SÁRKÖZY'S THEOREM

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1. Introduction

The purpose of this expository note is to give a self contained proof (modulo the Weyl inequality) of the following result, due independently to Sárközy [5] and Furstenberg [1].

Theorem 1. *Let $\delta > 0$, then there exists $N_0 = N_0(\delta)$ such that if $N \geq N_0$ and $A \subseteq [1, N]$ with $|A| \geq \delta N$, then A necessarily contains distinct elements a, b whose difference $a - b$ is a perfect square.*

We will closely follow the approach taken in Lyall and Magyar [3] and employ a *density increment strategy* to obtain the result above with

$$N_0 = \exp(C\delta^{-1} \log \delta^{-1}). \tag{1}$$

This is equivalent to the statement that if $A \subseteq [1, N]$ and $d^2 \notin A - A$ for any $d \neq 0$, then necessarily

$$\frac{|A|}{N} \leq C \frac{\log \log N}{\log N}. \tag{2}$$

2. Dichotomy between randomness and arithmetic structure

Our approach will be to obtain a dichotomy between randomness and structure of the following form.

Proposition 2. *Let $A \subseteq [1, N]$ and $\delta = |A|/N$. If $N \geq \delta^{-C}$ with C sufficiently large, then either*

$$\sum_{d=1}^{N} \left| A \cap \left(A + d^2 \right) \right| \geq \frac{1}{12} \, \delta |A| N^{1/2} \tag{3}$$

or there exists a square difference *arithmetic progression P in $[1, N]$ of length $|P| \geq c\delta^5 N$ such that*

$$|A \cap P| > \delta(1 + c\delta)|P|. \tag{4}$$

Proof that Proposition 2 implies Theorem 1 (with bound (1)). Suppose $A \subseteq [1, N]$, with $|A| \geq \delta N$, contains no square differences, then Proposition 2 allows us to perform an iteration. At the kth step of this iteration we will have a set $A_k \subseteq [1, N_k]$ of size $\delta_k N_k$. This set will be an appropriately rescaled version of a subset of A itself and hence will also contain no square differences.

Let $A_0 = A$, $N_0 = N$ and $\delta_0 = |A|/N \geq \delta$. Proposition 2 ensures that either

$$N_k \leq \delta_k^{-C} \tag{5}$$

or else the iteration proceeds allowing us to choose N_{k+1}, δ_{k+1} and A_{k+1} such that

$$N_{k+1} \geq c\delta_k^5 N_k$$

and

$$\delta_{k+1} \geq \delta_k + c\delta_k^2.$$

Now as long as the iteration continues we must have $\delta_k \leq 1$, and so after $O(\delta^{-1})$ iterations condition (5) must be satisfied, giving

$$\delta^{C\delta^{-1}} N \leq \delta^{-C} \quad \Longleftrightarrow \quad \log N \leq C\delta^{-1} \log \delta^{-1}. \qquad \square$$

The rest of this note is devoted to the proof of Proposition 2. As opposed to the standard L^∞ increment strategy of Roth, we will obtain the dichotomy in Proposition 2 by exploiting the concentration of the L^2 mass of the Fourier transform (this sometimes referred to as an energy increment strategy).

3. Setting the stage for the proof of Proposition 2

Let $A \subseteq [1,N]$, with $N \geq \delta^{-C}$, where $\delta = |A|/N$ and C is a sufficiently large constant. Our approach will be to assume that A exhibits neither of the two properties described in Proposition 2 and then seek a contradiction.

3.1. **A simple consequence of A being non-random.** If we were to suppose that A is non-random, in the sense that inequality (3) does not hold, then it would immediately follows that

$$\sum_{m,n\in\mathbb{Z}} 1_A(m)1_A(n)1_S(m-n) \leq \frac{1}{4}\,\delta|A||S| \tag{6}$$

where

$$S = \{d^2 \,:\, 1 \leq d \leq \sqrt{N/9}\}. \tag{7}$$

3.2. **A simple consequences of A being non-structured.** If we were to assume that A is *regular*, in the sense that A in fact satisfies the inequality

$$|A \cap P| \leq \delta(1+\varepsilon)|P|$$

for all arithmetic progressions $P \subseteq [1,N]$ of the form

$$P = \{m + \ell q^2 \,:\, 1 \leq \ell \leq L\} \tag{8}$$

with $L \geq \delta^4 \varepsilon N$, then the set $A \cap (N/9, 8N/9]$ must contain most of the elements of A. In particular

$$|A \cap (N/9, 8N/9]| \geq (3/4)|A|, \tag{9}$$

since if this were not the case we would immediately obtain a progression $P \subseteq [1,N]$ of the form (8) with $q = 1$ and $L \geq N/9$ such that $|A \cap P| \geq \delta(1+1/8)|P|$.

