ, while if $\rho \in A_1$ then T_α is bounded from L^1_ρ to $L^{1,\infty}_\rho$.*

(ii) *If $0 < \alpha \le \alpha_p := n\beta\left(\frac{1}{2} - \left|\frac{1}{p} - \frac{1}{2}\right|\right)$, $\gamma = (\alpha - \alpha_p)/\alpha_p$ and $\rho \in A_p$ then T_α is bounded on $L^p_{\rho^\gamma}$.*

These two results were established in [3], in the same paper it was also shown that if $1 < p < \infty$ and $\rho(x) = |x|^\lambda$, where $\lambda \le -n$ or $\lambda \ge n(p-1)$, then T_α is not bounded on L^p_ρ. For extensions to the prototype non-translation invariant setting discussed in §3.1.3 see [6].

3.3. **Two-weight estimates for Calderón-Zygmund singular integrals.** Two-weight inequalities for Calderón-Zygmund singular integrals have been studied in [28] and [8] (see also [16], [15], [23], [24], [10] Chapter 8, and [14]).

Theorem F. *Let $1 < p < \infty$ and K be a Calderón-Zygmund kernel. We put $v(x) = v_0(|x|)\rho(x)$ and $w(x) = w_0(|x|)\rho(x)$, where v_0 and w_0 are positive monotonic functions on $(0,\infty)$ and $\rho \in A_p$. If v_0 and w_0 are increasing and*

$$B_2(v,w) < \infty$$

or if v_0 and w_0 are decreasing and

$$B_2'(v,w) < \infty$$

then S is bounded from L^p_w to L^p_v.

Conversely, if the Hilbert transform H is to be bounded from L^p_w to L^p_v then the weights v and w must satisfy conditions $B_2(v,w) < \infty$ and $B_2'(v,w) < \infty$.

For the *two-weight* weak-type inequality we have the following, see [9] and [10].

Theorem G. *Let $1 \le p < \infty$ and K be a Calderón-Zygmund kernel. We put $v(x) = v_0(x)\rho(x)$ and $w(x) = w_0(x)\rho(x)$, where v_0 and w_0 are positive increasing functions on $(0,\infty)$ and $\rho \in A_1$. Now if the weights v and w satisfy*

$$B_2^{(p)}(v,w) < \infty$$

if $1 < p < \infty$, and

$$B_2^{(1)}(v,w) < \infty$$

if $p = 1$, then T is bounded from L^p_w to $L^{p,\infty}_v$.

Conversely, if the Hilbert transform H is to be bounded from L^p_w to $L^{p,\infty}_v$ then the weights v and w must satisfy conditions $B_2^{(p)}(v,w) < \infty$ and $B_2^{(1)}(v,w) < \infty$.

3.4. **Hardy operators.** Before presenting the proofs of the main results, we formulate some well known statements concerning two-weight norm estimates for Hardy-type transforms defined on $\mathbf{R}^n$. The two-weight problem for the classical Hardy operator

$$\mathcal{H}f(x) = \int_0^x f(y)\,dy$$

has been solve in [27], [2], [22], and [26].

Let

$$\mathcal{H}_{\alpha,d}f(x) := \frac{1}{|x|^{n+\alpha}} \int_{|y| \le |x|/d} f(y)\,dy$$

and

$$\mathcal{H}'_{\alpha,d}f(x) := \int_{|y|\ge d|x|} \frac{f(y)}{|y|^{n+\alpha}}\, dy$$

for measurable $f : \mathbf{R}^n \to \mathbf{R}$, where $d > 1$.

For the following *two-weight* strong-type and weak-type (p,p) estimates see [10], Chapter 1.

Theorem H. *Let $1 < p < \infty$ and $\alpha \ge 0$.*

(i) *$\mathcal{H}_{\alpha,d}$ is bounded from L^p_w to L^p_v if and only if*

$$D_{\alpha,d}(v,w) := \sup_{t>0} \left(\int\limits_{t\le|x|} v(x)|x|^{-(n+\alpha)p} dx \right)^{1/p} \left(\int\limits_{|x|\le t/d} w^{-p'/p}(x)\, dx \right)^{1/p'} < \infty.$$

Moreover, there exists constants c_1 and c_2 such that

$$c_1 D_{\alpha,d}(v,w) \le \|\mathcal{H}_{\alpha,d}\|_{L^p_w \to L^p_v} \le c_2 D_{\alpha,d}(v,w).$$

