

# VAN DER WAERDEN'S THEOREM

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An arithmetic progression (AP) of length  $k$  is a set of the form  $A = \{a, a+d, \dots, a+(k-1)d\}$  which we'll also denote by  $A = a + [0, k-1]d$ . A basic result due to Van der Waerden in 1927, says that if the natural numbers  $\mathbb{N}$  is colored with  $r$  colors, then there is a monochromatic AP of length  $k$ , for every  $k$ . The finite version is as follows.

**Theorem 1.** *Let  $r, k \in \mathbb{N}$  be given. Then there is a  $W(r, k) \in \mathbb{N}$  such that if  $N \geq W(r, k)$  and  $[1, N]$  is colored with  $r$  colors, then there is a monochromatic AP of length  $k$ .*

As before  $W(r, k)$  denotes the smallest such number. Let us discuss first the simplest non-trivial case  $r = 2, k = 3$ .

A set  $B$  of  $K$  consecutive numbers is called a block of size  $K$ . A translation of a block is a set  $B + d = \{x + d : x \in B\}$ . If  $c : [1, N] \rightarrow C$  is an  $r$  coloring of  $[1, N]$ , that is  $|C| = r$ , and if  $B \subset [1, N]$  is a block of size  $K$  then the  $K$ -tuple of colors  $(c(x) : x \in B)$  is called a derived coloring of  $B$ , and is denoted by  $c(B)$ . There are  $r^K$  possible coloring of a block.

Let us show that if  $[1, 325]$  is colored with 2 colors, then there is a monochromatic AP of length 3. Assume indirect, and divide  $[1, 325]$  into 65 blocks of length 5, say  $B_1, \dots, B_{65}$ . Each block can be colored  $2^5 = 32$  ways thus there are two blocks of the same color,  $B_i, B_{i+d}$ , such that  $1 \leq i < i + d \leq 33$ . Within the first 3 elements of  $B_i$  two of them  $a, a + e$  are of the same color, say Red. Then all elements of the set  $\{a, a + e, a + d, a + d + e\}$  are Red. Note that  $\{a, a + e, a + 2e\} \subset B_i$  and  $\{a + d, a + e + d, a + 2e + d\} \subset B_{i+d}$  thus both  $a + 2e$  and  $a + 2e + d$  must be Blue. Now, if  $a + 2e + 2d$  is Blue, then the AP  $\{a + 2e, a + 2e + d, a + 2e + 2d\}$  is Blue, while if it is Red then the AP  $\{a, a + e + d, a + 2e + 2d\}$  is Red. This contradicts our indirect assumption.

The bound 325 is extremely bad, in fact it is not hard to show that  $W(2, 3) = 9$ , see the proof in the textbook. In the above argument a key point was the existence of two AP's concentrated on  $a + 2e + 2d$ , whose first two elements were of the same color but different from each other. This idea works in the general case as we'll see in the next section.

## 1. N-TUPLES OF ARITHMETIC PROGRESSIONS.

A  $n$ -tuple of AP's with common base point  $a$ , is a set of the form  $A = (a + [0, k-1]d_1) \cup \dots \cup (a + [0, k-1]d_n)$ . We say it is *polychromatic* if each progression:  $a + [1, k-1]d_i$  ( $1 \leq i \leq n$ ) is monochromatic, but their colors are different from each other.

The key observation is that if all the translates  $A + d, \dots, A + (k - 1)d$  of such a set  $A$  are identically colored, then the set:  $A \cup (A + d) \cup \dots \cup (A + (k - 1)d)$  contains a polychromatic  $n + 1$ -tuple AP's or a monochromatic AP.

Indeed if the color of the base point  $a$  is the same as one of the AP's:  $a + [1, k - 1](d + d_i)$ , then  $a + [0, k - 1](d + d_i)$  is a monochromatic AP of length  $k$ . Otherwise it contains the  $n + 1$  tuple of AP's:  $a + [0, k - 1]d, a + [0, k - 1](d + d_1), \dots, a + [0, k - 1](d + d_n)$ , which is polychromatic.

The proof of Van der Waerden's theorem based on a double induction. The key element of the proof is the following

**Lemma 2.** *Let  $r, k \in \mathbb{N}$  and assume that  $W(r, k - 1)$  exists. Then for every  $n \in \mathbb{N}$  there exists a number  $W(r, n, k - 1)$  such that if  $N \geq W(r, n, k - 1)$  and if  $[1, N]$  is colored with  $r$  colors, then there is a polychromatic  $n$ -tuple of AP's of length  $k - 1$  or a monochromatic AP of length  $k$ .*

