# THE CAUCHY-SCHWARZ INEQUALITY AND SOME SIMPLE CONSEQUENCES

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ABSTRACT. This note has been taken almost verbatim from existing notes of Alex Iosevich.

# 1. The Cauchy-Schwarz inequality

Let x and y be points in the Euclidean space  $\mathbb{R}^n$  which we endow with the usual *inner* product and norm, namely

$$(x,y) = \sum_{j=1}^{n} x_j y_j$$
 and  $||x|| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$ 

# The Cauchy-Schwarz inequality:

(1) 
$$|(x,y)| \le ||x|| ||y||.$$

Here is one possible proof of this fundamental inequality.

*Proof.* We start with the seemingly innocent observation that if  $a, b \in \mathbb{R}$ , then  $(a - b)^2 \ge 0$ and hence

$$(2) ab \le \frac{a^2 + b^2}{2}$$

Using inequality (2) we see that

$$\sum_{j=1}^{n} x_j y_j = \|x\| \|y\| \sum_{j=1}^{n} \frac{x_j}{\|x\|} \frac{y_j}{\|y\|} \le \frac{\|x\| \|y\|}{2} \sum_{j=1}^{n} \left( \frac{x_j^2}{\|x\|^2} + \frac{y_j^2}{\|y\|^2} \right) = \|x\| \|y\|. \qquad \Box$$

• Let 1 and define the exponent <math>p' by the equation  $\frac{1}{p} + \frac{1}{p'} = 1$ . Can you adapt the argument above to show that

$$|(x,y)| \le \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |y_j|^{p'}\right)^{1/p'}$$

*Hint:* Prove that  $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$  and use this in place of (2) in the argument above. • Can you extend (1) above and establish the Cauchy-Schwarz inequality on  $\mathbb{C}^n$  endowed with the usual Hermitian inner product  $(z, w) = \sum_{j=1}^{n} z_j \overline{w_j}$ ?

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### 2. Projections

2.1. Two dimensional case. Let  $B_N$  denote any collection of N points in  $\mathbb{R}^2$ . In this situation we define the following projections; for  $x = (x_1, x_2) \in \mathbb{R}^2$  we let

$$\pi_1(x) = x_2$$
 and  $\pi_2(x) = x_1$ .

We will now prove by a simple geometric argument that one off the two projection of  $B_N$ , that is  $\pi_1(B_N)$  or  $\pi_2(B_N)$ , must contain at least  $\sqrt{N}$  points.

Define  $\chi_A = 1$  if  $x \in A$  and 0 otherwise.

• Verify that  $\chi_{B_N}(x) \leq \chi_{\pi_1(B_N)}(x_2)\chi_{\pi_2(B_N)}(x_1)$ .

Using this fact it immediately follows that

$$\sum_{x \in \mathbb{R}^2} \chi_{B_N}(x) \le \sum_{x_1, x_2 \in \mathbb{R}} \chi_{\pi_1(B_N)}(x_2) \chi_{\pi_2(B_N)}(x_1) = \sum_{x_2 \in \mathbb{R}} \chi_{\pi_1(B_N)}(x_2) \sum_{x_1 \in \mathbb{R}} \chi_{\pi_2(B_N)}(x_1),$$

and hence

$$N = |B_N| \le |\pi_1(B_N)| \cdot |\pi_2(B_N)| \le [\max_{j=1,2} |\pi_j(B_N)|]^2,$$

as claimed. This argument is essentially just the following. Suppose that  $|\pi_1(B_N)| < \sqrt{N}$ , by definition this means that  $B_N$  contains less than  $\sqrt{N}$  columns. However, since  $|B_N| = N$ we know that one of these columns must contain more than  $N/\sqrt{N}$  points. We conclude therefore that either  $|\pi_1(B_N)| \ge \sqrt{N}$  or  $|\pi_2(B_N)| \ge \sqrt{N}$ .

2.2. Three dimensional case. This case is not quite so easy, however the Cauchy-Schwarz inequality will come to our rescue. We now let  $B_N$  denote any collection of N points in  $\mathbb{R}^3$  and define the following three projections; for  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$  we let

$$\pi_1(x) = (x_2, x_3), \ \pi_2(x) = (x_1, x_3) \ \text{and} \ \pi_3(x) = (x_1, x_2).$$

**Claim:**  $\max_{j=1,2,3} |\pi_j(B_N)| \ge N^{2/3}$ .

We shall in fact establish that

(3) 
$$|B_N| = \sqrt{|\pi_1(B_N)|} \sqrt{|\pi_2(B_N)|} \sqrt{|\pi_3(B_N)|},$$

from which the claim clearly follows.

• Verify that in this case  $\chi_{B_N}(x) \leq \chi_{\pi_1(B_N)}(x_2, x_3)\chi_{\pi_2(B_N)}(x_1, x_3)\chi_{\pi_3(B_N)}(x_1, x_2)$ .