3.3. **The balance function.** We define the *balance function of A* to be

$$f_A = 1_A - \delta 1_{[1,N]}. \tag{10}$$

We note that f_A has mean value zero, that is $\sum f_A(m) = 0$, this will be critically important later.

It easy to verify that if A satisfies inequalities (6) and (9), then

$$\sum_{m,n\in\mathbb{Z}} f_A(m)f_A(n)1_S(m-n) \leq -\frac{1}{4}\,\delta|B||S|. \tag{11}$$

One can see this by simply expanding the sum into a sum of four sums, one involving only the function 1_A on which we can apply (6), two involving the functions 1_A and $-\delta 1_{[1,N]}$ on which we can apply (9), and one involving only the function $-\delta 1_{[1,N]}$ which can be estimated trivially.

3.4. **Fourier analysis on $\mathbb{Z}$.** If $f : \mathbb{Z} \to \mathbb{C}$ and has finite support, then we define its *Fourier transform* by

$$\widehat{f}(\alpha) = \sum_{m\in\mathbb{Z}} f(m)e^{-2\pi i m\alpha}.$$

The finite support assumption on f ensures that $\widehat{f}$ is a continuous function on the circle, and in this setting the Fourier inversion formula and Plancherel's identity, namely

$$f(m) = \int_0^1 \widehat{f}(\alpha)e^{2\pi i m\alpha}d\alpha \qquad \text{and} \qquad \int_0^1 |\widehat{f}(\alpha)|^2 d\alpha = \sum_{m\in\mathbb{Z}} |f(m)|^2$$

are simply immediate consequences of the familiar orthogonality relation

$$\int_0^1 e^{2\pi i m\alpha}d\alpha = \begin{cases} 1 & \text{if } \ m = 0 \\ 0 & \text{if } \ m \neq 0 \end{cases}.$$

It is then easy to verify that from inequality (11) we immediately obtain the estimate

$$\int_0^1 |\widehat{f_A}(\alpha)|^2 |\widehat{1_S}(\alpha)|\, d\alpha \geq \frac{1}{4}\,\delta|B||S| \tag{12}$$

where we recognize

$$\widehat{1_S}(\alpha) = \sum_{d=1}^{\sqrt{N/9}} e^{-2\pi i d^2 \alpha}, \tag{13}$$

as a classical Weyl sum.

3.5. **Estimates for Weyl sums.** Let $M = \sqrt{N/9}$. We note that whenever $|\alpha| \ll M^{-2}$ there can be no cancellation in the Weyl sum (13), in fact the same is also true when α is close to a rational with *small* denominator (i.e. there is no cancellation over sums in residue classes modulo q).

We now state a precise formulation of the fact that this is indeed the only obstruction to cancellation. For $\eta > 0$ we define

$$\mathbf{M}_q = \mathbf{M}_q(\eta) = \left\{ \alpha \in [0,1] \,:\, \left|\alpha - \frac{a}{q}\right| \leq \frac{1}{\eta^2 M^2} \text{ for some } a \in [1,q] \right\}. \tag{14}$$

Proposition 3. *Let $\eta > 0$.*

(i) (Minor arc estimate) *If $\alpha \notin \mathbf{M}_q$ for any $1 \leq q \leq \eta^{-2}$, then*

$$|\widehat{1_S}(\alpha)| \leq \eta M + O(M^{1-1/40}).$$

(ii) (Major arc estimate) *If $\alpha \in \mathbf{M}_q$ for some $1 \leq q \leq \eta^{-2}$, then*

$$|\widehat{1_S}(\alpha)| \leq C q^{-1/2} M + O(M^{1/2}).$$

The proof of this result is a straightforward (and presumably well known) consequence of the standard estimates for Weyl sums, for the sake of completeness we include these arguments in an appendix.

