EXERCISES

REU SUMMER 2005

1. Prove that $R(3,r) \leq er! + 1$ by using induction on r, by fine tuning the argument shown in class. In fact, show that
$$R_{r+1}(3) - 1 \leq (r+1)(R_r(3) - 1) + 1$$
and use the fact, that: $1 + \frac{1}{2!} + \ldots + \frac{1}{r!} \leq e$ for each r, where e denotes the Euler number.

2. Show that if $[1,N] \times [1,N]$ is colored with r colors, where $N \geq cr^2$, then there is a monochromatic rectangle. In fact find an explicit expression which is quadratic in r instead of cr^2.
   Note that, in class we showed if a rectangle of size $[r+1, r^3]$ is r colored, then it contains a monochromatic rectangle. So it is reasonable to expect the same for a square of the same area. The point is again in fine tuning the idea of "pairing". Let $\chi : [1,N] \times [1,N] \to [1,r]$ be a coloring.
   Assume that there is no monochromatic rectangle. We count the monochromatic pairs of points: $\{(i,j),(i,k)\}$ in two different ways.
   (i) Show that the number of such pairs is at most $r\, C_{N,2}$ which denotes the binomial coefficient "N choose 2".
   (ii) Use the Cauchy-Schwarz inequality to show that for each fixed i, there are at least $\frac{n^2}{2r} - \frac{n}{2}$ monochromatic pairs of points: $\{(i,j),(i,k)\}$.

3. Generalize the above argument to find a monochromatic $k \times k$ grid in every r coloring of $[1,N] \times [1,N]$. That is find such a number $N = N(k,r)$ for each r, k. Translate the problem to bipartite graphs.

4. Show that there are at least 2 monochromatic AP's of length 3 in every 2 coloring of [1,9].

5. Show that if [1,325] is 2-colored, then there is a monochromatic AP's of length 3, by completing the following steps.
   (i) Show that in any block of 5 consecutive numbers $[M, M+4]$ there is an AP of length 3, such that the color of the first two terms are the same.
   (ii) A coloring of a block is defined by the 5-tuple of its colors. Show that if [1,325] is divided into 65 blocks, then there is an arithmetic progression of three blocks, say $B = [M, M+4]$, $B+d = [M+d, M+d+4]$, $B+2d = [M+2d, M+2d+4]$,such that the coloring of the first two blocks are the same.

(i3) Use part (i) to find an AP $A = (a, a+e, a+2e)$ such that the first two element have the same color, and look at the corresponding AP's $A+d,\ A+2d$ in the blocks $B+d,\ B+2d$. Conclude that there must be a monochromatic AP in the union of $A \cup\ A+d\ \cup\ A+2d$, where $A+d = \{x+d:\ x \in A\}$ is the translated AP.

(i4) * You can start thinking on how would you generalize this argument for 3 colors, in preparation for Van der Waerden's theorem.