THE DETERMINANT II - KORANYI NORM

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ABSTRACT. In this wee note we calculate the determinant of the usual (non-vector field) mixed Hessian of our favourite phase function.

The Key Idea

First observe that if A and B are $d \times d$ matrices and $\operatorname{rank}(B) = 1$, then

$$\det(A+B) = \det(A) + \det\begin{pmatrix}\mathbf{b}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\\ \vdots\\ \mathbf{a}_d\end{pmatrix} + \det\begin{pmatrix}\mathbf{a}_1\\ \mathbf{b}_2\\ \mathbf{a}_3\\ \vdots\\ \mathbf{a}_d\end{pmatrix} + \cdots + \det\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_{d-1}\\ \mathbf{b}_d\end{pmatrix},$$

where $\mathbf{a}_j = (a_{j1}, \ldots, a_{jd})$ and $\mathbf{b}_j = (b_{j1}, \ldots, b_{jd})$.

1. Our Phase Function and its Mixed Hessian

We shall define our phase function Φ to be

$$\Phi(x,y) = \phi(x,y)^{-\frac{\beta}{4}},$$

where

$$\phi(x,y) = (u_1^2 + \cdots + u_{2n}^2)^2 + bt^2 =: s^2 + bt^2.$$

In the formula above

$$u_j = x_j - y_j \quad \text{for all } j = 1, \ldots, 2n$$

and

$$\begin{aligned} t &= x_{2n+1} - y_{2n+1} - 2a\big((x_1y_2 - x_2y_1) + \cdots + (x_{2n-1}y_{2n} - x_{2n}y_{2n-1})\big)\\ &= x_{2n+1} - y_{2n+1} + 2a\sum_{j=1}^{2n}(-1)^j x_j y_{j-(-1)^j}\end{aligned}$$

We therefore have

$$\Phi_{xy} = -\frac{1}{4}\beta\phi^{-\frac{\beta+8}{4}}\left[\phi(x,y)\phi_{xy} - \tfrac{\beta+4}{4}\phi_x\phi_y^t\right] =: -\frac{1}{4}\beta\phi^{-\frac{\beta+8}{4}}\left[\phi(x,y)A - \tfrac{\beta+4}{4}B\right].$$

One can then calculate that

$$A = -4(C + D + bE) \quad \text{and} \quad B = -8(sbtF + s^2D + b^2t^2E),$$

where

$$C = \begin{pmatrix} s & abt & 0 & 0 & \cdots & 0 & 0 & 0\\ -abt & s & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & s & abt & \cdots & 0 & 0 & 0\\ 0 & 0 & -abt & s & \cdots & 0 & 0 & 0\\ \vdots & \vdots & & & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \cdots & \cdots & s & abt & 0\\ 0 & 0 & \cdots & \cdots & \cdots & -abt & s & 0\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\end{pmatrix},$$

$$D = \begin{pmatrix} 2u_1u_1 & 2u_1u_2 & \cdots & 2u_1u_{2n-1} & 2u_1u_{2n} & 0 \\ 2u_1u_2 & 2u_2u_2 & \cdots & 2u_2u_{2n-1} & 2u_1u_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2u_1u_{2n} & \cdots & \cdots & 2u_{2n-1}u_{2n} & 2u_{2n}u_{2n} & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} 2a^2x_2y_2 & -2a^2x_1y_2 & \cdots & \cdots & 2a^2x_{2n}y_2 & -2a^2x_{2n-1}y_2 & -ay_2 \\ -2a^2x_2y_1 & 2a^2x_1y_1 & \cdots & \cdots & -2a^2x_{2n}y_2 & 2a^2x_{2n-1}y_2 & ay_1 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 2a^2x_2y_{2n} & -2a^2x_1y_{2n-1} & \cdots & \cdots & 2a^2x_{2n}y_{2n} & -2a^2x_{2n-1}y_{2n} & -ay_{2n} \\ -2a^2x_2y_{2n-1} & 2a^2x_1y_{2n-1} & \cdots & \cdots & -2a^2x_{2n}y_{2n-1} & 2a^2x_{2n-1}y_{2n-1} & ay_{2n-1} \\ -ax_2 & ax_1 & \cdots & \cdots & -ax_{2n} & ax_{2n-1} & \frac{1}{2} \end{pmatrix},$$

and

$$F = \begin{pmatrix} -2a(u_1x_2 + u_1y_2) & 2a(u_1x_1 - u_2y_2) & \cdots & u_1 \\ -2a(u_2x_2 - u_1y_1) & 2a(u_2x_1 + u_2y_1) & \cdots & u_2 \\ \vdots & \vdots & \ddots & \vdots \\ -2a(u_{2n-1}x_2 + u_1y_{2n}) & 2a(u_{2n-1}x_1 - u_2y_{2n}) & \cdots & u_{2n-1} \\ -2a(u_{2n}x_2 - u_1y_{2n-1}) & 2a(u_{2n}x_1 + u_2y_{2n-1}) & \cdots & u_{2n} \\ u_1 & u_2 & \cdots & 0 \end{pmatrix}.$$

