BERTRAND'S POSTULATE

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Bertrand's Postulate. *For every $n > 1$, there is a prime number p such that $n < p < 2n$.*

Our proof will be by contradiction. We will assume that there are no primes between n and $2n$ and conclude that the binomial coefficient $\binom{2n}{n}$ is then smaller than should be.

1. Preliminaries

The following observations are the key to our argument.

Theorem 1. *Let $n \geq 2$ be an integer, then*

$$\prod_{p \leq n} p < 4^n,$$

where the product on the left has one factor for each prime $p \leq n$.

Theorem 2. Legendre's Theorem. *The number $n!$ contains the prime factor p exactly*

$$\sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor$$

times. Here $\lfloor x \rfloor$ denotes the floor function of x.

Proof. Exactly $\lfloor n/p \rfloor$ of the factors of $n!$ are divisible by p, while exactly $\lfloor n/p^2 \rfloor$ of the factors of $n!$ are divisible by p^2,... $\square$

Corollary 3. *The binomial coefficient $\binom{2n}{n}$ contains the the prime factor p exactly*

$$s_p = \sum_{k \geq 1} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right) \leq \max\{r : p^r \leq 2n\}$$

times.

As a consequence of the identity in Corollary 3 we obtain the following key facts.

Key Facts. *If s_p is the largest power of p that divides $\binom{2n}{n}$, then*

(i) $p^{s_p} \leq 2n$
(ii) *If $\sqrt{2n} < p$, then $s_p \leq 1$*
(iii) *If $2n/3 < p \leq n$, then $s_p = 0$.*

2. Proof of Bertrand's Postulate

We start by assuming that there are no primes p with $n < p < 2n$. It follows from the **key facts** and Theorem 1 that

$$\binom{2n}{n} = \prod_{p \leq 2n/3} p^{s_p} \leq \prod_{p \leq \sqrt{2n}} p^{s_p} \cdot \prod_{p \leq 2n/3} p < (2n)^{\sqrt{2n}/2-1} 4^{2n/3}.$$

If we now use the fact that

$$2n \binom{2n}{n} \geq \sum_{k=0}^{n} \binom{2n}{k} = 4^n,$$

we can conclude that

$$4^{2n/3} < (2n)^{\sqrt{2n}}.$$

Taking logs of both sides and substituting $n = 2^{2k+1}$, for some integer k, we see that this necessarily implies

$$2^k < \frac{3}{2}(k+1),$$

which is clearly false whenever $k \geq 3$. This proves Bertrand's postulate for $n \geq 2^7 = 128$.

For the cases where $n < 128$ it suffices to check that

$$2, 3, 5, 7, 13, 23, 43, 83, 163$$

is a sequence of primes, where each is smaller than twice the previous one.

3. Proof of Theorem 1

The proof is by induction over n. For $n = 2$ we have $2 < 4^2$, which is certainly true. This provides a basis for the induction. Let us now assume that the statement has be proven for all integers small than n. We may assume that n is odd [why?], say $n = 2k+1$. Now since $\prod_{k+1<p\leq n} p$ is a divisor of $\binom{n}{k+1}$, we obtain

$$\prod_{p \leq n} p = \prod_{p \leq k+1} p \cdot \prod_{k+1 < p \leq n} p < 4^{k+1} \binom{n}{k+1} < 4^{k+1} 2^{n-1} = 4^n,$$

as required. Here we used the inductive hypothesis and the fact that

$$2\binom{2k+1}{k+1} = \binom{2k+1}{k} + \binom{2k+1}{k+1} \leq \sum_{j=0}^{2k+1} \binom{2k+1}{j} = 2^{2k+1}.$$

4. Exercises

1. Carefully verify Theorem 2 and how each of the **key facts** follow from it.
2. Fill in any details which are missing in the two proofs above, the presentation was admittedly a little brief.