(ii) *$\mathcal{H}'_{\alpha,d}$ is bounded from L^p_w to L^p_v if and only if*

$$D'_{\alpha,d}(v,w) := \sup_{t>0} \left(\int\limits_{|x|\le t/d} v(x)\, dx \right)^{1/p} \left(\int\limits_{t\le|x|} w^{-p'/p}(x)|x|^{-(n+\alpha)p'} dx \right)^{1/p'} < \infty.$$

Moreover, there exists constants c'_1 and c'_2 such that

$$c'_1 D'_{\alpha,d}(v,w) \le \|\mathcal{H}'_d\|_{L^p_w \to L^p_v} \le c'_2 D'_{\alpha,d}(v,w).$$

Remark 3. It is easy to see that for all $\alpha \ge 0$ one has

$$D_{\alpha,d}(v,w) \le B_{\alpha,d}(v,w) \quad \text{and} \quad D'_{\alpha,d}(v,w) \le B'_{\alpha,d}(v,w).$$

From Theorem H is easy to establish the following local result.

Corollary I. *Let $1 < p < \infty$, then the* two-weight *inequality*

$$\int_{|x-x_0|\le 1} \left|\mathcal{H}_{\alpha,d}|f_{x_0}|(x)\right|^p v(x-x_0)\, dx \le C_0 \int_{|x-x_0|\le 10} |f(x)|^p\, w(x-x_0)\, dx, \tag{7}$$

where $f_{x_0}(y) = f(y+x_0)$ holds if and only if $B^{\text{loc}}_{\alpha,d}(v,w) < \infty$. Moreover, there exists constants c_1 and c_2 such that if C_0 is the best possible constant in (7) then

$$c_1[B^{\text{loc}}_{\alpha,d}(v,w)]^p \le C_0 \le c_2[B^{\text{loc}}_{\alpha,d}(v,w)]^p.$$

Theorem J. *Let $1 \le p < \infty$ and $\alpha > -n$. $\mathcal{H}_{\alpha,d}$ is bounded from L^p_w to $L^{p,\infty}_v$ if and only if*

$$D^{(p)}_{\alpha,d}(v,w) := \sup_{0<t<\tau} \tau^{-n-\alpha} \left(\int\limits_{t\le|x|\le\tau} v(x)\, dx \right)^{1/p} \left(\int\limits_{|x|\le t/d} w^{-p'/p}(x)\, dx \right)^{1/p'} < \infty$$

when $1 < p < \infty$, and

$$D^{(1)}_{\alpha,d}(v,w) := \sup_{0<t<\tau} \tau^{-n-\alpha} \left(\int\limits_{t\le|x|\le\tau} v(x)\, dx \right) \left\| w^{-1} \right\|_{L^\infty(\{|\cdot|<t/d\})} < \infty$$

when $p = 1$ for some $d > 1$. Moreover, there exists constants c_1 and c_2 depending only on α and p such that

$$c_1 D_{\alpha,d}(v,w) \le \|\mathcal{H}_{\alpha,d}\|_{L^p_w \to L^{p,\infty}_v} \le c_2 D_{\alpha,d}(v,w).$$

This statement was proved in more generality in [10]. For *two-weight* weak-type estimates for the one-dimensional Hardy operator see [1].

Remark 4. For all $1 \le p < \infty$ and $\alpha \ge 0$ one has that

$$D^{(p)}_{\alpha,d}(v,w) \le B^{(p)}_{\alpha,d}(v,w).$$

4. Proof of Main Results

We shall need the following lemmata, they are easily established so we omit their proofs.

Lemma 1. *Let $1 < p < \infty$ and $\alpha \ge 0$. Suppose that $v(x) = v_0(|x|)$ and $w(x) = w_0(|x|)$, where v_0 and w_0 are positive monotonic functions on $(0,\infty)$.*

(i) *If v_0 and w_0 are increasing and*

$$B_{\alpha,d}(v,w) < \infty$$

for some $d > 1$, then there exists a positive constant C depending only on p, n, α, and d such that

$$v_0(dt) \le C[B_{\alpha,d}(v,w)]^p \; w_0(t)$$

for all $t > 0$.

(ii) *If v_0 and w_0 are decreasing and*

$$B'_{\alpha,d}(v,w) < \infty$$

for some $d > 1$, then there exists a positive constant C depending only on p, n, α, and d such that

$$v_0(t/d) \le C[B'_{\alpha,d}(v,w)]^p w_0(t)$$

for all $t > 0$.