*Proof.* The case  $n = 1$  is trivial. Assume the claim is true for  $n - 1$ . Let  $N_1 = W(r, n - 1, k - 1)$ , then each block of length  $N_1$  contains either a monochromatic AP of length  $k$  or polychromatic  $n - 1$ -tuple of AP's of length  $k - 1$ . In the first case we're done. In the second case, let  $N_2 = 2W(r^{N_1}, k - 1)$ , and consider  $N_2$  consecutive blocks of length  $N_1$ . Since there are  $r^{N_1}$  colorings of each block, there must be an arithmetic progression of length  $k - 1$  of identically colored blocks:  $B + d, \dots, B + (k - 1)d$ . Since  $|B| = W(r, n - 1, k - 1)$  there is an  $n - 1$  tuple of AP's, say  $A$  in  $B$  which is polychromatic. Then the set  $A + d \cup \dots \cup A + (k - 1)d$  contains a polychromatic  $n$ -tuple AP's of length  $k - 1$ , or a monochromatic AP of length  $k$ , by the remark we made earlier.  $\square$

Now it is easy to prove Van der Waerden's theorem. Obviously  $W(r, 2) = r + 1$  for every  $r$ . By induction, assume that  $W(r, k - 1)$  exists for each  $r$ . Then by the above lemma  $W(r, n, k - 1)$  exists for each  $n$ . Take  $n = r$ , then for any  $r$  coloring of  $[1, W(r, r, k - 1)]$  there is a polychromatic  $r$ -tuple of AP's. The color of the base point must agree with the color of one of the  $k - 1$  progressions as there are only  $r$  colors. Then again, it contains a monochromatic AP of length  $k$ . Thus one can take:  $W(r, k) = W(r, r, k - 1)$ . The bounds obtained this way, even for  $W(2, k)$  are enormous, and can be described in terms of the so-called Ackermann function, see the exercises.

## 2. N-TUPLES OF PATTERNS IN $\mathbb{Z}^d$ .

We'll see below that the above argument works without any essential change, to obtain a higher dimensional version of Van der Waerden's result, which also has a more geometric flavor.

A *pattern* of size  $k$ , is simply a set of vectors  $F = \{v_1, \dots, v_k\}$  in  $\mathbb{Z}^d$ . A *homothetic copy* of  $F$  is a subset of  $\mathbb{Z}^d$  of the form  $x + tF = \{x, x + tv_1, \dots, x + tv_k\}$ .

Indeed  $x + tF$  is obtained from  $F$  by a dilation with  $t > 0$ , and translation with  $x$ . A result due to Gallai says that if  $\mathbb{Z}^d$  is colored with finitely many colors, then there is monochromatic homothetic copy of every pattern  $F$ . More precisely one has

**Theorem 3.** *Let  $k, r, d \in \mathbb{N}$ , and let  $F = \{v_1, \dots, v_k\}$  be a pattern in  $\mathbb{Z}^d$ . Then there exists an  $N = N(k, d, r, v_1, \dots, v_k)$  such that if  $[1, N]^d \subset \mathbb{Z}^d$  is colored with  $r$  colors, then there exists a monochromatic set of the form  $x + tF$ , for some  $x \in \mathbb{Z}^d$  and  $t \in \mathbb{N}$ .*

Note that Van der Waerden's theorem is a special case of this theorem where  $d = 1$  and  $v_j = j$ , whose most natural  $d$  dimensional analogue is the case  $F = [1, k]^d$ . However the proof given in the previous section works exactly the same way for Gallai's theorem as well. We'll give a sketch of the notions and main ideas, the details are left for you as an exercise.

An  $n$ -tuple of a pattern  $F = \{v_1, \dots, v_k\}$  with common base point  $x$ , is a set of the form:  $A = (x + t_1 F) \cup \dots \cup (x + t_n F)$ . We say that  $A$  is *polychromatic*, if the sets:  $\{x + t_i v_1, \dots, x + t_i v_k\}$  are all monochromatic, but their colors are different. Note that the base point  $x$  is not included in these sets. The key points in the proof are

- (i) Show that if the translates  $A + tv_1, \dots, A + tv_k$  are all identically colored, of such a polychromatic  $n$ -tuple  $A$ , then the set:  $A' = A \cup (A + tv_1) \cup \dots \cup (A + tv_k)$  contains either monochromatic set of the form  $x + tF$ , or a polychromatic  $n + 1$  tuple of the pattern  $F$ .
- (ii) Prove by induction on  $n$ , that if Gallai's theorem is true for every pattern  $F$  of size  $k - 1$ , then there exists an  $N = N(k, d, r, n, F)$  such that the following holds. If  $[1, N]^d$  is  $r$ -colored and  $F$  is a pattern of size  $k$ , then either there exists a monochromatic set of the form  $x + tF$ , or a polychromatic  $n$ -tuple of  $F$ .  
The point is that the sets:  $\{x + t_i v_1, \dots, x + t_i v_k\}$  can be written as  $x_i + t_i F'$  where  $F' = \{v_2 - v_1, \dots, v_k - v_1\}$  is a pattern of size  $k - 1$ .
- (i3) Use that if  $n \geq r$  then a polychromatic  $n$ -tuple of a pattern  $F$  contains a monochromatic set  $x + t_i F$  to finish the proof.

