NOTES ON THE ALMOST EVERYWHERE CONVERGENCE OF BOCHNER–RIESZ MEANS IN $\mathbf{R}^n$

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Abstract. In this expository note we present a result of Carbery, Rubio de Francia and Vega [1] on the almost everywhere convergence of Bochner–Riesz means.

1. Introduction

We are interested in the pointwise convergence of Bochner–Riesz means T_R^λ in $\mathbf{R}^n$, these are defined in terms of the Fourier transform for $\lambda > 0$ and $0 < R < \infty$ by

$$\widehat{T_R^\lambda f}(\xi) = \big(1 - \tfrac{|\xi|^2}{R^2}\big)_+^\lambda \widehat{f}(\xi).$$

We of course need only consider values of λ below the critical index of $\frac{1}{2}(n-1)$. It follows from the uniform boundedness principle and scaling that convergence of T_R^λ in L^p is equivalent to the L^p boundedness of $T^\lambda = T_1^\lambda$, it is conjectured that this should hold for $0 < \lambda \leq \frac{1}{2}(n-1)$ if and only if

$$\frac{2n}{n+1+2\lambda} = p'_\lambda < p < p_\lambda = \frac{2n}{n-1-2\lambda}.$$

It is easy to show that this inequality is necessary and well known that the conjecture is indeed a theorem in $\mathbf{R}^2$; see [3]. There has been progress in higher dimensions but the problem is still open. The following result in $\mathbf{R}^n$ for $n \geq 2$ concerning almost everywhere convergence is due to Carbery, Rubio de Francia and Vega [1].

Theorem A. *If $2 \leq p < p_\lambda$ then $\lim_{R\to\infty} T_R^\lambda f(x) = f(x)$ almost everywhere for all $f \in L^p(\mathbf{R}^n)$.*

We naturally need to consider the maximal operator: $T_*^\lambda f(x) = \sup_{R>0} |T_R^\lambda f(x)|$. Now for $p > 2$ almost everywhere convergence is no longer equivalent to the L^p boundedness of a corresponding maximal operator. As a result we can avoid the hard problem of proving L^p boundedness, it will in fact suffice to instead establish the following weighted L^2 estimate.

Theorem B. *If $0 \leq \alpha < 1 + 2\lambda \leq n$ then $\|T_*^\lambda f\|_{L^2(|x|^{-\alpha})} \leq C_{\alpha,\lambda} \|f\|_{L^2(|x|^{-\alpha})}$.*

Note that $1 + 2\lambda = n(1 - \frac{2}{p_\lambda})$ and that Theorem B implies the almost everywhere convergence of $T_R^\lambda(x)$ for all $f \in L^2(|x|^{-\alpha})$ as Schwartz functions are dense in $L^2(|x|^{-\alpha})$.

The key idea is then to use the fact that $L^p \subseteq L^2 + L^2(|x|^{-\alpha})$ whenever $\alpha > n(1 - \frac{2}{p})$, which follows immediately from Hölder's inequality. Then for a fixed p such that $2 \leq p < p_\lambda$ we can certainly choose α such that $n(1 - \frac{2}{p}) < \alpha < 1 + 2\lambda$, almost everywhere convergence of $T_R^\lambda(x)$ for all $f \in L^p$ then follows from Theorem B.

2. Reduction to basic estimate

In order to prove Theorem B we are going to decompose the multipliers on dyadic annuli whose widths are approximately their distances to the sphere $|\xi| = 1$. To be precise: choose smooth functions φ supported where $\frac{1}{2} < t < 1$ such that $0 \le \varphi \le 1$, and $\sum_{k=1}^{\infty} \varphi_k(t) = 1$ for $\frac{1}{2} \le t \le 1$, where $\varphi_k(t) = \varphi(2^k t)$. Now define $\varphi_0(t) = 1 - \sum_{k=1}^{\infty} \varphi_k(t)$, for $0 \le t < \frac{1}{2}$ and $\varphi_0(t) = 0$ otherwise. Then we have

$$(1-|\xi|^2)_+^{\lambda} = \sum_{k=0}^{\infty} (1-|\xi|^2)^{\lambda} \varphi_k(1-|\xi|^2) = \sum_{k=0}^{\infty} 2^{-k\lambda} m^{2^{-k}}(|\xi|),$$

where

$$m^{2^{-k}}(|\xi|) = 2^{k\lambda}(1-|\xi|^2)^{\lambda}\varphi_k(1-|\xi|^2).$$

This allows us to decompose the operator

$$T_R^{\lambda} f(x) = \sum_{k=0}^{\infty} 2^{-k\lambda} \mathcal{F}^{-1}[m^{2^{-k}}(\tfrac{|\cdot|}{R})\widehat{f}\,](x). \tag{1}$$

For $k = 0$ and 1 the terms are controlled by the Hardy-Littlewood maximal operator which is bounded in $L^p(|x|^{-\alpha})$ for $n(1-p) < \alpha < n$[1]; see Appendix. We will therefore study operators S_t^{δ} defined by

$$\widehat{S_t^{\delta} f}(\xi) = m^{\delta}(t|\xi|)\widehat{f}(\xi) \quad \text{and} \quad S_*^{\delta} f(x) = \sup_{t>0} |S_t^{\delta} f(x)|,$$

for $\delta < \frac{1}{2}$. Notice that given a small $\delta > 0$, $m^{\delta}(t)$ is a smooth function supported in $[1-\delta, 1]$, we have that $0 \le m^{\delta}(t) \le 1$ and $|D^l m^{\delta}(t)| \le C\delta^{-l}$ for all $l \in \mathbf{N}$.

Lemma 1. *For $\delta > 0$ and $0 \le \alpha < n$ we have*

$$\int |S_*^{\delta} f(x)|^2 \frac{dx}{|x|^{\alpha}} \le C_{\alpha} A_{\alpha}(\delta) \int |f(x)|^2 \frac{dx}{|x|^{\alpha}},$$

where C_{α} is independent of δ and

$$A_{\alpha}(\delta) = \begin{cases} 1, & \text{if } 0 \le \alpha < 1, \\ |\log \delta|, & \text{if } \alpha = 1, \\ \delta^{1-\alpha}, & \text{if } 1 < \alpha < n. \end{cases}$$

Theorem B is an immediate consequence of Lemma 1; it is clear from (1) that

$$\|T_*^{\lambda} f\|_{L^2(|x|^{-\alpha})} \le C \sum_{k=0}^{\infty} 2^{-k\lambda} \|S_*^{2^k} f\|_{L^2(|x|^{-\alpha})},$$

so setting $\delta = 2^{-k}$ we see that T_*^{λ} is bounded on $L^2(|x|^{-\alpha})$ provided that $\lambda > 0$ (in the $0 \le \alpha < 1$ case) or that $\lambda > \frac{\alpha-1}{2}$ (in the case when $1 \le \alpha < n$).