Lemma 2. *Let $1 \le p < \infty$ and $\alpha \ge 0$. If $v(x) = v_0(|x|)$ and $w(x) = w_0(|x|)$, where v_0 and w_0 are positive increasing functions on $(0,\infty)$ satisfy, for some constant $d > 1$, the condition*

$$B^{(p)}_{\alpha,d}(v,w) < \infty$$

if $1 < p < \infty$, and

$$B^{(1)}_{\alpha,d}(v,w) < \infty$$

if $p = 1$, then there exists a positive constant C depending only on p, n, α, and d such that

$$v_0(dt) \le C[B^{(p)}_{\alpha,d}(v,w)]^p w_0(t)$$

for all $t > 0$.

When $\alpha = 0$ we have the following two lemmata, see [8], [9], and [10].

Lemma 3. *Let $1 < p < \infty$ and $\alpha = 0$. Suppose that $v(x) = v_0(|x|)\rho(x)$ and $w(x) = w_0(|x|)\rho(x)$, where v_0 and w_0 are positive monotonic functions on $(0,\infty)$ and $\rho \in A_p$.*

(i) *If v_0 and w_0 are increasing and*

$$B_d(v,w) < \infty$$

for some $d > 1$, then there exists a positive constant C depending only on p, n, and d such that

$$v_0(dt) \le C[B_d(v,w)]^p \; w_0(t)$$

for all $t > 0$.

(ii) *If v_0 and w_0 are decreasing and*

$$B'_d(v,w) < \infty$$

for some $d > 1$, then there exists a positive constant C depending only on p, n, and d such that

$$v_0(t/d) \le C[B'_d(v,w)]^p w_0(t)$$

for all $t > 0$.

Lemma 4. *Let $1 \le p < \infty$ and $\alpha = 0$. If $v(x) = v_0(|x|)\rho(x)$ and $w(x) = w_0(|x|)\rho(x)$, where v_0 and w_0 are positive increasing functions on $(0,\infty)$ and $\rho \in A_p$ satisfy, for some constant $d > 1$, the condition*

$$B_d^{(p)}(v,w) < \infty$$

if $1 < p < \infty$, and

$$B_d^{(1)}(v,w) < \infty$$

if $p = 1$, then there exists a positive constant C depending only on p, n, and d such that

$$v_0(dt) \le C[B_d^{(p)}(v,w)]^p w_0(t)$$

for all $t > 0$.

Proof of Theorem 1. We shall assume that v_0 and w_0 are increasing. Without loss of generality we can assume that the weight $v(x) = v_0(|x|)$ has the form

$$v(x) = v(0) + \int_0^{|x|} \varphi(t)\,dt, \tag{8}$$

where $\varphi \ge 0$ and $v(0) := \lim_{|x|\to 0} v(x)$. In fact there exists a sequence of absolutely continuous functions v_k such that

$$v_k(x) \le v(x) \quad \text{and} \quad \lim_{k\to\infty} v_k(x) = v(x),$$

that are given by

$$v_k(x) = v(0) + k\int_0^{|x|} [v_0(t) - v_0(t - \tfrac{1}{k})]\,dt.$$

Now using representation (8) we have

$$\int |Tf(x)|^p v(x)\,dx = \int |Tf(x)|^p v(0)\,dx + \int |Tf(x)|^p \Big(\int_0^{|x|} \varphi(t)\,dt\Big)\,dx =: I_1 + I_2.$$

Now if $v(0) = 0$ then $I_1 = 0$, while if $v(0) \ne 0$ it follows from the L^p boundedness of T and Lemma 1 (part (i)) that

$$I_1 \le v(0)\|T\|^p_{L^p\to L^p} \int |f(x)|^p dx \le C[B_{\alpha,d}(v,w)]^p \|T\|^p_{L^p\to L^p} \int |f(x)|^p w(x)\,dx.$$

For I_2 we have that

$$\begin{aligned} I_2 &= \int_0^\infty \varphi(t) \Big(\int_{|x|\ge t} |Tf(x)|^p dx\Big)\,dt \\ &\le 2^{p-1}\Bigg[\int_0^\infty \varphi(t)\Big(\int_{|x|\ge t} |Tf_{1,t}(x)|^p dx\Big)\,dt + \int_0^\infty \varphi(t)\Big(\int_{|x|\ge t} |Tf_{2,t}(x)|^p dx\Big)\,dt\Bigg] \\ &= I_{2,1} + I_{2,2}, \end{aligned}$$

where

$$f_{1,t}(x) = f(x)\chi_{\{|x|\geq t/d\}}(x) \ \text{ and } \ f_{2,t}(x) = f(x) - f_{1,t}(x).$$