4. The proof of Proposition 2

In the previous section we established that inequalities (6) and (9) would be immediate consequences of A not exhibiting either of the two properties described in Proposition 2. We now present the two lemmas from which we will obtain our desired contradiction.

In both lemmas below we assume that $A \subseteq [1,N]$ and $N \geq \delta^{-C}$, where $\delta = |A|/N$ and C is a sufficiently large constant. Moreover, we set $\eta = c\delta$, where c is a sufficiently small constant.

Lemma 4. *If A is neither random nor structured, in the sense outlined in Proposition 2, then there exists $1 \leq q \leq \eta^{-2}$ such that*

$$\frac{1}{\delta|A|} \int_{\mathbf{M}_q} |\widehat{f_A}(\alpha)|^2 d\alpha \geq c\delta. \tag{15}$$

The second lemma is a precise quantitative formulation, in our setting, of the standard L^2 density increment lemma.

Lemma 5. *Let $\varepsilon \leq \eta^2/4\pi$. If A is* regular*, in the sense that*

$$|A \cap P| \leq \delta(1+\varepsilon)|P|$$

for all progressions $P \subseteq [1,N]$ of the form (8) with $q^2 L \geq \eta^2 \varepsilon N$, then

$$\frac{1}{\delta|A|} \int_{\mathbf{M}_q} |\widehat{f_A}(\alpha)|^2 d\alpha \leq 12\varepsilon. \tag{16}$$

We therefore obtain a contradiction if $\varepsilon \leq c\delta$, proving Proposition 2.

4.1. **Proof of Lemma 4.** It follows from the minor arc estimate of Proposition 3 (since $N \geq \delta^{-C}$ for C sufficiently large) and Plancherel's identity that

$$\int\limits_{\text{minor arcs}} |\widehat{f_A}(\alpha)|^2 |\widehat{1_S}(\alpha)|\, d\alpha \leq C\eta M|A|.$$

Therefore, if $\eta = \delta/8C$, it follows from estimate (12) and the major arc estimate of Lemma 3 that

$$\sum_{q=1}^{\eta^{-2}} q^{-1/2} \int_{\mathbf{M}_q} |\widehat{f_A}(\alpha)|^2 d\alpha \geq \eta|A|.$$

It therefore follows that

$$\max_{1\leq q\leq \eta^{-2}} \int_{\mathbf{M}_q} |\widehat{f_A}(\alpha)|^2 d\alpha \geq \eta^2|A|$$

as required. □

4.2. **Proof of Lemma 5.** We fix q and L so that $q^2L = \eta^2\varepsilon N$ and define

$$P = \{-\ell q^2 \,|\, 1 \leq \ell \leq L\}.$$

Claim 1. *If $\alpha \in \mathbf{M}_q$, then $|\widehat{1_P}(\alpha)| \geq |P|/2$.*

Proof of Claim 1. Since

$$L\|q^2\alpha\| \leq Lq^2\eta^{-2}M^{-2} = \eta^{-2}\varepsilon,$$

for all $\alpha \in \mathbf{M}_q$, where $\|\cdot\|$ denotes the distance to the nearest integer, it follows that

$$|\widehat{1_P}(\alpha)| \geq |P| - \sum_{\ell=1}^{L} \big|e^{2\pi i\ell q^2\alpha} - 1\big| \geq |P|\Big(1 - 2\pi L\|q^2\alpha\|\Big) \geq |P|/2,$$

for all $\alpha \in \mathbf{M}_q$, provided $\varepsilon \leq \eta^2/4\pi$. □

We now note that Plancherel's identity (applied to the function $f_A * 1_P$) and Claim 1 imply that

$$\frac{1}{\delta|A|}\int_{\mathbf{M}_q} |\widehat{f_A}(\alpha)|^2 d\alpha \leq \frac{4}{\delta|A||P|^2}\sum_{m\in\mathbb{Z}} |f_A * 1_P(m)|^2. \tag{17}$$

The conclusion of Lemma 5 will therefore be an immediate consequence of the following.