Let $L_k f$ be the usual Littlewood–Paley operator, defined by $\widehat{L_k f}(\xi) = \phi(2^k|\xi|)\widehat{f}(\xi)$ where $\operatorname{supp} \phi \subset [\frac{1}{4}, 4]$ and $\phi(t) = 1$ for $\frac{1}{2} \le t \le 2$. If $n(1-p) < \alpha < n$ then we have, see Appendix, that

$$C_1 \|f\|_{L^p(|x|^{-\alpha})} \le \left\| \Big(\sum_{k=0}^{\infty} |L_k f(x)|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(|x|^{-\alpha})} \le C_2 \|f\|_{L^p(|x|^{-\alpha})}.$$

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[1] This condition ensures that $|x|^{-\alpha}$ is an A_p weight.

Using this fact we can reduce matters to establishing the local maximal operator estimate

$$\big\| \sup_{1\le t\le 2} |S_t^\delta f(x)| \big\|^2_{L^2(|x|^{-\alpha})} \le C_\alpha A_\alpha(\delta) \big\| f\big\|^2_{L^2(|x|^{-\alpha})}. \tag{2}$$

By homogeneity (2) also holds for S_{Rt}^δ for any $R>0$. Lets now see that this estimate in fact implies Lemma 1.

$$\begin{aligned}
\big\|\sup_{t>0} |S_t^\delta f(x)|\big\|^2_{L^2(|x|^{-\alpha})} &= \big\|\sup_k \sup_{2^{k-1}\le t\le 2^k} |S_t^\delta f(x)|\big\|^2_{L^2(|x|^{-\alpha})} \\
&\le \Big\| \Big(\sum_k \sup_{2^{k-1}\le t\le 2^k} |S_t^\delta f(x)|^2\Big)^{\frac12}\Big\|^2_{L^2(|x|^{-\alpha})} \\
&= \Big\| \Big(\sum_k \sup_{2^{k-1}\le t\le 2^k} |S_t^\delta (L_k f)(x)|^2\Big)^{\frac12}\Big\|^2_{L^2(|x|^{-\alpha})} \\
&= \sum_k \Big\| \sup_{1\le t\le 2} |S_t^\delta (L_k f)(x)|\Big\|^2_{L^2(|x|^{-\alpha})} \\
&\le C_\alpha A_\alpha(\delta) \sum_k \big\|L_k f\big\|^2_{L^2(|x|^{-\alpha})} \\
&= C_\alpha A_\alpha(\delta) \Big\| \Big(\sum_k |L_k f|^2\Big)^{\frac12}\Big\|^2_{L^2(|x|^{-\alpha})} \\
&\le C_\alpha A_\alpha(\delta) \|f\|^2_{L^2(|x|^{-\alpha})}.
\end{aligned}$$

We are therefore left with verifying estimate (2). Let $F(t) = |S_t^\delta f(x)|$, then by the Fundamental Theorem of Calculus we have

$$\sup_{1\le t\le 2} F(t) \le F(1) + c\|F\|_2^{\frac12}\|F'\|_2^{\frac12}.$$

Therefore

$$\Big(\int \sup_{1\le t\le 2} |F(t)|^2 \frac{dx}{|x|^\alpha}\Big)^{\frac12} \le \Big(\int |F(1)|^2 \frac{dx}{|x|^\alpha}\Big)^{\frac12} + c\Big(\int \|F\|_2\|F'\|_2 \frac{dx}{|x|^\alpha}\Big)^{\frac12}.$$

Of course by definition

$$\|F(1)\|_{L^2(|x|^{-\alpha})} = \|S^\delta f\|_{L^2(|x|^{-\alpha})},$$

while

$$\begin{aligned}
\int \|F\|_2\|F'\|_2 \frac{dx}{|x|^\alpha} &= \int \Big(\int_1^2 |S_t^\delta f(x)|^2 dt\Big)^{\frac12}\Big(\int_1^2 |\tfrac{d}{dt} S_t^\delta f(x)|^2 dt\Big)^{\frac12} \frac{dx}{|x|^\alpha} \\
&\le \Big\| \Big(\int_1^2 |S_t^\delta f|^2 dt\Big)^{\frac12}\Big\|_{L^2(|x|^{-\alpha})} \Big\| \Big(\int_1^2 |\tfrac{d}{dt} S_t^\delta f|^2 dt\Big)^{\frac12}\Big\|_{L^2(|x|^{-\alpha})} \\
&= I_1 \cdot I_2
\end{aligned}$$

Argument for $\alpha = 0$: Then we of course have by Plancherel that

$$\|F(1)\|_2 = \|S^\delta f\|_2 = \|m^\delta(|\cdot|)\widehat{f}\|_2 \le C\|f\|_2.$$

Notice also that

$$I_1^2 \le \int_1^2 \|S_t^\delta f\|_2^2 dt = \int_1^2 \int |m^\delta(t|\xi|)\widehat{f}(\xi)|^2 d\xi dt = \int |\widehat{f}(\xi)|^2 \int_1^2 |m^\delta(t|\xi|)|^2 dt d\xi.$$

Now if for fixed ξ the t integrand is non–zero then necessarily $[(1-\delta)|\xi|^{-1}, |\xi|^{-1}] \cap [1,2]$ must be non–empty, which implies that $1 \le |\xi|^{-1} \le 4$ and therefore that the effective size of the region of integration is in fact bounded by 4δ. It therefore follows that

$$I_1^2 \le \int |\widehat{f}(\xi)|^2 \int_1^2 \eta_\xi(t)|m^\delta(t|\xi|)|^2 dt d\xi \le \int \tfrac{\delta}{|\xi|}\tilde{\eta}(|\xi|)|\widehat{f}(\xi)|^2 d\xi \le C\delta \|f\|_2^2.$$

Now for I_2 notice that $|\frac{d}{dt}m^\delta(t|\xi|)| \le C|\xi|\delta^{-1}$ so arguing as above we get that

$$I_2^2 \le \int \tfrac{|\xi|}{\delta}\tilde{\eta}(|\xi|)|\widehat{f}(\xi)|^2 d\xi \le C\delta^{-1}\|f\|_2^2,$$

we therefore have that $I_1 \cdot I_2 \le C\|f\|_2^2$, this establishes Lemma 1 in the special case where $\alpha = 0$.