Using again the L^p boundedness of T and Lemma 1 (part (i)) it follows that

$$\begin{aligned}
I_{2,1} &\leq \|T\|^p_{L^p\to L^p}\int_0^\infty \varphi(t)\Big(\int_{|x|\geq t/d}|f(x)|^p dx\Big)\,dt\\
&= \|T\|^p_{L^p\to L^p}\int |f(x)|^p\Big(\int_0^{d|x|}\varphi(t)\,dt\Big)\,dx\\
&\leq C[B_{\alpha,d}(v,w)]^p\|T\|^p_{L^p\to L^p}\int |f(x)|^p w(x)\,dx.
\end{aligned}$$

Using the fact that if $|x|\geq t$ and $|y|\leq t/d$ then $(d-1)|x|/d\leq |x-y|$ and Theorem H (part (i)) we see that

$$\begin{aligned}
I_{2,2} &\leq CA^p\int_0^\infty \varphi(t)\left(\int_{|x|\geq t}|x|^{-(n+\alpha)p}\Big(\int_{|y|\leq t/d}|f(y)|\,dy\Big)^p dx\right)\,dt\\
&\leq CA^p\int_0^\infty \varphi(t)\left(\int_{|x|\geq t}|x|^{-(n+\alpha)p}\Big(\int_{|y|\leq |x|/d}|f(y)|\,dy\Big)^p dx\right)\,dt\\
&\leq CA^p\int |x|^{-(n+\alpha)p}\Big(\int_{|y|\leq|x|/d}|f(y)|\,dy\Big)^p\Big(\int_0^{|x|}\varphi(t)\,dt\Big)\,dx\\
&\leq CA^p\int v(x)|x|^{-(n+\alpha)p}\big|\mathcal{H}_{\alpha,d}|f|(x)\big|^p dx\\
&\leq C[B_{\alpha,d}(v,w)]^pA^p\int |f(x)|^p w(x)\,dx.
\end{aligned}$$

This completes the proof in the case when v_0 and w_0 are increasing. The proof in the decreasing case follows in exactly then same manner using the representation

$$v(x) = v(\infty) + \int_{|x|}^\infty \varphi(t)\,dt,\ \ \varphi\geq 0,\ \ v(\infty) := \lim_{|x|\to\infty} v(x), \tag{9}$$

and part (ii) of both Theorem H and Lemma 1. □

Proof of Theorem 2. Using representation (8) we have

$$\int_{\{|Tf(x)|>\lambda\}} v(x)\,dx = v(0)|\{x : |Tf(x)|>\lambda\}| + \int_{\{|Tf(x)|>\lambda\}}\Big(\int_0^{|x|}\varphi(t)\,dt\Big)\,dx =: I_1 + I_2.$$

Now if $v(0)=0$ then $I_1=0$, while if $v(0)\neq 0$ it follows from the assumption that T is of weak-type (p,p) and Lemma 2 that

$$I_1 \leq v(0)\|T\|^p_{L^p\to L^{p,\infty}}\frac{1}{\lambda^p}\int |f(x)|^p dx \leq C[B^{(p)}_{\alpha,d}(v,w)]^p\|T\|^p_{L^p\to L^{p,\infty}}\frac{1}{\lambda^p}\int |f(x)|^p w(x)\,dx.$$

To estimate I_2 we introduce the following notation:

$$\begin{aligned}
J_t(\lambda) &= \{x : |Tf(x)|>\lambda\}\cap\{x : |x|\geq t\}\\
J_{1,t}(\lambda) &= \{x : |Tf_{1,t}(x)|>\lambda/d\}\cap\{x : |x|\geq t\}\\
J_{2,t}(\lambda) &= \{x : |Tf_{2,t}(x)|>\lambda/d\}\cap\{x : |x|\geq t\},
\end{aligned}$$

where again

$$f_{1,t}(x) = f(x)\chi_{\{|x|\geq t/d\}}(x) \ \text{ and } \ f_{2,t}(x) = f(x) - f_{1,t}(x).$$

Now it is easy to see that

$$I_2 = \int_0^\infty \varphi(t)\,|J_t(\lambda)|\,dt \le \int_0^\infty \varphi(t)\,|J_{1,t}(\lambda)|\,dt + \int_0^\infty \varphi(t)\,|J_{2,t}(\lambda)|\,dt = I_{2,1} + I_{2,2}.$$