Claim 2. *As a consequence of the assumptions in Lemma 5 if follows that*

$$\sum_{m\in\mathbb{Z}} |f_A * 1_P(m)|^2 \leq 3\varepsilon\,\delta|A||P|^2.$$

Proof of Claim 2. We let

$$\mathcal{M} = \{m \in \mathbb{Z} \,|\, m - P \subseteq [1, N]\}$$
$$\mathcal{E} = ([1, N] + P) \setminus \mathcal{M}$$

and write

$$\sum_{m\in\mathbb{Z}} |f_A * 1_P(m)|^2 = \sum_{m\in\mathcal{M}} |f_A * 1_P(m)|^2 + \sum_{m\in\mathcal{E}} |f_A * 1_P(m)|^2.$$

We note that since

$$f_A * 1_P(m) = |A \cap (m - P)| - \delta|[1, N] \cap (m - P)|$$

it follows from our *regularity* assumption on A that if $m \in \mathcal{M}$, then

$$-\delta L \leq f_A * 1_P(m) \leq \delta\varepsilon L,$$

while for $m \in \mathcal{E}$ we can only conclude that

$$|f_A * 1_P(m)| \leq L.$$

Now since f_A has mean value zero the convolution

$$f_A * 1_P(m) = \sum_n f_A(n)1_P(m - n)$$

also has mean value zero. Thus, using the fact that $|g| = 2g_+ - g$, where $g_+ = \max\{g, 0\}$ denotes the *positive-part* function, and the trivial size estimate $|\mathcal{M}| \leq N$, we can deduce that

$$\begin{aligned}\sum_{m\in\mathcal{M}} |f_A * 1_P(m)|^2 &\leq 2\left(\sup_{m\in\mathcal{M}} |f_A * 1_P(m)|\right) \sum_{m\in\mathcal{M}} (f_A * 1_P)_+(m)\\ &\leq 2(\delta L)(\delta\varepsilon L)|\mathcal{M}|\\ &\leq 2\delta^2\varepsilon L^2 N.\end{aligned}$$

Using the fact that $|\mathcal{E}| \leq 2q^2L = 2\eta^2\varepsilon N$, it follows that

$$\sum_{m\in\mathcal{E}} |f_A * 1_P(m)|^2 \leq |P|^2|\mathcal{E}| \leq \frac{1}{2}\sum_{m\in\mathcal{M}} |f_A * 1_P(m)|^2,$$

provided $2\eta^2 \leq \delta^2$.

This concludes the proof of Claim 2 and establishes Lemma 5. □

Appendix A. Weyl Sum Estimates: Proof of Proposition 3

A.1. **Standard estimates for Weyl sums.** Let

$$S_M(\alpha) = \sum_{d=1}^{M} e^{-2\pi i d^2\alpha}.$$

Proposition 6 (The Weyl inequality)**.** *If $|\alpha - a/q| \leq q^{-2}$ and $(a, q) = 1$, then*

$$|S_M(\alpha)| \leq 20M\log M(1/q + 1/M + q/M^2)^{1/2}.$$

The proof of this result is completely standard, see for example [4] or [2]. We remark that this gives a non-trivial estimate whenever $M^\mu \leq q \leq M^{2-\mu}$ for some $0 < \mu < 1$.

Let

$$\mathbf{M}'_{a/q} = \left\{\alpha \in [0, 1] : \left|\alpha - \frac{a}{q}\right| \leq \frac{1}{M^{2-1/10}}\right\}. \tag{18}$$

We say that α is in a *minor arc* if $\alpha \notin \mathbf{M}'_{a/q}$ for any $(a, q) = 1$ with $1 \leq q \leq M^{1/10}$.

Lemma 7 (Minor arc estimate I)**.** *If $\alpha \notin \mathbf{M}'_{a/q}$ for any $(a, q) = 1$ with $1 \leq q \leq M^{1/10}$, then*

$$|S_M(\alpha)| \leq CM^{1-1/40}. \tag{19}$$

While on the complement of these minor arcs (the major arcs) we have the follow important estimate.

Lemma 8 (Major arc estimate)**.** *If $\alpha \in \mathbf{M}'_{a/q}$ for some $(a, q) = 1$ with $1 \leq q \leq M^{1/10}$, then*

$$|S_M(\alpha)| \leq CMq^{-1/2}(1 + M^2|\alpha - a/q|)^{-1/2} + O(M^{1/2}). \tag{20}$$

For the proofs of these two lemmas see Section A.3.