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Using this fact and the Cauchy-Schwarz inequality we see that

$$\sum_{x \in \mathbb{R}^{3}} \chi_{B_{N}}(x) \leq \sum_{x_{1}, x_{2}, x_{3} \in \mathbb{R}} \chi_{\pi_{1}(B_{N})}(x_{2}, x_{3}) \chi_{\pi_{2}(B_{N})}(x_{1}, x_{3}) \chi_{\pi_{3}(B_{N})}(x_{1}, x_{2})$$

$$= \sum_{x_{1}, x_{2} \in \mathbb{R}} \chi_{\pi_{3}(B_{N})}(x_{1}, x_{2}) \sum_{x_{3} \in \mathbb{R}} \chi_{\pi_{1}(B_{N})}(x_{2}, x_{3}) \chi_{\pi_{2}(B_{N})}(x_{1}, x_{3})$$

$$\leq \left(\sum_{x_{1}, x_{2} \in \mathbb{R}} \chi_{\pi_{3}(B_{N})}^{2}(x_{1}, x_{2})\right)^{1/2} \left(\sum_{x_{1}, x_{2} \in \mathbb{R}} \left(\sum_{x_{3} \in \mathbb{R}} \chi_{\pi_{1}(B_{N})}(x_{2}, x_{3}) \chi_{\pi_{2}(B_{N})}(x_{1}, x_{3})\right)^{2}\right)^{1/2}$$

Now it is easy to see that the first sum above, namely

$$\sum_{x_1,x_2 \in \mathbb{R}} \chi^2_{\pi_3(B_N)}(x_1,x_2) = |\pi_3(B_N)|,$$

while the second sum is simply

$$\sum_{x_1,x_2\in\mathbb{R}} \left( \sum_{x_3\in\mathbb{R}} \chi_{\pi_1(B_N)}(x_2,x_3)\chi_{\pi_2(B_N)}(x_1,x_3) \right)^2$$
  
=  $\sum_{x_1,x_2\in\mathbb{R}} \sum_{x_3\in\mathbb{R}} \sum_{x'_3\in\mathbb{R}} \chi_{\pi_1(B_N)}(x_2,x_3)\chi_{\pi_2(B_N)}(x_1,x_3)\chi_{\pi_1(B_N)}(x_2,x'_3)\chi_{\pi_2(B_N)}(x_1,x'_3)$   
 $\leq \sum_{x_1,x_2\in\mathbb{R}} \sum_{x_3\in\mathbb{R}} \sum_{x'_3\in\mathbb{R}} \chi_{\pi_1(B_N)}(x_2,x_3)\chi_{\pi_2(B_N)}(x_1,x'_3)$   
=  $|\pi_1(B_N)| \cdot |\pi_2(|B_N)|,$ 

this establishes (3).

• Let  $\Omega$  be a convex set in  $\mathbb{R}^3$ , show that

$$\max_{j=1,2,3} |\pi_j(\Omega)| \ge |\Omega|^{2/3}.$$

Can you prove the stronger statement that  $|\Omega| \leq |\pi_1(\Omega)|^{1/2} \cdot |\pi_2(\Omega)|^{1/2} \cdot |\pi_3(\Omega)|^{1/2}$ ?

• Can you generalize (3)? By which I mean replace three dimensions with n and projections onto coordinate planes by projections onto k-dimensional coordinate planes, with  $1 \le k \le n-1$ . Formulate what the correct generalization is and then try to prove it.

### 3. Incidences

Consider the set of N lines and N points in the plane. Define an incidence to be a pair  $(p, \ell)$ , where p is one of our points,  $\ell$  is one of our lines, and the point p lies on  $\ell$ . Let I(N) denote the total number of incidences determined by a given set of N points and a given set of N lines. To make our lives a little easier let us further assume that every one of our points lies on at least one of our lines and every one of our lines contains at least one of our points. How large can I(N) be?

# Szemerédi-Trotter Incidence Theorem.

$$I(N) \le CN^{4/3}.$$

This result is sharp in the sense that one can construct a set of N points and N lines such that the number of incidences is approximately  $N^{4/3}$ , up to a constant.

We will not prove this result here, however using the Cauchy-Schwarz inequality and matrices we will establish the weaker upper bound

$$I(N) \le \sqrt{2}N^{3/2}.$$

We define a matrix A as follows. Enumerate our N points and our N lines. Let  $a_{ij} = 1$  is the *i*th point is on the *j*th line, and 0 otherwise.

• Observe that if j and j' are fixed, with  $j \neq j'$ , then

$$a_{ij} \cdot a_{ij'} = 1$$

for at most one value of i.

It follows from the Cauchy-Schwarz inequality that

$$I(N) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} = \sum_{i=1}^{N} \left( \sum_{j=1}^{N} a_{ij} \right) \cdot 1 \le \sqrt{N} \left( \sum_{i=1}^{N} \left( \sum_{j=1}^{N} a_{ij} \right)^2 \right)^{1/2}$$

However

$$\sum_{i=1}^{N} \left(\sum_{j=1}^{N} a_{ij}\right)^{2} = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{j'=1}^{N} a_{ij} a_{ij'} = \sum_{i} \sum_{j} \sum_{j'\neq j} a_{ij} a_{ij'} + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^{2} \le N^{2} - N + N^{2} \le 2N^{2},$$

from which it (4) follows.

- Show that the estimate  $I(N) \leq CN^{3/2}$  we just obtained for points and lines in the plane is actually best possible for points and lines in  $\mathbb{Z}_p^2$ .
- Let  $D_N$  denote the number of distinct distances between a given set of N points in the plane. Use (4) to show that  $|D_N| \ge C\sqrt{N}$ . It is conjectured that  $|D_N| \ge C\frac{N}{\log N}$ .
- If we now let  $D_N$  denote the number of distinct distances between a given set of N points in  $\mathbb{R}^n$ . Show that in this case  $|D_N| \ge CN^{1/n}$ . It is conjectured that when  $n \ge 3$  that  $|D_N| \ge CN^{2/d}$ .
- Prove that N points and N spheres of the same radius in  $\mathbb{R}^n$ ,  $n \ge 4$ , can have  $N^2$  incidences. Use what we have learned to show that when n = 2 the number of incidences is  $\le CN^{3/2}$ . What can you say about the case where n = 3.