Using again that T is of weak-type (p,p) and Lemma 2 it follows that

$$\begin{aligned} I_{2,1} &\le \|T\|^p_{L^p\to L^{p,\infty}} \frac{1}{\lambda^p} \int_0^\infty \varphi(t) \Big(\int_{|x|\ge t/d} |f(x)|^p dx \Big)\,dt \\ &= \|T\|^p_{L^p\to L^{p,\infty}} \int |f(x)|^p \Big(\int_0^{d|x|} \varphi(t)\,dt \Big)\,dx \\ &\le C[B^{(p)}_{\alpha,d}(v,w)]^p \|T\|^p_{L^p\to L^{p,\infty}} \int |f(x)|^p w(x)\,dx. \end{aligned}$$

Using, as in the proof of Theorem 1, the fact that $|x| \ge t$ and $|y| \le t/d$ ensures $(d-1)|x|/d \le |x-y|$ and Theorem J we see that

$$\begin{aligned} I_{2,2} &\le \int_0^\infty \varphi(t)\,\big|\{x : |x| \ge t\} \cap \{x : \mathcal{H}_{\alpha,d}|f|(x) > \lambda/d'\}\big|\,dt \\ &= \int_{\{\mathcal{H}_{\alpha,d}|f|(x) > \lambda/d'\}} \Big(\int_0^{|x|} \varphi(t)\,dt \Big)\,dx \\ &\le \int_{\{\mathcal{H}_{\alpha,d}|f|(x) > \lambda/d'\}} v(x)\,dx \\ &\le C[B^{(p)}_{\alpha,d}(v,w)]^p \frac{1}{\lambda^p} \int |f(x)|^p w(x)\,dx, \end{aligned}$$

where $d' = d\left(\frac{d}{d-1}\right)^{n+\alpha}$. □

The proofs of Theorems 4 and 6 are similar to those for Theorems 1 and 2 above, one simply instead uses the *one-weight* strong-type and weak-type (p,p) assumptions respectively together with Lemmata 3 and 4.

Arguing as in the proof of Theorem 1 and using Corollary I one can easily obtain Theorem 3.

Before proving Theorems 7 and 8, we present the following Lemma.

Lemma 5. *If* $|x| \ge \dfrac{4nA_0}{A_1}t$, *then*

$$|Tf(x)| \ge \frac{A_1}{4}|x|^{-n} \int_{|y|\le t} f(y)\,dy \tag{10}$$

for all non-negative f *supported in* $B(0,t)$.

Proof. It follows from (4a) and (4b) that

$$\left|\widetilde{K}(x-y) - \widetilde{K}(x)\right| \le \frac{A_1}{4}|x|^{-n} \tag{11}$$

whenever $|x| \ge \frac{4nA_0}{A_1}|y|$ and that either

$$\left|\operatorname{Re}\widetilde{K}(x)\right| \ge \frac{A_1}{2}|x|^{-n} \quad \text{or} \quad \left|\operatorname{Im}\widetilde{K}(x)\right| \ge \frac{A_1}{2}|x|^{-n}.$$

Lets assume that $\big|\operatorname{Re}\widetilde{K}(x)\big| \geq \dfrac{A_1}{2}|x|^{-n}$, then it follows from (11) that

$$\Big|\big|\operatorname{Re}\widetilde{K}(x-y)\big| - \big|\operatorname{Re}\widetilde{K}(x)\big|\Big| \leq \Big|\widetilde{K}(x-y) - \widetilde{K}(x)\Big| \leq \frac{1}{2}\big|\widetilde{K}(x)\big|,$$

whenever $|x| \geq \frac{4nA_0}{A_1}|y|$ and thus that

$$\frac{1}{2}\big|\operatorname{Re}\widetilde{K}(x)\big| \leq \big|\operatorname{Re}\widetilde{K}(x-y)\big| \leq \frac{3}{2}\big|\operatorname{Re}\widetilde{K}(x)\big| \tag{12}$$

It is then immediate from the continuity of $\widetilde{K}$ on $\mathbf{R}^n \setminus \{0\}$ that $\operatorname{Re}\widetilde{K}(x-y)$ does not change sign for $|y| \leq \dfrac{A_1}{4nA_0}|x|$.