A.2. **Refinement of the major arcs.** We now let $0 < \eta \leq 1$ and define

$$\mathbf{M}_{a/q} = \left\{\alpha : \left|\alpha - \frac{a}{q}\right| \leq \frac{1}{\eta^2 M^2}\right\}. \tag{21}$$

Lemmas 7 and 8 combine to give the following result from which Proposition 3 is an immediate consequence.

Lemma 9 (Minor arc estimate II)**.** *If $\alpha \notin \mathbf{M}_{a/q}$ for any $(a, q) = 1$ with $1 \leq q \leq \eta^{-2}$, then*

$$|S_M(\alpha)| \leq \eta M + O(M^{1-1/40}).$$

Proof. It follow from Lemma 8 that on $\mathbf{M}'_{a/q}$ we have

$$|S_M(\alpha)| \leq \eta M$$

provided $(a, q) = 1$ and either $\eta^{-2} \leq q \leq N^{1/10}$ or $\eta^{-2}M^{-2} \leq |\alpha - a/q| \leq M^{-2+1/10}$. □

A.3. **Proof of Lemmas 7 and 8.** Before launching into this, we make the important observation that the major arcs (and hence the refined major arcs) are a union of (necessarily short) pairwise disjoint intervals.

Lemma 10. *If $a/q \neq a'/q'$ with $1 \leq q, q' \leq M^{1/10}$, then $\mathbf{M}'_{a/q} \cap \mathbf{M}'_{a'/q'} = \emptyset$.*

Proof. Suppose that $\mathbf{M}'_{a/q} \cap \mathbf{M}'_{a'/q'} \neq \emptyset$. Using the fact that $aq' - a'q \neq 0$, we see that

$$\frac{2}{M^{2-1/10}} \geq \Big|\frac{a}{q} - \frac{a'}{q'}\Big| = \Big|\frac{aq' - a'q}{qq'}\Big| \geq \frac{1}{qq'} \geq \frac{1}{M^{1/5}},$$

a contradiction. □

Proof of Lemma 7. It follows from the Dirichlet principle and the fact that α is in a minor arc that there exists a reduced fraction a/q with

$$M^{1/10} \leq q \leq M^{2-1/10}$$

such that $|\alpha - a/q| \leq q^{-2}$. It therefore follows from the Weyl inequality that

$$|S_M(\alpha)| \leq 30M^{1-1/20}\log M \leq CM^{1-1/40}.$$ □

Key to the proof of Lemma 8 is the following approximation.

Lemma 11. *If $\alpha \in \mathbf{M}'_{a/q}$ with $1 \leq q \leq M^{1/10}$, then*

$$S_M(\alpha) = q^{-1}S(a,q)I_M(\alpha - a/q) + O(M^{1/5}) \tag{22}$$

where

$$S(a,q) := \sum_{r=0}^{q-1} e^{-2\pi i a r^2/q} \quad \text{and} \quad I_M(\beta) := \int_0^M e^{-2\pi i \beta x^2} dx.$$

Proof. We can write $\alpha = a/q + \beta$ where $|\beta| \leq 1/M^{2-1/10}$ and $1 \leq q \leq M^{1/10}$. We can also write each $1 \leq d \leq M$ uniquely as $d = mq + r$ with $1 \leq r \leq q$ and $0 \leq m \leq M/q$. It then follows that

$$\begin{aligned} S_M(\alpha) &= \sum_{r=1}^{q}\sum_{m=0}^{M/q} e^{-2\pi i(a/q+\beta)(mq+r)^2} + O(q) \\ &= \sum_{r=1}^{q} e^{-2\pi i a r^2/q} \sum_{m=0}^{M/q} e^{-2\pi i\beta(mq+r)^2} + O(q). \end{aligned}$$