The interest in Gallai's theorem is in investigating the numbers  $N(k, d, r, F)$  for a specific pattern  $F$ . Also while the density version of Van der Waerden's theorem had a combinatorial proof already in 1975 by Szemerédi, and a Fourier analytic proof was given by Gowers in 1998, however for the density version of Gallai's theorem combinatorial proof was given only in 2003 by Gowers and Solymosi, and no proof exists yet, based on Fourier analysis. Such a proof would be highly desirable as it would lead far better bounds for the numbers  $N(k, d, r, F)$ . We'll investigate certain coloring and density questions for simple patterns such as isosceles right triangles later. Next we give yet another version of the proof of Van der Waerden's result mainly to introduce the notion:

### 3. N-DIMENSIONAL ARITHMETIC PROGRESSIONS.

An  $n$ -dimensional arithmetic progression (AP) of length  $k$  is a set  $A = \{a + s_1 d_1 + \dots + s_n d_n : 0 \leq s_i \leq k - 1, 1 \leq i \leq n\}$ . We shall use the notation  $A = a + [0, k - 1]d_1 + \dots + [0, k - 1]d_n$ . We call  $a$  the base point, and  $\{d_1, \dots, d_n\}$  the difference set of  $A$ .

Note that a 1-dimensional AP is just an ordinary AP, and an  $n$ -dimensional AP is just the sum of  $n$  ordinary AP's;  $A = A_1 + \dots + A_n$  where  $A_1 = a_1 + [0, k-1]d_1$  and  $A_i = [0, k-1]d_i$  if  $i > 1$ . The sum of two sets  $A$  and  $B$  is defined by  $A+B = \{a+b : a \in A, b \in B\}$ . If  $A$  is an  $n$ -dimensional AP then the set  $A' = A \cup (A+d) \cup \dots \cup (A+(k-1)d$  is an  $n+1$ -dimensional AP.

It is useful to associate to such a set  $A$  the  $n$ -dimensional cube  $[0, k-1]^n = \{(s_1, \dots, s_n) : 0 \leq s_i \leq k-1, 1 \leq i \leq n\}$ . Indeed, the arithmetic progressions corresponding to the diagonal lines of the faces of  $[0, k-1]^n$  will play a role in the proof.

An  $n$ -dimensional AP of length  $k$  is called *polychromatic* if the following holds; The colors of two elements

$$x = a + \sum_{i=1}^n s_i d_i \quad \text{and} \quad y = a + \sum_{i=1}^n t_i d_i$$

is the same, if for every  $1 \leq i \leq n$ , either  $s_i = t_i$  or  $s_i < t_i < k-1$ . Let us derive some simple observations

- (i) The  $n$ -dimensional AP of length  $k-1$ :  $A' = \{a + [0, k-2]d_1 + \dots + [0, k-2]d_n$  is monochromatic. More generally the  $m$ -dimensional AP's:  $A'_i = a + (k-1)(d_1 + \dots + d_{n-m}) + [0, k-2]d_{n-m+1} + \dots + [0, k-2]d_n$  are all monochromatic.
- (ii) If  $A$  is polychromatic of length  $k$  then either  $A$  contains a monochromatic AP of length  $k$  or the elements  $x_i = a + (k-1)(d_1 + \dots + d_i)$  ( $1 \leq i \leq n$ ) have distinct colors. Indeed, if  $i < j$  then consider the AP:  $A_{ij} = \{a + (k-1)(d_1 + \dots + d_i) + [0, k-1](d_{i+1} + \dots + d_j)\}$ . The first  $k-1$  elements of  $A_{ij}$  has same color as  $x_i$ , thus its last element  $x_j$  must have a different color.
- (i3) If  $A$  is a polychromatic AP of dimension  $n$  and length  $k$ , and if its translates  $A+d, \dots, A+(k-1)d$  are identically colored, then the set

$$A' = A \cup (A+d) \cup \dots \cup (A+(k-1)d)$$

is a polychromatic AP of dimension  $n+1$  and length  $k$ .

The proof of van der Waerden's theorem goes along the same lines as before. Assume that  $W(r, k-1)$  exists, then we show that there exists a number  $N = N(r, N, k)$  such that every  $r$  coloring of  $[1, N]$  contains either a monochromatic AP of length  $k$  or a polychromatic AP of dimension  $n$  and length  $k$ . The case  $n=1$  is immediate. By induction on  $n$ , assume that  $N_1 = N(r, n-1, k)$  exists. Consider  $N_2 = 2W(r^{N_1}, k-1)$  consecutive blocks of length  $N_1$ . Then there exists an arithmetic progression of blocks:  $B+d, \dots, B+(k-1)d$  which are identically colored. There is a polychromatic AP  $A \subset B$  of dimension  $n-1$  and length  $k$ . If  $n=r+1$  then such a set must contain a monochromatic AP of length  $k$  since otherwise the elements  $x_1, \dots, x_{r+1}$  defined in (ii), would have distinct colors which is not possible.