We of course wish to obtain this result for $0 \le \alpha < n$, we claim that proving estimate (2) holds boils down to establishing the following result.

Lemma 2. *For $\delta > 0$ and $0 \le \alpha < n$ we have*

$$\int |S^\delta f(x)|^2 \frac{dx}{|x|^\alpha} \le C_\alpha A_\alpha(\delta) \int |f(x)|^2 \frac{dx}{|x|^\alpha}.$$

This clearly takes care of the $F(1)$ term, we claim that it also implies $I_1 \cdot I_2 \le CA_\alpha(\delta)\|f\|^2_{L^2(|x|^{-\alpha})}$.

Claim. Lemma 2 implies that

$$I_1^2 \le C\delta A_\alpha(\delta)\|f\|^2_{L^2(|x|^{-\alpha})} \quad \text{and} \quad I_2^2 \le C\delta^{-1}A_\alpha(\delta)\|f\|^2_{L^2(|x|^{-\alpha})}.$$

Proof of Claim. We shall first consider I_1, we wish to show that Lemma 2 implies

$$\int\int_1^2 |S_t^\delta f(x)|^2 dt \frac{dx}{|x|^\alpha} \le C_\alpha \delta A_\alpha(\delta) \int |f(x)|^2 \frac{dx}{|x|^\alpha}. \tag{3}$$

It follows from duality that this is equivalent to

$$\int \left| \int_1^2 S_t^\delta f_t(x)\, dt \right|^2 |x|^\alpha dx \le C_\alpha \delta A_\alpha(\delta) \int\int_1^2 |f_t(x)|^2 dt |x|^\alpha dx. \tag{4}$$

Lets see this: let $T := S_t^\delta$ and $G(x) := \{g_t(x)\}$, then $T : L^2(|x|^{-\alpha}) \to L^2_{x,t}(|x|^{-\alpha})$ so

$$\begin{aligned}
\langle Tf, G\rangle_{L^2_{x,t}(|x|^{-\alpha})} &= \int\int_1^2 S_t^\delta f(x)\overline{g_t(x)} dt \frac{dx}{|x|^\alpha} \\
&= \int\int_1^2 \int K_t^\delta(x-y) f(y) dy\, \overline{g_t(x)} dt \frac{dx}{|x|^\alpha} \\
&= \int f(y) \int_1^2 \int K_t^\delta(x-y)\overline{g_t(x)} \frac{dx}{|x|^\alpha} dt dy \\
&= \int f(y) \int_1^2 |y|^\alpha \overline{S_t^\delta\big[\tfrac{g_t(\cdot)}{|\cdot|^\alpha}\big]} dt \frac{dy}{|y|^\alpha} \\
&= \langle f, T^*G\rangle_{L^2(|x|^{-\alpha})},
\end{aligned}$$

where (since K_t^δ is even)

$$T^*G(x) = \int_1^2 |x|^\alpha S_t^\delta\big[\tfrac{g_t(\cdot)}{|\cdot|^\alpha}\big](x) dt.$$

So estimate (3) is equivalent to

$$\int \big|T^*G(x)\big|^2 \frac{dx}{|x|^\alpha} \le C_\alpha \delta A_\alpha(\delta) \int\int_1^2 |g_t(x)|^2 dt \frac{dx}{|x|^\alpha},$$

that is

$$\int \Big|\int_1^2 S_t^\delta\big[\tfrac{g_t(\cdot)}{|\cdot|^\alpha}\big](x)|x|^\alpha dt\Big|^2 \frac{dx}{|x|^\alpha} \le C_\alpha \delta A_\alpha(\delta) \int\int_1^2 |g_t(x)|^2 dt \frac{dx}{|x|^\alpha},$$

so if we let $f_t(x) = g_t(x)|x|^{-\alpha}$, this is equivalent to

$$\int \Big|\int_1^2 S_t^\delta f_t(x)|x|^\alpha dt\Big|^2 \frac{dx}{|x|^\alpha} \le C_\alpha \delta A_\alpha(\delta) \int\int_1^2 |f_t(x)|^2 |x|^{2\alpha} dt \frac{dx}{|x|^\alpha}.$$

So we have reduced matters to showing that Lemma 2 implies estimate (4).

For $0 < \alpha < 2$ we let

$$\mathcal{D}^{\frac{\alpha}{2}} f(x) = \left(\int_{\mathbf{R}^n} \frac{|f(x+y) - f(x)|^2}{|y|^\alpha} \frac{dy}{|y|^n}\right)^{\frac{1}{2}},$$

if $\alpha = 2$ we replace f with ∇f and then for $2 < \alpha < 4$ define $\mathcal{D}^{\frac{\alpha}{2}} f$ as above but with f replaced by ∇f, etc. Then a simple application of Plancherel's theorem (see [2], p. 139) shows that

$$\|\mathcal{D}^{\frac{\alpha}{2}} f\|_2^2 \sim \int |\widehat{f}(\xi)|^2 |\xi|^\alpha d\xi.$$

By Plancherel, estimate (4) is equivalent to

$$\Big\|\mathcal{D}^{\frac{\alpha}{2}} \int_1^2 m^\delta(t|\cdot|)\widehat{f}_t(\cdot)dt\Big\|_2^2 \le C_\alpha \delta A_\alpha(\delta) \int_1^2 \int |\mathcal{D}^{\frac{\alpha}{2}} \widehat{f}_t(\xi)|^2 d\xi dt. \tag{5}$$

We shall now argue as we did in the model case where $\alpha = 0$, we see that the left hand side of estimate (5)

$$\int \Big|\mathcal{D}^{\frac{\alpha}{2}} \int_1^2 m^\delta(t|\xi|)\widehat{f}_t(\xi)dt\Big|^2 d\xi = \int\int \Big|\int_1^2 [\eta_{\xi+y}(t)m^\delta(t|\xi+y|)\widehat{f}_t(\xi+y) - \eta_\xi(t)m^\delta(t|\xi|)\widehat{f}_t(\xi)]dt\Big|^2 |y|^{-n-\alpha} dy d\xi.$$