Since

$$\Big|e^{-2\pi i(mq+r)^2\beta} - e^{-2\pi i m^2 q^2\beta}\Big| \leq \Big|e^{-2\pi i(2mqr+r^2)\beta} - 1\Big| \leq Cdr|\beta| \leq CqM^{-1+1/10}$$

and

$$\begin{aligned} \Big|\sum_{m=0}^{M/q} e^{-2\pi i m^2 q^2\beta} - \int_0^{M/q} e^{-2\pi i x^2 q^2\beta} dx\Big| &\leq \sum_{m=0}^{M/q}\int_m^{m+1} \Big|e^{-2\pi i m^2 q^2\beta} - e^{-2\pi i x^2 q^2\beta}\Big| dx \\ &\leq \sum_{m=0}^{M/q} 2\pi(2m+1)q^2|\beta| \\ &\leq 20M^{1/10} \end{aligned}$$

it follows that

$$\Big|S_M(\alpha) - \frac{1}{q}S(a,q)I_M(\beta)\Big| \leq CM^{1/5}.$$ □

Lemma 8 then follows almost immediately from the two basic lemmas below.

Lemma 12 (Gauss sum estimate)**.** *If $(a,q) = 1$, then $|S(a,q)| \leq \sqrt{2q}$. More precisely,*

$$|S(a,q)| = \begin{cases} \sqrt{q} & \text{if } q \text{ odd} \\ \sqrt{2q} & \text{if } q \equiv 0 \mod 4 \\ 0 & \text{if } q \equiv 2 \mod 4 \end{cases}.$$

Lemma 13 (Oscillatory integral estimate)**.** *For any* $\lambda \geq 0$

$$\Big|\int_0^1 e^{2\pi i\lambda x^2}dx\Big| \leq C(1+\lambda)^{-1/2}.$$

Proof of Lemma 8. Lemmas 12 and 13 imply that the main term in (22)

$$q^{-1}S(a,q)I_M(\alpha - a/q) \leq Mq^{-1/2}(1+M^2|\alpha - a/q|)^{-1/2},$$

and since $q^{-1/2} \geq M^{-1/20}$ and $M^2(|\alpha - a/q| \leq M^{1/10}$, it follows that

$$Mq^{-1/2}(1+M^2|\alpha - a/q|)^{-1/2} \geq M^{9/10} \gg M^{1/5}. \qquad \square$$

Proof of Lemma 12. Squaring-out $S(a,q)$ we obtain

$$|S(a,q)|^2 = \sum_{s=0}^{q-1}\sum_{r=0}^{q-1} e^{2\pi i a(r^2-s^2)/q}.$$

Letting $r = s+t$ and using the fact that $(a,q)=1$ and

$$\sum_{s=0}^{q-1} e^{2\pi i a(2st)/q} = \begin{cases} q & \text{if} \quad 2at \equiv 0 \mod q \\ 0 & \text{otherwise} \end{cases}$$

it follows that

$$|S(a,q)|^2 = \sum_{t=0}^{q-1} e^{2\pi i a t^2/q}\sum_{s=0}^{q-1} e^{2\pi i a(2st)/q} = \begin{cases} q & \text{if } \ q \text{ odd} \\ q\left(e^{2\pi i a(q/4)}+1\right) & \text{if } \ q \text{ even} \end{cases}. \qquad \square$$

Proof of Lemma 13. We need only consider the case when $\lambda \geq 1$. We write

$$\int_0^1 e^{2\pi i\lambda x^2}dx = \int_0^{\lambda^{-1/2}} e^{2\pi i\lambda x^2}dx + \int_{\lambda^{-1/2}}^1 e^{2\pi i\lambda x^2}dx =: I_1 + I_2.$$

It is then easy to see that $|I_1| \leq \lambda^{-1/2}$, while integration by parts gives that

$$\begin{aligned} |I_2| &= \left|\int_{\lambda^{-1/2}}^1 \frac{1}{4\pi i\lambda x}\Big(\frac{d}{dx}e^{2\pi i\lambda x^2}\Big)dx\right| \\ &\leq \frac{1}{4\pi\lambda}\left|\left[\frac{1}{x}\,e^{2\pi i\lambda x^2}\right]_{\lambda^{-1/2}}^1 + \int_{\lambda^{-1/2}}^1 \frac{1}{x^2}\,e^{2\pi i\lambda x^2}dx\right| \\ &\leq C\lambda^{-1/2}. \end{aligned} \qquad \square$$

References

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