We now for convenience introduce a $2n \times 2n$ matrix G with rows $\mathbf{g}_j = \mathbf{f}_j - 2u_j\mathbf{e}_{2n+1}$, that is

$$G = \begin{pmatrix} -2au_1y_2 & \cdots & -2au_{2n}y_2 \\ 2au_1y_1 & \cdots & 2au_{2n}y_1 \\ \vdots & \ddots & \vdots \\ -2au_1y_{2n} & \cdots & -2au_{2n}y_{2n} \\ 2au_1y_{2n-1} & \cdots & 2au_{2n}y_{2n-1} \end{pmatrix}.$$

2. Calculating the Determinant

2.1. **The Big Reduction.** We note that $\operatorname{rank}(B) = 1$ and consequently

$$\det\big(\phi(x,y)A - \tfrac{\beta+4}{4}B\big) = \phi^{2n+1}\det(A) - \tfrac{\beta+4}{4}\phi^{2n}\left\{\sum_{j=1}^{2n}\det\begin{pmatrix}\mathbf{a}_1 \\ \vdots \\ \mathbf{b}_j \\ \vdots \\ \mathbf{a}_{2n} \\ \mathbf{a}_{2n+1}\end{pmatrix} + \det\begin{pmatrix}\mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_{2n} \\ \mathbf{b}_{2n+1}\end{pmatrix}\right\}.$$

Now since $\operatorname{rank}(E) = 1$ it is easy to see that

$$\det(A) = -4^{2n+1}b\det\begin{pmatrix}\mathbf{c}_1 + \mathbf{d}_1 \\ \vdots \\ \mathbf{c}_{2n} + \mathbf{d}_{2n} \\ \mathbf{e}_{2n+1}\end{pmatrix}.$$

Using again the fact that rank$(E) = 1$ we see that for $j = 1, \dots, 2n$

$$\det\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{b}_j\\ \vdots\\ \mathbf{a}_{2n}\\ \mathbf{a}_{2n+1}\end{pmatrix} = -2{\cdot}4^{2n+1}\det\begin{pmatrix}\mathbf{c}_1+\mathbf{d}_1+b\mathbf{e}_1\\ \vdots\\ bst\mathbf{f}_j+s^2\mathbf{d}_j+b^2t^2\mathbf{e}_j\\ \vdots\\ \mathbf{c}_{2n}+\mathbf{d}_{2n}+b\mathbf{e}_{2n}\\ b\mathbf{e}_{2n+1}\end{pmatrix} = -2{\cdot}4^{2n+1}bs\det\begin{pmatrix}\mathbf{c}_1+\mathbf{d}_1\\ \vdots\\ bt\mathbf{f}_j+s\mathbf{d}_j\\ \vdots\\ \mathbf{c}_{2n}+\mathbf{d}_{2n}\\ \mathbf{e}_{2n+1}\end{pmatrix} = -4^{2n+1}bs\det\begin{pmatrix}\tilde{\mathbf{c}}_1+\tilde{\mathbf{d}}_1\\ \vdots\\ bt\mathbf{g}_j+s\tilde{\mathbf{d}}_j\\ \vdots\\ \tilde{\mathbf{c}}_{2n}+\tilde{\mathbf{d}}_{2n}\end{pmatrix}.$$

Finally, and of course again using the fact that rank$(E) = 1$, we see that

$$\det\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_{2n}\\ \mathbf{b}_{2n+1}\end{pmatrix} = -2{\cdot}4^{2n+1}bt\det\begin{pmatrix}\mathbf{c}_1+\mathbf{d}_1+b\mathbf{e}_1\\ \vdots\\ \mathbf{c}_{2n}+\mathbf{d}_{2n}+b\mathbf{e}_{2n}\\ s\mathbf{f}_{2n+1}+bt\mathbf{e}_{2n+1}\end{pmatrix} = -2{\cdot}4^{2n+1}bt\left\{bt\det\begin{pmatrix}\mathbf{c}_1+\mathbf{d}_1\\ \vdots\\ \mathbf{c}_{2n}+\mathbf{d}_{2n}\\ \mathbf{e}_{2n+1}\end{pmatrix} + bs\sum_{j=1}^{2n}\det\begin{pmatrix}\mathbf{c}_1+\mathbf{d}_1\\ \vdots\\ \mathbf{e}_j\\ \vdots\\ \mathbf{c}_{2n}+\mathbf{d}_{2n}\\ \mathbf{f}_{2n+1}\end{pmatrix}\right\}.$$