Now we shall define $\chi_E(\xi, y)$ to be the characteristic function of the set

$$E = \{(\xi, y) : |\xi| \le (1-\delta)|\xi + y|\} \cup \{(\xi, y) : |\xi + y| \le (1-\delta)|\xi|\},$$

and notice that

$$\operatorname{supp} \eta_{\xi+y} \cap \operatorname{supp} \eta_\xi = \emptyset \Longleftrightarrow (\xi, y) \in E.$$

With this in mind we write

$$\begin{aligned}
&\int\Big|\mathcal{D}^{\frac{\alpha}{2}}\int_1^2 m^\delta(t|\xi|)\widehat{f}_t(\xi)dt\Big|^2 d\xi\\
&\quad=\int\int[\chi_E+(1-\chi_E)](\xi,y)\Big|\int_1^2[\eta_{\xi+y}(t)m^\delta(t|\xi+y|)\widehat{f}_t(\xi+y)-\eta_\xi(t)m^\delta(t|\xi|)\widehat{f}_t(\xi)]dt\Big|^2|y|^{-n-\alpha}dyd\xi\\
&\quad=C\int\int\chi_E(\xi,y)\Big|\int_1^2[\eta_{\xi+y}(t)+\eta_\xi(t)][m^\delta(t|\xi+y|)\widehat{f}_t(\xi+y)-m^\delta(t|\xi|)\widehat{f}_t(\xi)]dt\Big|^2|y|^{-n-\alpha}dyd\xi\\
&\qquad+\int\int(1-\chi_E(\xi,y))\Big|\int_1^2[\tilde{\eta}_\xi(t)][m^\delta(t|\xi+y|)\widehat{f}_t(\xi+y)-m^\delta(t|\xi|)\widehat{f}_t(\xi)]dt\Big|^2|y|^{-n-\alpha}dyd\xi\\
&\quad\le C\int\int\chi_E(\xi,y)\int_1^2[\eta_{\xi+y}(t)+\eta_\xi(t)]^2dt\cdot\int_1^2[m^\delta(t|\xi+y|)\widehat{f}_t(\xi+y)-m^\delta(t|\xi|)\widehat{f}_t(\xi)]^2dt|y|^{-n-\alpha}dyd\xi\\
&\qquad+\int\int(1-\chi_E(\xi,y))\int_1^2[\tilde{\eta}_\xi(t)]^2dt\cdot\int_1^2[m^\delta(t|\xi+y|)\widehat{f}_t(\xi+y)-m^\delta(t|\xi|)\widehat{f}_t(\xi)]^2dt|y|^{-n-\alpha}dyd\xi\\
&\quad\le C\int\int\chi_E(\xi,y)[\tfrac{\delta}{|\xi+y|}\tilde{\eta}(|\xi+y|)+\tfrac{\delta}{|\xi|}\tilde{\eta}(|\xi|)]\int_1^2[m^\delta(t|\xi+y|)\widehat{f}_t(\xi+y)-m^\delta(t|\xi|)\widehat{f}_t(\xi)]^2dt|y|^{-n-\alpha}dyd\xi\\
&\qquad+\int\int(1-\chi_E(\xi,y))[\tfrac{\delta}{|\xi|}\tilde{\tilde{\eta}}(|\xi|)]\int_1^2[m^\delta(t|\xi+y|)\widehat{f}_t(\xi+y)-m^\delta(t|\xi|)\widehat{f}_t(\xi)]^2dt|y|^{-n-\alpha}dyd\xi\\
&\quad\le C\delta\int\int\int_1^2|m^\delta(t|\xi+y|)\widehat{f}_t(\xi+y)-m^\delta(t|\xi|)\widehat{f}_t(\xi)|^2dt|y|^{-n-\alpha}dyd\xi\\
&\quad\le C\delta\int_1^2\int|\mathcal{D}^{\frac{\alpha}{2}}[m^\delta(t|\xi|)\widehat{f}_t(\xi)]|^2d\xi\,dt.
\end{aligned}$$

So we need to show that $m^\delta(t|\cdot|)$ is a pointwise multiplier of the homogeneous Sobolev space $L^2_{\frac{\alpha}{2}}=\{f:\|\mathcal{D}^{\frac{\alpha}{2}}f\|_2<\infty\}$ with a constant $\le C_\alpha A_\alpha(\delta)^{\frac12}$, that is

$$\int|\mathcal{D}^{\frac{\alpha}{2}}[m^\delta(t|\xi|)\widehat{f}_t(\xi)]|^2d\xi\le C_\alpha A_\alpha(\delta)\int|\mathcal{D}^{\frac{\alpha}{2}}\widehat{f}_t(\xi)|^2d\xi,\tag{6}$$

uniformly in $1\le t\le 2$. By homogeneity it suffices to prove (6) for $t=1$. Now by Plancherel again estimate (6) is equivalent to

$$\int|S^\delta f(x)|^2|x|^\alpha dx\le C_\alpha A_\alpha(\delta)\int|f(x)|^2|x|^\alpha dx,$$

and this follows from Lemma 2 by duality. Now for the integral I_2 we note that

$$t\frac{d}{dt}m^\delta(t|\xi|)=s\frac{d}{ds}m^\delta(s)\Big|_{s=t|\xi|}.$$

So if we define

$$\widetilde{m}^\delta(s)=s\delta\tfrac{d}{ds}m^\delta(s),$$

it is easy to see that $\widetilde{m}^\delta$ satisfies the same estimates as m^δ, if we now define

$$\widehat{\widetilde{S}^\delta_t}f(\xi)=\widetilde{m}^\delta(t|\xi|)\widehat{f}(\xi),$$

then we have $\widetilde{S}^\delta_t f(x)=t\delta\frac{d}{dt}S^\delta_t f(x)$. Now since $\widetilde{S}^\delta_t$ satisfies the same estimates as S^δ_t the argument above runs through with a lose of δ^2. $\square$

3. Proof of Lemma 2

In the proof of Lemma 2 we shall separate the cases where $0 \le \alpha < 1$ and where $1 < \alpha < n$. In both cases the proof relies on the following two lemmas that we shall for the moment assume.

Lemma 3. *For $0 < \delta \le \frac{1}{2}$ we have*

$$\int_{\left|1-|\xi|\right|\le\delta} |\widehat{f}(\xi)|^2 d\xi \le C_\alpha A_\alpha(\delta)\delta^\alpha \int |f(x)|^2|x|^\alpha dx,$$

where C_α is independent of δ.