If we now let $0 < t \leq \dfrac{A_1}{4nA_0}|x|$ and

$$f_t(y) = f(y)\chi_{\{|y|\leq t\}},$$

from (12) it then follows that

$$|Tf_t(x)| \geq \int_{|y|\leq t} f(y)\big|\operatorname{Re}\widetilde{K}(x-y)\big|\,dy \geq \frac{A_1}{4}|x|^{-n}\int_{|y|\leq t} f(y)\,dy.$$

Arguing in a similar manner for the case where $|\operatorname{Im}\widetilde{K}(x)| \geq \dfrac{A_1}{2}|x|^{-n}$ we obtain the same conclusion.

□

Proof of Theorems 7 and 8. Let us first prove Theorem 8. We consider the case $p > 1$, the case $p = 1$ is similar. We claim that if the operator T is bounded from L^p_w to $L^{p,\infty}_v$, then

$$I(r) := \int_{|x|<r} w^{-p'/p}(x)dx < \infty \tag{13}$$

for all $r > 0$.

Indeed, first observe that $I(r) = \|w^{-1/p}\chi_{|\cdot|<r}\|^{p'}_{L^p}$. If $I(r) = \infty$ for some $r > 0$, then by the duality properties there exists non-negative $g \in L^p$ supported in $B(0,r)$ such that $\int_{|\cdot|<r} gw^{-1/p} = \infty$.

Let us take the function $f_r(y) = g(y)w^{-1/p}(y)\chi_{\{|y|<r\}}$. Then by Lemma 5 we have

$$|T_rf(x)| \geq \frac{A_1}{4}|x|^{-n}\int_{|y|\leq r} g(y)w^{-1/p}(y)dy = \infty,$$

whenever $|x| > \frac{4nA_0}{A_1}r$.

Due to *two-weight* weak-type inequality and the latter estimate we have

$$\int_{|x|>\frac{4nA_0}{A_1}r} v(x)dx \leq \int_{\{x:|Tf_r(x)|>\lambda\}} v(x)dx \leq \frac{c}{\lambda^p}\int_{|y|<r} g(y)dy < \infty$$

for all positive λ. Consequently, passing λ to ∞ we find that the left-hand side of the latter inequality is equal to 0 which contradicts the assumption that the weight v is positive almost everywhere.

Now let us derive the condition $B^{(p)}_{\frac{4nA_0}{A_1}}(v,w) < \infty$.

Applying Lemma 5 we conclude that

$$|Tf(x)| \geq \frac{A_1}{4}|x|^{-n}\int_{|y|<\frac{A_1}{4nA_0}t} w^{-p'/p}(y)dy \geq \frac{A_1}{4}\tau^{-n}I\big(\frac{A_1}{4A_0n}t\big) \tag{14}$$

whenever $0 < t \le |x| < \tau$ and $f(y) = w^{-p'/p}(y)\chi_{\{|y|<\frac{A_1}{4nA_0}t\}}(y)$.

The *two-weight* weak-type inequality for T leads to the estimates

$$\int_{t<|x|<\tau} v(x)dx \le \int_{\{x:|Tf(x)|\ge (A_1\tau^{-n}/4)I\left(\frac{A_1}{4nA_0}\right)\}}$$

$$\le \left(\frac{4\tau^n\|T\|_{L^p_w\to L^{p,\infty}}}{A_1}\right)^p \frac{1}{I^p\left(\frac{A_1}{4nA_0}t\right)} I\left(\frac{A_1}{4nA_0}t\right) < \infty$$

for all t, τ, $0 < t < \tau < \infty$. This completes the proof of Theorem 8.

To prove Theorem 7 we observe that due to (12) which is true also for all $|x| \ge t$ because of Lemma 5, we have

$$\|Tf\|^p_{L^p_v} \ge \int_{|x|>t} |Tf(x)|^p v(x)dx \ge \frac{A_1}{4}\left(\int_{|x|>t} |x|^{-np}v(x)dx\right)\left(\int_{|y|<\frac{A_1}{4A_0 n}t} w^{-p'/p}(y)dy\right)^p. \tag{15}$$

On the other hand, by (11) we have

$$\|f\|^p_{L^p_w} = \int_{|x|<\frac{A_1}{4nA_0}} w^{-p'/p}(x)dx < \infty.$$

Finally, from the boundedness of T from L^p_w to L^p_v we conclude that $B_{\frac{4A_0 n}{A_1}}(v,w) < \infty$. ☐

Appendix

Here we shall verify the statement made in Remark 1. We first note that if the measure

$$w^{-p'/p}(E) = \int_E w^{-p'/p}(x)\,dx$$

is doubling then it also satisfies the reverse doubling condition: that there exists constants η_1, $\eta_2 > 1$ such that for all $t > 0$ the inequality

$$\int_{|x|\le \eta_1 t} w^{-p'/p}(x)\,dx \ge \eta_2 \int_{|x|\le t} w^{-p'/p}(x)\,dx$$

holds, see [35] page 21.