Therefore

$$\det\big(\phi A - \tfrac{\beta+4}{4}B\big)$$
$$= (4\phi)^{2n}b\left\{2\big((\beta+2)bt^2-2s^2\big)\det\begin{pmatrix}\mathbf{c}_1+\mathbf{d}_1\\ \vdots\\ \mathbf{c}_{2n}+\mathbf{d}_{2n}\\ \mathbf{e}_{2n+1}\end{pmatrix} + (\beta+4)s\sum_{j=1}^{2n}\left\{\det\begin{pmatrix}\tilde{\mathbf{c}}_1+\tilde{\mathbf{d}}_1\\ \vdots\\ bt\mathbf{g}_j+s\tilde{\mathbf{d}}_j\\ \vdots\\ \tilde{\mathbf{c}}_{2n}+\tilde{\mathbf{d}}_{2n}\end{pmatrix} + 2bt\det\begin{pmatrix}\mathbf{c}_1+\mathbf{d}_1\\ \vdots\\ \mathbf{e}_j\\ \vdots\\ \mathbf{c}_{2n}+\mathbf{d}_{2n}\\ \mathbf{f}_{2n+1}\end{pmatrix}\right\}\right\}.$$

2.2. **Lets get down to business.** There are three calculations that must now be carried out. We now introduce the notation

$$\widetilde{C} = \{c_{ij}\}_{i,j=1,\dots,2n} \text{ and } \widetilde{D} = \{d_{ij}\}_{i,j=1,\dots,2n}.$$

Key to these arguments is the fact that rank$(\widetilde{D}) = 1$.

2.2.1. *The Easy One.* Since rank$(\widetilde{D}) = 1$ it follows that

$$\det\begin{pmatrix}\mathbf{c}_1+\mathbf{d}_1\\ \vdots\\ \mathbf{c}_{2n}+\mathbf{d}_{2n}\\ \mathbf{e}_{2n+1}\end{pmatrix} = \frac{1}{2}\det(\widetilde{C}+\widetilde{D}) = \frac{1}{2}\left\{\det(\widetilde{C}) + \sum_{j=1}^{2n}\det\begin{pmatrix}\tilde{\mathbf{c}}_1\\ \vdots\\ \tilde{\mathbf{d}}_j\\ \vdots\\ \tilde{\mathbf{c}}_{2n}\end{pmatrix}\right\}.$$

Now it is easy to see that

$$\det(\widetilde{C}) = (s^2 + a^2b^2t^2)^n,$$

while a more careful calculations shows

$$\det\begin{pmatrix}\tilde{\mathbf{c}}_1\\ \vdots\\ \tilde{\mathbf{d}}_j\\ \vdots\\ \tilde{\mathbf{c}}_{2n}\end{pmatrix} = (s^2+a^2b^2t^2)^{n-1}\big(2su_j^2 + (-1)^{j+1}2abtu_ju_{j-(-1)^j}\big).$$

Therefore

$$\det\begin{pmatrix}\mathbf{c}_1+\mathbf{d}_1\\ \vdots\\ \mathbf{c}_{2n}+\mathbf{d}_{2n}\\ \mathbf{e}_{2n+1}\end{pmatrix}=\frac{1}{2}\left\{(s^2+a^2b^2t^2)^n+(s^2+a^2b^2t^2)^{n-1}\sum_{j=1}^{2n}2su_j^2+(-1)^{j+1}2abtu_ju_{j-(-1)^j}\right\}$$
$$=\frac{1}{2}\left\{(s^2+a^2b^2t^2)^n+(s^2+a^2b^2t^2)^{n-1}2s^2\right\}$$
$$=\frac{1}{2}(s^2+a^2b^2t^2)^{n-1}(3s^2+a^2b^2t^2).$$

2.2.2. *The First Hard One.* Using the fact that $\operatorname{rank}(\widetilde{D})=1$ we see that

$$\det\begin{pmatrix}\tilde{\mathbf{c}}_1+\tilde{\mathbf{d}}_1\\ \vdots\\ bt\mathbf{g}_j+s\tilde{\mathbf{d}}_j\\ \vdots\\ \tilde{\mathbf{c}}_{2n}+\tilde{\mathbf{d}}_{2n}\end{pmatrix}=bt\det\begin{pmatrix}\tilde{\mathbf{c}}_1\\ \vdots\\ \mathbf{g}_j\\ \vdots\\ \tilde{\mathbf{c}}_{2n}\end{pmatrix}+s\det\begin{pmatrix}\tilde{\mathbf{c}}_1\\ \vdots\\ \tilde{\mathbf{d}}_j\\ \vdots\\ \tilde{\mathbf{c}}_{2n}\end{pmatrix}=\big(s+(-1)^j\frac{1}{u_j}abty_{j-(-1)^j}\big)\det\begin{pmatrix}\tilde{\mathbf{c}}_1\\ \vdots\\ \tilde{\mathbf{d}}_j\\ \vdots\\ \tilde{\mathbf{c}}_{2n}\end{pmatrix},$$

