Lecture 9

Relations equivalent to the Prime Number Theorem

It was observed over 200 years ago by Legendre and Gauss (independently) that the density of primes around x was $(\log x)^{-1}$, precisely they postulated the following:

<u>Prime Number Theorem</u>: If $\pi(x) := \sum_{p \le x} 1$, then as $x \to \infty$

$$\pi(x) \sim \frac{x}{\log x}$$

Gauss observed that an even better approximation to $\pi(x)$ is given by the integral

$$Li(x) := \int_2^\infty \frac{dt}{\log t}.$$

<u>Exercise ①</u>: Show that $Li(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$.

Later, Chebyshev realized (and we have seen) that it is simpler to count the primes p with the weight $\log p$, so he investigated

$$\theta(x) := \sum_{p \le x} \log p$$

rather than $\pi(x)$. It is still more convenient to evaluate the average of the von Mangoldt function

$$\psi(x) := \sum_{n \le x} \Lambda(n).$$

Theorem 1

$$\psi(x) \sim x \overset{(*)}{\iff} \theta(x) \sim x \iff \pi(x) \sim \frac{x}{\log x}$$

Proof: Equivalence (*) follows immediately from the fact that

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \underbrace{\sum_{p \le x} \log p}_{=:\,\theta(x)} + \sum_{k \ge 2} \sum_{p \le x^{1/k}} \log p$$

$\le \psi(\sqrt{x}) \le 2\sqrt{x}\log x$

Only $\frac{\log x}{\log 2}$ terms

$$= \theta(x) + O(\sqrt{x}(\log x)^2)$$

Partial summation gives that

$$\theta(x) = \sum_{p \le x} \log p = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\,dt$$

Since splitting integral at $\sqrt{x}$ shows $\int_2^x \frac{1}{\log t}dt \ll \frac{x}{\log x}$.

and hence if $\pi(x) \sim \frac{x}{\log x}$, then $\theta(x) \sim x + O\left(\int_2^x \frac{1}{\log t}dt\right) \sim x$.

Partial summation also shows that

$$\pi(x) = \sum_{p \le x} \log p \frac{1}{\log p} = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2}dt$$

Since splitting integral at $\sqrt{x}$ shows $\int_2^x \frac{1}{(\log t)^2}dt \ll \frac{x}{(\log x)^2}$.

and hence if $\theta(x) \sim x$, then

$$\pi(x) \sim \frac{x}{\log x} + O\left(\int_2^x \frac{1}{(\log t)^2}dt\right) \sim \frac{x}{\log x}$$

$\square$

We saw in Supplement 2 that the von Mangoldt and Möbius functions are related by

$$\Lambda = \mu * \log \iff \Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right)$$

[Since $1 * \mu(n) = \sum_{d|n} \mu(d) = \delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$ & $\log 1 = 0$

it follows that

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d = -1 * \mu\log$$

and hence that

$$-\mu * \Lambda = \mu\log.$$

We'll use this later.]

It is therefore natural to ask if the average value of $\mu(n)$ has a simple interpretation in terms of the asymptotic behaviour of the function $\psi(x)$ (and hence also $\theta(x)$ & $\pi(x)$).

Landau, in 1909, provided a complete answer to this question:

Theorem 2: $\psi(x) \sim x \iff M(x) := \sum_{n \le x} \mu(n) = o(x)$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$\frac{\psi(x) - x}{x} \to 0 \qquad \frac{1}{x}\sum_{n \le x} \mu(n) \to 0$$

$$(\text{as } x \to \infty) \qquad (\text{as } x \to \infty).$$

Proof of Theorem 2

$M(x) = o(x) \Rightarrow \psi(x) - x = o(x)$:

Since $\psi(x) - x = \sum_{n\le x} (\Lambda(n) - 1) + O(1)$ it suffices to show that

$$\sum_{n\le x} (\Lambda(n)-1) = o(x) .$$

Since $\Lambda = \mu * \log$ & $1 = \mu * \tau$ (since $1*1 = \tau$)

$$\Rightarrow \Lambda - 1 = \mu * (\log - \tau) = \mu * f - 2\gamma\delta \qquad (*)$$

where $f(n) := \log n - \tau(n) + 2\gamma$

[Recall (from Supplement 3) that

$$\Delta(x) := \sum_{n\le x} f(n) = O(\sqrt{x}) .$$]