In particular, for $1 < \alpha < n$ we have

$$\int_{|\xi|=1} |\widehat{f}(\xi)|^2 d\xi \le C_\alpha \int |f(x)|^2|x|^\alpha dx,$$

and since $\widehat{f}(R\xi) = R^{-n}\widehat{f(\frac{\cdot}{R})}(\xi)$, that

$$\int_{|\xi|=1} |\widehat{f}(R\xi)|^2 d\xi \le C_\alpha R^{-2n} \int |f(\tfrac{x}{R})|^2|x|^\alpha dx = C_\alpha R^{\alpha-n} \int |f(x)|^2|x|^\alpha dx.$$

In the same way we can, for $0 \le \alpha < n$, rescale the $\delta = \frac{1}{2}$ case to obtain

$$\int_{\frac{1}{2}R\le|\xi|\le\frac{3}{2}R} |\widehat{f}(\xi)|^2 d\xi = R^n \int_{\frac{1}{2}\le|\xi|\le\frac{3}{2}} |\widehat{f}(R\xi)|^2 d\xi \le C_\alpha R^\alpha \int |f(x)|^2|x|^\alpha dx.$$

We now let $K = K^\delta$ be the kernel such that $\widehat{K} = m^\delta(|\xi|)$ and dyadically decompose our kernel $K(x) = \sum_j K_j(x)$, where for each $j \ge 1$ we have that K_j is supported where $|x| \sim 2^j\delta^{-1}$ and K_0 is supported where $|x| \le C\delta^{-1}$. Then we see that the main contribution to K comes from K_0. The following Lemma makes this precise.

Lemma 4. *For $0 \le \alpha \le n$ and all $m \in \mathbf{N}$ we have*

$$|\widehat{K_j}(\xi)| \le C_{m,\alpha}2^{-mj} \quad \textit{and} \quad \int_0^\infty |\widehat{K_j}(r)|r^{\alpha-1}dr \le C_{m,\alpha}2^{-mj}\delta.$$

3.1. Proof of Lemma 2 for $1 < \alpha < n$.

First notice that $A_\alpha(\delta) = \delta^{1-\alpha} \ge 1$ and that by Plancherel we trivially have

$$\int |K_j * f(x)|^2 dx \le C2^{-j} \int |f(x)|^2 dx.$$

We of course would like to divide both sides by $|x|^\alpha$ and we can do this if $|x|$ is about a non–zero constant. With this in mind we shall divide $\mathbf{R}^n$ into disjoint cubes $\{Q_i\}_{i=0}^\infty$ with sidelength $2^j\delta^{-1}$ each centered at x_i with $x_0 = 0$. It is immediate from the support properties of K_j that

$$\int_{|x-x_i|\le 2^j\delta^{-1}} |K_j * f(x)|^2 dx \le C2^{-j} \int_{|x-x_i|\le 10\cdot 2^j\delta^{-1}} |f(x)|^2 dx.$$

Therefore for $|x| \gg 2^j\delta^{-1}$ we have

$$\int |K_j * f(x)|^2 \frac{dx}{|x|^\alpha} \le C2^{-j} \int |f(x)|^2 \frac{dx}{|x|^\alpha}.$$

Now for $|x| \leq C2^j\delta^{-1}$ we use the fact that

$$\begin{aligned}\int |K_j * f|^2 dx &= \int |\widehat{K_j}(\xi)|^2 |\widehat{f}(\xi)|^2 d\xi \\ &= \int_0^\infty \int_{S^{n-1}} |\widehat{f}(r\omega)|^2 d\omega |\widehat{K_j}(r)|^2 r^{n-1} dr \\ &\leq C_\alpha \int |f(x)|^2 |x|^\alpha dx \int_0^\infty |\widehat{K_j}(r)|^2 r^{\alpha-1} dr,\end{aligned}$$

by Lemma 3. Now $\|\widehat{K_j}\|_\infty \leq 1$, and so Lemma 4 gives that for each $m \in \mathbf{N}$,

$$\int |K_j * f|^2 dx \leq C_{m,\alpha} 2^{-mj} \delta \int |f(x)|^2 |x|^\alpha dx,$$

and hence, by duality

$$\int |K_j * f|^2 \frac{dx}{|x|^\alpha} \leq C_{m,\alpha} 2^{-mj} \delta \int |f(x)|^2 dx.$$

Now using the fact that $|x|^\alpha \leq 2^{j\alpha}\delta^{-\alpha}$, we see that

$$\int |K_j * f|^2 \frac{dx}{|x|^\alpha} \leq C_{m,\alpha} 2^{(\alpha-m)j} \delta^{1-\alpha} \int |f(x)|^2 \frac{dx}{|x|^\alpha}.$$

If we now pick $m \geq \alpha + 1$, as we are free to do, then we have

$$\int |K_j * f|^2 \frac{dx}{|x|^\alpha} \leq C_\alpha 2^{-j} \delta^{1-\alpha} \int |f(x)|^2 \frac{dx}{|x|^\alpha}. \tag{7}$$

We are therefore done modulo verifying Lemmas 3 and 4.

3.2. **Proof of Lemma 2 for $0 < \alpha \leq 1$.** By (6) for $t = 1$ it shall suffice to show that

$$\|\mathcal{D}^{\frac{\alpha}{2}} m^\delta \widehat{f}\|_2^2 \leq C_\alpha A_\alpha(\delta) \|\mathcal{D}^{\frac{\alpha}{2}} \widehat{f}\|_2^2.$$

Now using the following Leibniz rule for $\mathcal{D}^\beta$, namely

$$\mathcal{D}^\beta[gh](x) \leq \|g\|_\infty \mathcal{D}^\beta h(x) + |h(x)| \mathcal{D}^\beta g(x),$$

we see that

$$\|\mathcal{D}^{\frac{\alpha}{2}} m^\delta \widehat{f}\|_2^2 \leq \|m\|_\infty \|\mathcal{D}^{\frac{\alpha}{2}} \widehat{f}\|_2^2 + \|\widehat{f} \mathcal{D}^{\frac{\alpha}{2}} m^\delta\|_2^2.$$

It therefore suffices to show that

$$\int |\widehat{f}(\xi)|^2 |\mathcal{D}^{\frac{\alpha}{2}} m^\delta(\xi)|^2 d\xi \leq C_\alpha A_\alpha(\delta) \int |\mathcal{D}^{\frac{\alpha}{2}} \widehat{f}(\xi)|^2 d\xi. \tag{8}$$