Using this fact we find that

$$\int\limits_{\eta_1^k t\le|x|\le\eta_1^{k+1}t} w^{-p'/p}(x)\,dx = \int\limits_{|x|\le\eta_1^{k+1}t} w^{-p'/p}(x)\,dx \;-\int\limits_{|x|\le\eta_1^k t} w^{-p'/p}(x)\,dx \ge (\eta_2-1)\eta_2^k \int\limits_{|x|\le t} w^{-p'/p}(x)\,dx,$$

and hence

$$\int\limits_{|x|\le t} w^{-p'/p}(x)\,dx \le \frac{1}{(\eta_2-1)\eta_2^k} \int\limits_{\eta_1^k t\le|x|\le\eta_1^{k+1}t} w^{-p'/p}(x)\,dx. \tag{16}$$

Arguing as in the proof of Corollary 5 leads to the following string of inequalities

$$\begin{aligned}
B_{\alpha,d}(v,w) &= \sup_{t>0}\Bigg(\int\limits_{t\le|x|} v(x)(|x|^{-\alpha}+1)^p|x|^{-np}dx\Bigg)^{1/p}\Bigg(\int\limits_{|x|\le t/d} w^{-p'/p}(x)\,dx\Bigg)^{1/p'}\\
&= \sup_{t>0}\sum_{k=0}^{\infty}\Bigg(\int\limits_{\eta_1^k t\le|x|<\eta_1^{k+1}t} v(x)|x|^{-np}dx\Bigg)^{1/p}\Bigg(\int\limits_{|x|\le t/d} w^{-p'/p}(x)\,dx\Bigg)^{1/p'}\\
&\le \sup_{t>0}\sum_{k=0}^{\infty}\frac{(\eta_1^k t)^{-n}[(\eta_1^k t)^{-\alpha}+1]}{[(\eta_2-1)\eta_2^k]^{1/p'}}\Bigg(\int\limits_{\eta_1^k t\le|x|<\eta_1^{k+1}t} v(x)\,dx\Bigg)^{1/p}\Bigg(\int\limits_{\eta_1^k t\le|x|<\eta_1^{k+1}t} w^{-p'/p}(x)\,dx\Bigg)^{1/p'}\\
&\le A_\alpha(v,w)\sum_{k=0}^{\infty}\frac{1}{[(\eta_2-1)\eta_2^k]^{1/p'}}\\
&\le CA_\alpha(v,w).
\end{aligned}$$

Acknowledgements

The second and third authors would like to express their gratitude to Prof. Fulvio Ricci and thank him for his warmth, generosity and support.

References

[1] K. Andersen and B. Muckenhoupt, *Weighted weak type inequalities with applications to Hilbert transforms and maximal functions*, Studia Math., 72 (1982), pp. 9–26.
[2] J. Bradley, *Hardy inequality with mixed norms*, Canad. Math. Bull., 21 (1978), pp. 405–408.
[3] S. Chanillo, *Weighted norm inequalities for strongly singular convolution operators*, Trans. Amer. Math. Soc., 281 #1 (1984), pp. 77–107.
[4] S. Chanillo and M. Christ, *Weak* $(1,1)$ *bounds for oscillatory singular integrals*, Duke Math. J., 55 (1987), pp. 141–155.
[5] S. Chanillo, D. Kurtz, and G. Sampson, *Weighted weak* $(1,1)$ *and weighted* L^p *estimates for oscillatory kernels*, Trans. Amer. Math. Soc., 295 (1986), pp. 127–145.
[6] S. Chanillo and A. Torchinsky, *Sharp function and weighted* L^p *estimates for a class of pseudodifferential operators*, Ark. Mat., 24 #1 (1986), pp. 1–25.
[7] R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math., 51 (1974), pp. 241–249.
[8] D. Edmunds and V. Kokilashvili, *Two-weight inequalities for singular integrals*, Canadian Math. Bull., 38 (1995), pp. 119–125.
[9] D. Edmunds, V. Kokilashvili, and A. Meskhi, *Two-weight estimates for singular integrals defined on spaces of homogeneous type*, Canadian J. Math., 52 #3 (2000), pp. 468–502.
[10] ———, *Bounded and compact integral operators*, Kluwer, Dordrecht, Boston, London, 2002.
[11] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math., 124 (1970), pp. 9–36.
[12] ———, L^p *bounds for pseudo-differential operators*, Israel J. Math., 14 (1973), pp. 413–417.
[13] J. Garcia-Cuerza and J. Rubio de Francia, *Weighted norm inequalities and related topics*, North Holland, Amsterdam, New York, Oxford, 1985.
[14] I. Genebashvili, A. Gegatishvili, V. Kokilashvili, and M. Krbec, *Weight theory for integral transforms on spaces of homogeneous type* Pitman Monographs and surveys in Pure and Applied Mathematics ***92***, Longman, Harlow, 1992.
[15] V. Guliev, *Two-weight* L^p *inequality for singular integral operators on Heisenberg groups*, Georgian Math. J., 1 #4 (1994), pp. 367–376.
[16] E. Gusseinov, *Singular integrals in the space of functions summable with monotone weight (Russian)*, Mat. Sb., 132 (174) #1 (1977), pp. 28–44.

