EXERCISES 2

REU SUMMER 2005

1. Use the argument described in the notes to obtain a bound for $W(3, n, r)$ and $W(3, r)$, here r denotes the number of colors. Use the observation that are at most $(Nr)^{n+1}$ polychromatic n tuples of 3-AP's, in a block of size N.
2. Find a bound for $W(4, r)$ along the same lines.
3. If $F = \{v_1, \ldots, v_k\} \subset \mathbb{Z}^d$ is a pattern in the standard d-dimensional lattice, we define the Gallai number $N(F, r)$ to be the smallest N, such that for any r coloring of $[1, N]^d$, there is a monochromatic set of the form $F' = \{x + lv_1, \ldots, x + lv_k\}$ for some $x \in \mathbb{Z}^d$ and $l \in \mathbb{N}$.
   (i) Let $T = \{0, e_1, e_2\} \subset \mathbb{Z}^2$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$. Determine $N(T, 2)$.
   (ii) Use the proof of Gallai's theorem to obtain an upper bound for $N(T, r)$.
4. Use the probabilistic argument to show that $W(k, 2) \geq 2^{k/2}\sqrt{\frac{k-1}{2}}$.
5. Let p be a prime and let F_{2^p} be the finite field with 2^p elements. Note that F_{2^p} can be viewed as an p dimensional vector space over the field F_2.
   (i) Let $x \in F_{2^p}$ not equal to 0 or 1. Show that the elements $1, x, \ldots, x^{p-1}$ are linearly independent over F_2.
   Hint: Let d be the least integer such that $1, x, \ldots x^d$ are linearly dependent; then these elements generate a subfield of G of F_{2^p} of cardinality 2^d, but this is not possible as p is a prime.
   (ii) Let x be a primitive element, that is $x^a \neq 1$ for $1 \leq a < 2^p - 1$ Let V be any hyperplane in F_{2^p} (which may not pass through the origin). Show that for any proper AP: the elements $\{x^a, x^{a+r}\, ldots, x^{a+(p-1)r}\}$ cannot be all contained in V.
   Hint: If $0 \in V$, use part (i). Otherwise, let P the polynomial of smallest degree such that $P(x^r) = 0$. Show that $P(1) = 0$, which contradicts the irreducibility of P.