Lemma 5. *For $0 < \alpha < 2$ we have*

$$|\mathcal{D}^{\frac{\alpha}{2}} m^\delta(\xi)|^2 \leq C_\alpha \begin{cases} \delta^{-\alpha} & \text{if } \ |1 - |\xi|| \leq 2\delta, \\ \delta |1 - |\xi||^{-\alpha-1} & \text{if } \ 0 \leq |\xi| \leq 2, \\ \delta |\xi|^{-\alpha-n} & \text{if } \ |\xi| \geq 2. \end{cases}$$

Assuming Lemma 5 for the moment we see that the left hand side of equation (8) is dominated by

$$I_1 + I_2 + I_3 = \delta^{-\alpha} \int_{|1-|\xi||\leq 2\delta} |\widehat{f}(\xi)|^2 d\xi + \delta \int_{|\xi|\leq 2} |1 - |\xi||^{-\alpha-1} |\widehat{f}(\xi)|^2 d\xi + \delta \int_{|\xi|\geq 2} |\xi|^{-\alpha-n} |\widehat{f}(\xi)|^2 d\xi.$$

Now it clearly follows from Lemma 3 that

$$I_1 \le CA_\alpha(\delta)\int |f(x)|^2|x|^\alpha dx.$$

While

$$I_2 \le C\delta\sum_{k=1}^\infty 2^{k(\alpha+1)}\int_{|1-|\xi||\le 2^{-k}}|\widehat{f}(\xi)|^2 d\xi + C\delta\Big(\int_{0\le|\xi|\le\frac12}|\widehat{f}(\xi)|^2 d\xi + \int_{\frac32\le|\xi|\le 2}|\widehat{f}(\xi)|^2 d\xi\Big),$$

and

$$I_3 \le C\delta\sum_{k=1}^\infty 2^{-k(\alpha+n)}\int_{|\xi|\sim 2^k}|\widehat{f}(\xi)|^2 d\xi.$$

It then follows from Lemma 3 and the remarks proceeding it that

$$\sum_{k=1}^\infty 2^{k(\alpha+1)}\int_{|1-|\xi||\le 2^{-k}}|\widehat{f}(\xi)|^2 d\xi \le \sum_{k=1}^\infty 2^k A_\alpha(2^{-k})\int |f(x)|^2|x|^\alpha dx,$$

$$\int_{0\le|\xi|\le\frac12}|\widehat{f}(\xi)|^2 d\xi \le C\sum_{k=1}^\infty\int_{|\xi|\sim 2^{-k}}|\widehat{f}(\xi)|^2 d\xi \le C\sum_{k=1}^\infty 2^{-k\alpha}\int |f(x)|^2|x|^\alpha dx,$$

$$\int_{\frac32\le|\xi|\le 2}|\widehat{f}(\xi)|^2 d\xi \le \int_{\frac12 2\le|\xi|\le\frac32 2}|\widehat{f}(\xi)|^2 d\xi \le C_\alpha\int |f(x)|^2|x|^\alpha dx,$$

and

$$\sum_{k=1}^\infty 2^{-k(\alpha+n)}\int_{|\xi|\sim 2^k}|\widehat{f}(\xi)|^2 d\xi \le C\sum_{k=1}^\infty 2^{-kn}\int |f(x)|^2|x|^\alpha dx.$$

It therefore follows that

$$I_2 \le CA_\alpha(\delta)\int |f(x)|^2|x|^\alpha dx \qquad\text{and}\qquad I_3 \le C\delta\int |f(x)|^2|x|^\alpha dx,$$

and so (8) is established. This completes the proof of Lemma 2 modulo proving Lemma 3, 4 and 5.

4. Proofs of Lemma 3, 4 and 5

4.1. **Proof of Lemma 3.** By the usual duality argument it suffice to show that

$$\int |\widehat{g}(\xi)|^2\frac{d\xi}{|\xi|^\alpha} \le C_\alpha A_\alpha(\delta)\delta^\alpha\int_{|1-|x||\le\delta}|g(x)|^2 dx,$$

where $\operatorname{supp} g \subseteq \{x : |1-|x|| \le \delta\}$. Now if $\alpha \ne 0$, then

$$\begin{aligned}\int |\widehat{g}(\xi)|^2\frac{d\xi}{|\xi|^\alpha} &= \int \widehat{g*\tilde{g}}(\xi)\frac{d\xi}{|\xi|^\alpha}\\ &= C\int g*\tilde{g}(x)|x|^{\alpha-n}dx\\ &= C\iint_{\substack{|1-|x||\le\delta\\ |1-|y||\le\delta}} g(x)\overline{g(y)}|x-y|^{\alpha-n}dx\,dy\\ &\le C\|g\|_2^2\sup_x\int_{|1-|y||\le\delta}|x-y|^{\alpha-n}dy,\end{aligned}$$

by Schur's Lemma. Changing variables we see that

$$\int_{|1-|y||\le\delta} |x-y|^{\alpha-n}dy = \int_{|1-|x-y||\le\delta} |y|^{\alpha-n}dy.$$

Now here we are integrating over an annulus centered at x and of width δ. The integral is clearly majorized if the origin falls inside the annulus, it is then controlled by

$$\int\limits_{\substack{|v_n|\le\delta\\|v'|\le 1}} |v|^{\alpha-n}dv = \int\limits_{\substack{|v_n|\le\delta\\|v'|\le\delta}} |v|^{\alpha-n}dv + \int\limits_{\substack{|v_n|\le\delta\\\delta\le|v'|\le 1}} |v|^{\alpha-n}dv = I_1 + I_2,$$

this is easily justified by a switch to tangential and normal coordinates and some error analysis. Now

$$|I_1| \le C\int_0^\delta r^{\alpha-1}dr = C\delta^\alpha \le CA_\alpha(\delta)\delta^\alpha,$$

and

$$|I_2| \le C\int_0^\delta dv_n \int_{|v'|\ge\delta} |v'|^{\alpha-n}dv' = C\delta\int_\delta^1 r^{\alpha-2}dr = CA_\alpha(\delta)\delta^\alpha.$$

4.2. **Proof of Lemma 4.** Recall that $K = K^\delta$ satisfies $\widehat{K}(\xi) = m^\delta(\xi)$. We shall now make the decomposition of K precise, we define

$$h_j(x) = \begin{cases} \phi(|x|), & \text{if } j = 0, \\ \phi(2^{-j}|x|) - \phi(2^{1-j}|x|), & \text{if } j \ge 1, \end{cases}$$

where ϕ is a smooth function with $\operatorname{supp}\phi \subseteq [\frac{1}{2}, 2]$ and $\phi(t) \equiv 1$ for $\frac{3}{4} \le t \le \frac{3}{2}$. We decompose our kernel K as

$$K(x) = \sum_{j=0}^{\infty} K_j(x) \qquad \text{where} \qquad K_j(x) = K(x)h_j(\delta x).$$