From $(*)$ it follows that

$$\sum_{n\le x} (\Lambda(n)-1) = \sum_{n\le x} \mu * f(n) \underbrace{-2\gamma}_{\text{clearly } o(x)} .$$

Apply the Dirichlet Hyperbola method we obtain, for each $y \geq 2$,

$$\sum_{n\le x} \mu * f(n) = \sum_{md\le x} \mu(d) f(m) = \sum_{d\le x/y} \mu(d)\Delta\left(\frac{x}{d}\right) + \sum_{m\le y} f(m) M\left(\frac{x}{m}\right) - \Delta(y) M\left(\frac{x}{y}\right)$$

Since

$$\frac{1}{x}\sum_{d\le x/y}\mu(d)\Delta\left(\frac{x}{d}\right) \ll \frac{1}{x}\sum_{d\le x/y}\sqrt{\frac{x}{d}} \ll \frac{1}{\sqrt{y}}$$

Supplement 3

and for any fixed y

- $$\limsup_{x\to\infty}\frac{1}{x}\sum_{m\le y} f(m)M\left(\frac{x}{m}\right) \le \sum_{m\le y}\frac{f(m)}{m}\limsup_{x\to\infty}\frac{M(x/m)}{x/m} = 0$$

- $$\lim_{x\to\infty}\frac{\Delta(y)M(x/y)}{x} = \frac{\Delta(y)}{y}\lim_{x\to\infty}\frac{M(x/y)}{x/y} = 0$$

it follows that

$$\limsup_{x\to\infty}\left|\frac{1}{x}\sum_{n\le x}\mu * f(n)\right| \ll \frac{1}{\sqrt{y}} .$$

Since y can be chosen arbitrary large it follows that

$$\sum_{n\le x}(\Lambda(n)-1) = o(x)$$

as required.

<u>$\psi(x)-x = o(x) \Rightarrow M(x) = o(x)$:</u>

Since $\sum_{n\le x}(\log x - \log n) = x\log x + O(\log x) - x\log x + O(x) = O(x)$

it follows that $M(x)\log x = \sum_{n\le x}\mu(n)\log n + O(x)$

and that it suffices to estabish that $\sum_{n\le x}\mu(n)\log n = o(x\log x)$.

Since

$$\mu \log = -\mu * \Lambda = \mu * (1-\Lambda) - \delta$$

it follows that

$$1 + \sum_{n \leq x} \mu(n) \log n = \sum_{n \leq x} \sum_{d \mid n} \mu(d) \left(1 - \Lambda\left(\tfrac{n}{d}\right)\right) = \sum_{d \leq x} \mu(d) \left(\left[\tfrac{x}{d}\right] - \psi\left(\tfrac{x}{d}\right)\right).$$

We know that for any $\varepsilon > 0$, there is a large number $C = C(\varepsilon)$ such that if $d \leq x/C$, then $\left| \psi\left(\frac{x}{d}\right) - \left[\frac{x}{d}\right] \right| \leq \varepsilon \frac{x}{d}$. Thus

$$\left| \sum_{d \leq x/C} \mu(d) \left(\left[\tfrac{x}{d}\right] - \psi\left(\tfrac{x}{d}\right)\right) \right| \leq \sum_{d \leq x/C} \varepsilon \frac{x}{d} \ll \varepsilon x \log x .$$

The remaining range we treat trivially:

$$\sum_{\frac{x}{C} < d \leq x} \mu(d) \left(\left[\tfrac{x}{d}\right] - \psi\left(\tfrac{x}{d}\right)\right) \ll \sum_{\frac{x}{C} < d \leq x} \frac{x}{d} \ll x \log C .$$

Since ε can be taken arbitrarily small, we see that

$$\lim_{x \to \infty} \frac{1}{x \log x} \sum_{n \leq x} \mu(n) \log n = 0$$

i.e. $$\sum_{n \leq x} \mu(n) \log n = o(x \log x)$$

as required. $\square$