EXERCISES 3

REU SUMMER 2005

1. Use Roth' theorem to show that to every $0 < \delta < 1$ there is a number $c_\delta > 0$ depending only on δ, such that if $A \subset [1, N]$ and $|A| \geq \delta N$ then A contains at least $c_\delta N^2$ 3-progressions.
2. Let $0 < \delta < 1$, and let $0 < \alpha < \delta^2$ be given. Suppose $A \subset \mathbb{Z}_N$ such that $|A| = \delta N$. This exercise aims to show that if A is α- uniform, then it behaves like a random set when considering the size of its intersection with its translates: $A \cap (A + k)$. We expect $|A \cap (A + k)| = \delta^2 N$ in a random set.

   a) Let $a_k = |A \cap (A + k)|$. Show that

   $$s = \frac{1}{N} \sum_{k=0}^{N-1} a_k = \delta^2 N$$

   b) Show that

   $$\sum_{k=0}^{N-1} a_k^2 = \frac{1}{N} \sum_{r=0}^{N-1} |\hat{\chi}_A(r)|^4 \leq (\delta^4 + \alpha^2)$$

   c) Show that

   $$\sum_{k=0}^{N-1} (a_k - s)^2 \leq \alpha^2 N^3$$

   d) Conclude that

   $$|\{k \in \mathbb{Z}_N : \ |a_k - s| \geq \sqrt{\alpha} N\}| \leq \alpha N$$

   This means that for all but $\leq \alpha^2 N$ values of k:

   $$(\delta^2 - \sqrt{\alpha})N \leq |A \cap (A + k)| \leq (\delta^2 + \sqrt{\alpha})N$$

3.* Generalize the proof of Roth' theorem to the following situation. Let $a_1, \ldots, a_k$ be given integers, such that $a_1 + \ldots + a_k = 0$.

   Prove that for any $0 < \delta < 1$ there exists a constant $C > 0$ (which may only depend on the numbers a_i), such that $N > exp\, exp\, (\delta^{-\frac{1}{k-2}})$ and if $A \subset [1, N]$ with $|A| = \delta N$ then A contains k mutually distinct numbers: $x_1, \ldots, x_k$, such that

   $$a_1 x_1 + \ldots + a_k x_k = 0$$

   Note: This is (the translation invariant case of) Rado's theorem for a single equation. Roth' theorem is the case: $a_1 = a_2 = 1, \ a_3 = -2$.

4. This is partly a computational problem. For given $0 < \delta < 1$, define the Roth number $R(\delta)$ as follows;

   Let $R(\delta)$ be the smallest number R, such that if $N \geq R$ then for any $A \subset [1, N]$ with $|A| = \delta N$, A contains a 3-progression.

   a) Explain, why $R(1/m) \geq W(3, m)$. Find $R(2/3)$.

   b) Find a combinatorial argument to show the existence of $R(1/2)$

   Hint: Assume A contains no 3-progression. Let A_0 be the set of even elements of A, and A_1 be the set odd elements of A, then one of them has at least $|A|/2$ elements, say $|A_0| \geq |A|/2$. Let b be the smallest element of A_0 and c be the largest.

   Define the sets $B = \{(b + a)/2 : a \in A_0\}$, $C = \{(a + c)/2 : a \in A,\ a \neq b\}$. Show, that A, B, C are disjoint sets, and moreover none of them contains a 3-progression. On the other hand $|A \cup B \cup C| \geq 2|A| - 3 \geq N - 3$ and use Van der Waerden's theorem to get a contradiction if N is large enough.

   c)* Can you modify the argument to show the existence of $R(1/3)$?

   d)* Use a computer (if necessary) to find $R(1/2)$ and $R(1/3)$.

5. A set S is called an d- dimensional square, if there exist: $a, k_1, \ldots, k_d$ such that

$$S = \{a + \varepsilon_1 k_1 + \ldots + \varepsilon_d k_d : \ \varepsilon_i = 0 \text{ or } \varepsilon_i = 1 \text{ for all } 1 \leq i \leq d\}$$

   a) Use induction on d to show that if $A \subset Z_N$, and $|A| = \delta N$ then A contains at least $\delta^{2^d} N^{d+1}$ d- dimensional cubes.

   b) Show that if A *subset*$[1, N]$ and $|A| = \delta N$, then A contains at least $\delta^{2^d} N$ d-dimensional cubes.

   Hint: The idea of the induction in both cases, is that if $A \cap (A + k)$ contains a d- dimensional cube, then A contains a $d + 1$ dimensional cube. The case $d = 2$ on $\mathbb{Z}_N$ was already discussed. For $A \subset [1, N]$, show that there is a k such that: $|A \cap (A + k)| \geq \delta^2/2\ N$, by considering the "most popular difference" in A.