We therefore have $\widehat{K_j}(\xi) = m^\delta * \widehat{h_j(\delta\cdot)}(\xi)$. If we now let $h(x) = \phi(|x|) - \phi(2|x|)$, then we get that

$$\widehat{K_j}(\xi) = \int m^\delta(\xi - 2^{-j}\delta\eta)\widehat{h}(\eta)d\eta.$$

Now since $h \equiv 0$ in a neighborhood of 0 it follows that $\int \eta^\beta \widehat{h}(\eta)d\eta = 0$, for any multi–index β. Therefore, expanding m^δ in a Taylor series about 0 we get

$$\widehat{K_j}(\xi) = \int R_m(\xi,\eta)\widehat{h}(\eta)d\eta,$$

where $|R_m(\xi,\eta)| \le \sum_{|\beta|=m} \|D^\beta m^\delta\|_\infty |2^{-j}\delta\eta|^m \le 2^{-jm}|\eta|^m$. Now since $\widehat{h} \in \mathcal{S}(\mathbf{R}^n)$, this implies

$$|\widehat{K_j}(\xi)| \le C_m 2^{-mj},$$

for all $m \ge 0$ and $\xi \in \mathbf{R}^n$. However, looking at the definition of $\widehat{K_j}(\xi)$ and the fact that $\operatorname{supp} m^\delta \subseteq [1-\delta, 1]$ it follows that if $|\xi| < \frac{1}{2}$, then necessarily $|\eta| > 2^{j-1}\delta^{-1}$ and it follows that

$$|\widehat{K_j}(\xi)| \le C\int (1+|\eta|)^{-m-n-1}d\eta \le C2^{-mj}\delta^m,$$

for any $m \geq 0$ and $|\xi| < \frac{1}{2}$. Thus

$$\int_0^{\frac{1}{2}} |\widehat{K_j}(r)| r^{\alpha-1} dr \leq C_{\alpha,m} 2^{-mj} \delta^m.$$

On the other hand consider the set $S = \{\xi \in \mathbf{R}^n : 1 - 2\delta < |\xi| < 1 + 2\delta\}$, and look at

$$\begin{aligned}
\int_{\frac{1}{2}}^{\infty} |\widehat{K_j}(r)| r^{\alpha-1} dr &\leq C \int_{\mathbf{R}^n} |\widehat{K_j}(\xi)| d\xi \\
&= C \int_S |\widehat{K_j}(\xi)| d\xi + C \int_{\mathbf{R}^n \setminus S} |\widehat{K_j}(\xi)| d\xi \\
&\leq C_m 2^{-mj} \delta + \int_{|\eta| > 2^j} |\widehat{h}(\eta)| \int m^\delta(\xi - 2^{-j}\delta\eta) d\xi \, d\eta \\
&\leq C_m 2^{-mj} \delta + C \|m^\delta\|_\infty \int_{|\eta| > 2^j} |\widehat{h}(\eta)| d\eta \\
&\leq C_m 2^{-mj} \delta.
\end{aligned}$$

4.3. **Proof of Lemma 5.** Let us first consider the case when $|1 - |\xi|| \leq 2\delta$; if $|y| \geq \delta$, then

$$\int_{|y| \geq \delta} |m^\delta(\xi + y) - m^\delta(\xi)|^2 |y|^{-n-\alpha} dy \leq \int_{|y| \geq \delta} |y|^{-n-\alpha} dy \leq \delta^{-\alpha},$$

while if $|y| \leq \delta$, then

$$\int_{|y| \leq \delta} |m^\delta(\xi + y) - m^\delta(\xi)|^2 |y|^{-n-\alpha} dy \leq \delta^{-2} \int_{|y| \geq \delta} |y|^{2-n-\alpha} dy \leq \delta^{-\alpha},$$

since $0 < \alpha < 2$. Let us now consider the case when $|\xi| \geq 2$; now this implies that $m^\delta(\xi) = 0$, so if the integrand is to be non–zero we must have that

$$|\xi + y| \sim 1 \iff |y| \sim |\xi| \pm 1 \implies |y|^{-n-\alpha} \leq C|\xi|^{-n-\alpha},$$

therefore we have

$$|\mathcal{D}^{\frac{\alpha}{2}} m^\delta(\xi)|^2 = \int |m^\delta(\xi + y) - m^\delta(\xi)|^2 |y|^{-n-\alpha} dy \leq C|\xi|^{-n-\alpha}.$$

Finally we must consider the case where $|\xi| \leq 2$ and $|1 - |\xi|| \geq 2\delta$; we are looking at

$$I(\xi) = \int_{\{y : |1 - |\xi + y|| \leq \delta\}} |y|^{-n-\alpha} dy.$$

Now $2\delta \leq |1 - |\xi|| \leq 1$ so we can break $I(\xi)$ into dyadic pieces where $|1 - |\xi|| \sim 2^{-j}$ and $\delta \leq 2^{-j}$. Consider the contribution from the annuli $|y| \sim 2^{-j+r}$, it is straightforward to see that

$$|\{y : |1 - |\xi + y|| \leq \delta\} \cap \{|y| \sim 2^{-j+r}\}| \leq C\delta 2^{-(j-r)(n-1)},$$

and hence that $I_j(\xi) \leq C\delta 2^{-(j-r)(\alpha+1)}$, if we now sum in r it follows that

$$\int_{\{y : |1 - |\xi + y|| \leq \delta\}} |y|^{-n-\alpha} dy \leq C\delta |1 - |\xi||^{-\alpha-1}.$$

Appendix

Here we prove that the Hardy–Littlewood Maximal function and the Littlewood–Paley square function are bounded on $L^p(|x|^{-\alpha})$ whenever $n(1-p)<\alpha<n$. These two results will be essentially a consequence of the following weighted L^p mapping property of singular integrals.

Proposition 6. *Suppose that $|K(x)|\leq C|x|^{-n}$ and that $Tf=f*K$ is bounded on L^p, then*

$$\|Tf\|_{L^p(|x|^{-\alpha})}\leq C\|f\|_{L^p(|x|^{-\alpha})}\quad \text{for}\ \ n(1-p)<\alpha<n.$$

Proof. We shall smoothly break our operator into two pieces; a conic neighborhood of the diagonal $x=y$ of aperture ϵ and the complement of this. Inside the conic region we use the fact that T is bounded on L^p and off the diagonal we observe that $|K(x-y)|\leq C|x-y|^{-n}\approx(|x|+|y|)^{-n}$.

