# Note on a Generalization of Roth's Theorem

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## Abstract

We give a simple proof that for sufficiently large $N$, every subset of $[N]^2$ of size at least $\delta N^2$ contains three points of the form $\{(a,b),(a+d,b),(a,b+d)\}$.

In this note we give a simple proof for a theorem of Ajtai and Szemerédi [1]. In their proof Ajtai and Szemerédi used and iterated Szemerédi's theorem about long arithmetic progressions in dense sets of integers [8]. A more general theorem of Fürstenberg and Katznelson also implies Theorem 1, but does not give bound on $N$ as it uses ergodic theory [2]. After improving the bound in Szemerédi's theorem, Gowers asked for a quantitative proof of Theorem 1 [3, 4].

**Theorem 1** *(Ajtai-Szemerédi) For any real number $\delta > 0$ there is a natural number $N_0$ such that for $N > N_0$ every subset of $[N]^2$ of size at least $\delta N^2$ contains a triple of the form $\{(a,b),(a+d,b),(a,b+d)\}$ for some integer $d \neq 0$.*

The key of the proof is a lemma of Ruzsa and Szemerédi [7]. A subgraph of a graph $G$ is a *matching* if every vertex has degree one. A matching $M$ is an *induced matching* if there are no other edges of $G$ between the vertices of $M$.

**Lemma 2** *(Ruzsa-Szemerédi) If $G_n$ is the union of $n$ induced matchings, then $e(G_n) = o(n^2)$.*

The lemma, with a simple proof deduced from Szemerédi's Regularity Lemma, can be also found in a survey paper of Komlós and Simonovits [5].

*Proof of Theorem 1*: Let $S$ be a subset of the grid $[N]^2$ of size at least $\delta N^2$. We refer to a point of the grid with its coordinates, which are pairs $(i,j); i,j \in \{1,2,\ldots,N\}$. Let us define a bipartite graph $G(A,B)$ with vertex sets $A = \{v_1,\ldots,v_N\}$ and $B = \{w_1,\ldots,w_N\}$. Two vertices $v_i$ and $w_j$ are connected by an edge iff $(i,j) \in S$ (see Fig. 1).

Let us partition the edges of $G$ according to their length, $(v_i,w_j) \sim (v_l,w_m)$ iff $i+j = l+m$. Every partition class is a matching, so we can apply Lemma 2 to $G$. If $N$ is large enough, then at least one matching is not induced. A triple

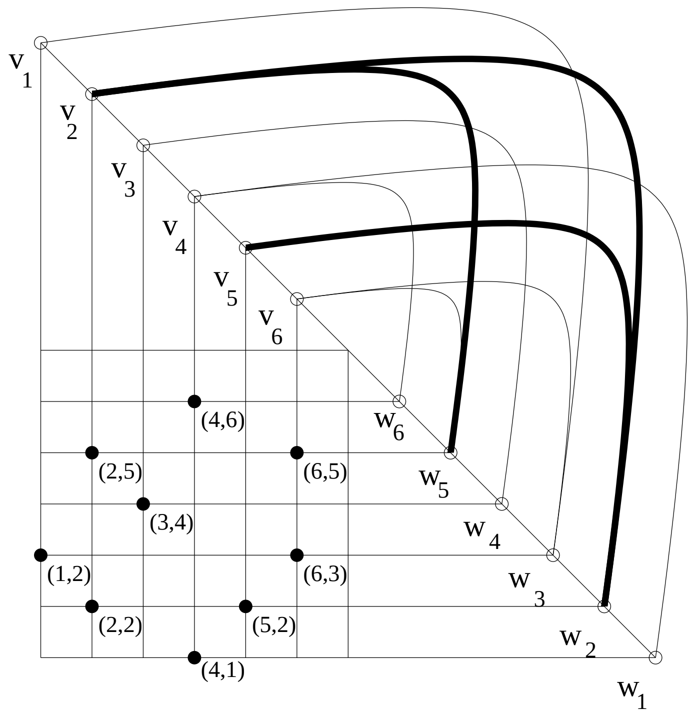


Figure 1: Converting points into edges

of edges $(v_i, w_m), (v_i, w_j), (v_l, w_m)$ such that $(v_i, w_j) \sim (v_l, w_m)$ guarantees a triple in $S$, $\{(a, b), (a + d, b), (a, b + d)\}$ (see bold edges in Fig.1). □

The only known proof of Lemma 2 uses Szemerédi's Regularity Lemma [8], so while the proof is quantitative, it gives a tower-type bound on $N_0 = N_0(\delta^{-1})$. It would be very important to find another, maybe analytical proof for Lemma 2 to get a better bound.

## References


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