and hence

$$\sum_{j=1}^{2n}\det\begin{pmatrix}\tilde{\mathbf{c}}_1+\tilde{\mathbf{d}}_1\\ \vdots\\ t\mathbf{g}_j+s\tilde{\mathbf{d}}_j\\ \vdots\\ \tilde{\mathbf{c}}_{2n}+\tilde{\mathbf{d}}_{2n}\end{pmatrix}=2(s^2+a^2b^2t^2)^{n-1}\sum_{j=1}^{2n}s^2u_j^2-(-1)^jsu_jabt(u_{j-(-1)^j}-y_{j-(-1)^j})-a^2b^2t^2u_{j-(-1)^j}y_{j-(-1)^j}$$
$$=2(s^2+a^2b^2t^2)^{n-1}\Big\{s^3-sabt\big((u_1y_2-u_2y_1)+\cdots+(u_{2n-1}y_{2n}-u_{2n}y_{2n-1})\big)-$$
$$-a^2b^2t^2(u_1y_1+\cdots+u_{2n}y_{2n})\Big\}.$$

2.2.3. *The Second Hard One.* Using once more the fact that $\operatorname{rank}(\widetilde{D})=1$ we see that

$$\sum_{j=1}^{2n}\det\begin{pmatrix}\mathbf{c}_1+\mathbf{d}_1\\ \vdots\\ \mathbf{e}_j\\ \vdots\\ \mathbf{c}_{2n}+\mathbf{d}_{2n}\\ \mathbf{f}_{2n+1}\end{pmatrix}=\sum_{j=1}^{2n}(-1)^{j+1}ay_{j-(-1)^j}\frac{1}{2u_j}\det\begin{pmatrix}\tilde{\mathbf{c}}_1\\ \vdots\\ \tilde{\mathbf{d}}_j\\ \vdots\\ \tilde{\mathbf{c}}_{2n}\end{pmatrix}$$
$$=2(s^2+a^2b^2t^2)^{n-1}\sum_{j=1}^{2n}(-1)^{j+1}ay_{j-(-1)^j}\frac{1}{2u_j}\big(su_j^2+(-1)^{j+1}abtu_ju_{j-(-1)^j}\big)$$
$$=(s^2+a^2b^2t^2)^{n-1}a\sum_{j=1}^{2n}y_{j-(-1)^j}\big)\big(abtu_{j-(-1)^j}+(-1)^{j+1}su_j\big)$$
$$=(s^2+a^2b^2t^2)^{n-1}a\Big\{abt(u_1y_1+\cdots+u_{2n}y_{2n})+s\big((u_1y_2-u_2y_1)+\cdots$$
$$\cdots+(u_{2n-1}y_{2n}-u_{2n}y_{2n-1})\big)\Big\}.$$

3. Conclusion

Lets now put everything together, doing so we see that

$$\begin{aligned}\det\bigl(\phi A - \tfrac{\beta+4}{4}B\bigr) &= (4\phi)^{2n} b(s^2 + a^2b^2t^2)^{n-1}\left\{\bigl((\beta+2)bt^2 - 2s^2\bigr)\bigl(3s^2 + a^2b^2t^2\bigr) + 2(\beta+4)s^4\right\}\\ &= (4\phi)^{2n} b(s^2 + a^2b^2t^2)^{n-1}\left\{2(\beta+1)s^4 + \bigl(3(\beta+2)b - 2a^2b^2\bigr)s^2t^2 + (\beta+2)a^2b^3t^4\right\}.\end{aligned}$$

By analyzing the discriminant

$$\Delta = 4a^4b^2 - 4(\beta+2)(2\beta+5)a^2b + 9(\beta+2)^2,$$

we see that our Hessian will be *non-degenerate* provided either

$$2a^2b \le 3(\beta+2) \qquad \text{or} \qquad |2a^2b - (2\beta+5)(\beta+2)| < (\beta+2)\sqrt{(2\beta+5)^2 - 9},$$

which reduces simply to the condition that

$$2a^2b < (\beta+2)\left(2\beta + 5 + \sqrt{(2\beta+5)^2 - 9}\right).$$

Some Remarks:

1. Note that some condition on the size of a^2b is forced upon us.
2. When $a^2b = 1$ the corresponding pseudo-norms are in fact norms.