(a) Inside the conic neighborhood Γ_ϵ:

$$\begin{aligned}\Big(\int|Tf(x)|^p\frac{dx}{|x|^\alpha}\Big)^{\frac1p}&=\Big(\int\Big|\int f(y)K(x-y)dy\Big|^p\frac{dx}{|x|^\alpha}\Big)^{\frac1p}\\&=\Big(\sum_j\int\Big|\chi\big(\tfrac{|x|}{2^j}\big)\int f(y)K(x-y)\chi\big(\tfrac{|x-y|}{\epsilon2^j}\big)dy\Big|^p\frac{dx}{|x|^\alpha}\Big)^{\frac1p}\\&=\Big(\sum_j\int\Big|\chi\big(\tfrac{|x|}{2^j}\big)\int f(y)K(x-y)\chi\big(\tfrac{|x-y|}{\epsilon2^j}\big)\tilde\chi\big(\tfrac{|y|}{2^j}\big)dy\Big|^p\frac{dx}{|x|^\alpha}\Big)^{\frac1p}\\&\leq C\Big(\sum_j2^{-j\alpha}\int\Big|[f\tilde\chi\big(\tfrac{|\cdot|}{2^j}\big)]*[K\chi\big(\tfrac{|\cdot|}{\epsilon2^j}\big)\Big|^pdx\Big)^{\frac1p}.\end{aligned}$$

Now since T is bounded on L^p it follows that $f\mapsto f*K\chi\big(\frac{|\cdot|}{\epsilon2^j}\big)$ is also bounded on L^p provided that $\widehat\chi\in L^1$. We therefore have that

$$\begin{aligned}\Big(\int|Tf(x)|^p\frac{dx}{|x|^\alpha}\Big)^{\frac1p}&\leq C\Big(\sum_j2^{-j\alpha}\int|f(x)|^p\tilde\chi\big(\tfrac{|x|}{2^j}\big)dx\Big)^{\frac1p}\\&\leq C\Big(\int|f(x)|^p\frac{dx}{|x|^\alpha}\Big)^{\frac1p}.\end{aligned}$$

(b) Away from the conic neighborhood Γ_ϵ: here we have that

$$\Big(\int|Tf(x)|^p\frac{dx}{|x|^\alpha}\Big)^{\frac1p}\leq C\Big(\int\Big(\int|f(y)||x-y|^{-n}dy\Big)^p\frac{dx}{|x|^\alpha}\Big)^{\frac1p}.$$

Now, of course, there are two main possibilities, where $|x-y|^{-n}\approx|x|^{-n}$ and $|x-y|^{-n}\approx|y|^{-n}$.

(i) $|x-y|^{-n} \approx |x|^{-n}$; here we have

$$\Big(\int |Tf(x)|^p \frac{dx}{|x|^\alpha}\Big)^{\frac{1}{p}} \le C\Big(\int\Big(\int_{|y|\le\frac{1}{2}|x|} |f(y)||x|^{-n-\frac{\alpha}{p}}dy\Big)^p dx\Big)^{\frac{1}{p}}$$
$$= C\Big(\int\Big(\sum_{\ell\ge 1}\int_{|y|\sim 2^{-\ell}|x|} |f(y)||x|^{-n-\frac{\alpha}{p}}dy\Big)^p dx\Big)^{\frac{1}{p}}$$
$$\le C\sum_{\ell\ge 1}\Big(\int |x|^{-np-\alpha}\Big(\int_{|y|\sim 2^{-\ell}|x|} |f(y)|dy\Big)^p dx\Big)^{\frac{1}{p}}$$
$$= C\sum_{\ell\ge 1}\Big(\int |x|^{-np-\alpha}\Big(2^{-\ell n}|x|^n\int_{|y|\sim 2^{-\ell}|x|} |f(y)|\tfrac{dy}{2^{-\ell n}|x|^n}\Big)^p dx\Big)^{\frac{1}{p}}.$$

Now it follow immediately from Hölder's inequality that

$$\int_{|y|\sim 2^{-\ell}|x|} |f(y)|\tfrac{dy}{2^{-\ell n}|x|^n} \le \Big(\int_{|y|\sim 2^{-\ell}|x|} |f(y)|^p\tfrac{dy}{2^{-\ell n}|x|^n}dx\Big)^{\frac{1}{p}},$$

therefore

$$\Big(\int |Tf(x)|^p \frac{dx}{|x|^\alpha}\Big)^{\frac{1}{p}} \le C\sum_{\ell\ge 1}\Big(\int |x|^{-np-\alpha}2^{-\ell np}|x|^{np}\int_{|y|\sim 2^{-\ell}|x|} |f(y)|^p\tfrac{dy}{2^{-\ell n}|x|^n}dx\Big)^{\frac{1}{p}}$$
$$\le C\sum_{\ell\ge 1}2^{-\ell n(1-\frac{1}{p})}\Big(\int |f(y)|^p\int_{|x|\sim 2^{\ell}|y|} |x|^{-n-\alpha}dx\,dy\Big)^{\frac{1}{p}}$$
$$\le C\sum_{\ell\ge 1}2^{-\ell\frac{1}{p}(np-n+\alpha)}\Big(\int |f(y)|^p|y|^{-\alpha}dy\Big)^{\frac{1}{p}}$$
$$\le C\Big(\int |f(y)|^p|y|^{-\alpha}dy\Big)^{\frac{1}{p}},$$

provided $\alpha > n(1-p)$.

(ii) $|x-y|^{-n} \approx |y|^{-n}$; here we argue similarly to above and obtain the restriction that $\alpha < n$. □

Remark. The above argument applies to our operators, as

(1) $f \mapsto M_{HL}f = \sup_{r>0}|f| * \frac{\chi_{B_r}}{|B_r|}$ and $\Big|\frac{\chi_{B_r}}{|B_r|}\Big| \le C|x|^{-n}$.

(2) $f \mapsto \Big(\sum_k |L_kf|^2\Big)^{\frac{1}{2}} \le \sum_k |L_kf| \le |f| * \sum_k \frac{2^{kn}}{(1+2^k|x|)^N}$ and $\sum_k \frac{2^{kn}}{(1+2^k|x|)^N} \le C|x|^{-n}$.

Note also that the ℓ^∞ and ℓ^2 norms respectively do not effect the argument inside Γ_ϵ.

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