[17] I. I. Hirschman, *Multiplier Transforms I*, Duke Math. J., 26 (1956), pp. 222–242.
[18] S. Hoffman, *Singular integrals with power weights*, Proc. Amer. math. Soc., 110 (1990), pp. 343–353.
[19] Y. Hu, *A weighted norm inequality for oscillatory singular integrals,* Harmonic Analysis *(Tianjin 1988), Lecture notes in Math.*, Springer Verlag, Berlin, 1994.
[20] Y. Hu and Y. Pan, *Boundedness of oscillatory singular integrals on Hardy spaces*, Ark. Mat., 30 #2 (1992), pp. 311–320.
[21] R. Hunt, B.Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the congugate function and Hilbert transform*, Trans. Amer. Math. Soc., 176 (1973), pp. 227–251.
[22] V. Kokilashvili, *On Hardy's inequalities in weighted spaces (Russian)*, Soobsch. Akad. Nauk Gruz. SSR, 96 (1979), pp. 37–40.
[23] V. Kokilashvili and A. Meskhi, *Two-weight inequalities for singular integrals defined on homogeneous groups*, Proc. A. Razmzdze Math. Inst., 112 (1997), pp. 57–90.
[24] ———, *Two-weight inequalities for singular integrals defined on homogeneous groups*, in Lecture Notes in Pure and Applied Mathematics, 213, *Function Spaces V*, Proceedings of the Conference, Poznań, Poland, M. Mudzik and L. Skrzypczak, eds., Marcel Dekker, 2000.
[25] N. Lyall, *A class of strongly singular Radon transforms on the Heisenberg group*. Preprint, 2004.
[26] V. Maz'ya, *Sobolev spaces*, Springer, Berlin, 1985.
[27] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math., 44 (1972), pp. 31–38.
[28] B. Muckenhoupt and R. Wheeden, *Two-weight function norm inequalities for the Hardy-Littlewood maximal function and Hilbert transform*, Studia Math., 55 #3 (1976), pp. 279–294.
[29] Y. Pan, *Oscillatory singular integrals on L^p and Hardy spaces*, Proc. Amer. Math. Soc., 124 #9 (1996), pp. 2821–2825.
[30] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals I. Oscillatory integrals*, J. Funct. Anal., 73 (1987), pp. 179–194.
[31] S. Sato, *Weighted weak type* $(1, 1)$ *estimates for oscillatory singular integrals*, Studia Math., 141 #1 (2000), pp. 1–24.
[32] F. Soria and G. Weiss, *A remark on singular integrals and power weights*, Indiana Univ. Math. J., 93 #1 (1994), pp. 187–204.
[33] E. M. Stein, *A note on singular integrals*, Proc. Amer. Math. Soc., 8 (1957), pp. 250–254.
[34] ———, *Harmonic Analysis: Real–Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, 1993.
[35] J. O. Strömberg and A. Torchinsky, *Weighted Hardy spaces, Lecture Notes in Math. 1381*, Springer Verlag, Berlin, 1989.
[36] S. Wainger, *Special Trigonometric Series in k Dimensions,* Memoirs of the AMS ***59***, American Math. Soc., 1965.

Authors' addresses:

V. Kokilashvili and A. Meskhi: A. Razmadze Mathematical Institute, Georgian Academy of Sciences, 1, M. Aleksidze St., 0193 Tbilisi, Georgia

N. Lyall: Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

A. Meskhi's current address: Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

N. Lyall's current address: Department of Mathematics, University of Georgia, Athens GA 30602 USA
e-mail: lyall@math.uga.edu